Proceedings of the Tenth ERME Topic Conference (ETC 10) on Mathematics Education in the Digital Age (MEDA), 16-18 September 2020 in Linz, Austria

Ana Donevska-Todorova, Eleonora Faggiano, Jana Trgalova, Zsolt Lavicza, Robert Weinhandl, Alison Clark-Wilson, Hans-Georg Weigand

10th ERME TOPIC CONFERENCE (ETC10)

Mathematics Education in the Digital Age (MEDA)

16-18 September 2020 in Linz, Austria

PROCEEDINGS

Edited by:

Ana Donevska-Todorova, Eleonora Faggiano, Jana Trgalova, Zsolt Lavicza, Robert Weinhandl, Alison Clark-Wilson, and Hans-Georg Weigand

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Introduction

The fifth ERME Topic Conference for Mathematics Education in the Digital Age (MEDA), held in September 2018 in Copenhagen was inspired by the contributions to the Thematic Working Groups 15 and 16 at CERME 10 in Dublin, which highlighted the diversity of current research and its overlaps with other TWG themes. MEDA was an interdisciplinary, multifaceted collaboration that brought together participants who would normally attend a range of CERME Thematic Working Groups to provide the opportunity for further in-depth discussion and debate. The successful conference experience resulted in an intensive communication and collaboration, which continued through the collegial work that culminated in the publication of a post-conference book in the ERME Series published by Routledge. Moreover, inspired by the contributions to the Thematic Working Groups 15 and 16 in the last CERME 11 in Utrecht, the second conference, MEDA2, provides the opportunity for further in-depth discussion and debate. In particular, MEDA2 is of interest to the following TWGs:

TWG 18	Mathematics Teacher Education and Professional Development
TWG 22	Curricular Resources and Task Design in Mathematics Education
TWG 21	Assessment in Mathematics Education
TWG17	Theoretical Perspectives and Approaches in Mathematics Education Research

The conference welcomed theoretical, methodological, empirical or developmental papers (8 pages) and poster proposals (2 pages) in relation to the following themes:

Theme 1:	Mathematics teacher education and professional development in the digital age
Theme 2:	Mathematics curriculum development and task design in the digital age
Theme 3:	Assessment in mathematics education in the digital age
Theme 4:	Theoretical perspectives and methodologies/approaches for researching mathematics education in the digital age

Theme 1 - Mathematics teacher education and professional development in the digital age

- The specific knowledge, skills and attributes required for efficient/effective mathematics teaching with digital resources, to include digital mathematics resources, which we define as resources that afford or embed mathematical representations that teachers and learners can interact with by acting on objects in mathematical ways.
- The design and evaluation of mathematics teacher education and professional development programmes that embed the knowledge, skills and attributes to teach mathematics with digital resources.

Theme 2 - Mathematics curriculum development and task design in the digital age

- The design of resources and tasks (e.g. task features, design principles and typologies for e-textbooks);
- The evaluation and analysis of resources and tasks (e.g. determining quality criteriafor curricular material, resources and methods of analysis);

- The interactions of teachers and students with digital curriculum materials (e.g. appropriation, amendment, re-design), both individually or collectively. This includes the consideration of teacher learning/professional development in their work with digital resources.

Theme 3 - Assessment in mathematics education in the digital age

- New possibilities of assessment (formative, summative, etc.) in mathematics education brought by digital technology
- Use of digital technology to support students to gain a better awareness of their own learning
- Assessment of learners' mathematical activity in digital environment

Theme 4 - Theoretical perspectives and methodologies/approaches for researching mathematics education in the digital age

- Theories for research on technology use in mathematics education (e.g. design theories, prescriptive theories, theories linking research and practice, theories addressing the transfer of learning arrangements to other learning conditions etc.)
- The linking of theoretical and methodological approaches and the identification of conditions for productive dialogue between theorists, within mathematics education and beyond (e.g. developing collaborative research with educationalists, including teachers and educational technologists).

The conference particularly welcomed contributions linking some of these four themes at any level of mathematics education: pre-school, primary, lower- and upper-secondary or tertiary.

The Conference Proceedings of the 10th ERME Topic Conference MEDA 2020 are rich in the variety of content-formats and are therefore structured in two parts. They include the contributions of the plenary speakers and all the 67 reviewed and accepted submissions from participants, organised as four chapters according to the aforementioned themes.

Ana Donevska-Todorova, Eleonora Faggiano, Jana Trgalova, Zsolt Lavicza, Robert Weinhandl, Alison Clark-Wilson, and Hans-Georg Weigand

Table of Contents

Plenary Talks
by

Mariam Haspekian

Teaching practices in digital era:
some theoretical and methodological perspectives

Paola Iannone

Assessment of mathematics in the digital age:
The case of university mathematics

Birgit Pepin

Quality of (digital) resources for curriculum innovation

Teaching practices in digital era: some theoretical and methodological perspectives

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In the spirit of MEDA 2 whose objective is to interlink the CERMEs' topic working group (TWG) on digital technologies with other TWG, the MEDA-2 conference organizers wished a lecture on the theme of theories. The aim is to make the audience think about theories and digital tools, while giving questions for the discussions.

INTRODUCTION CHOICES AND REASONS

The topic of the "theories" especially relates to the Theme 4 of the conference: *Theoretical Perspectives and Methodologies/Approaches to Conduct Research in Mathematics Education in the Digital Age*, which covers two issues: *Theories to conduct research on using technology in mathematics education*; *Linking theoretical and methodological approaches as well as identifying conditions to create productive dialogue between theorists, as part of mathematics education and beyond*". The subject is thus vast and requires making choices.

Pointing the only context of *digital* age and restraining to theories still leads to a huge body of issues, about which one only person can hardly be well-informed.

Following up investigations on student learning, numerous research on teachers and classroom practices have emerged, then considerably developed over time. Among this teacher-oriented research, a growing body focuses on issues specifically related to technologies. It is therefore interesting to stop now and take stock.

Presented in *Annex* 1, a more detailed retrospective panorama, based on Drijvers et al. 2010 historical overview, further enlightens the reasons of this choice.

Hence, I propose here to limit the *theory* issue to research *on teachers and mathematics teaching practices* in digital age (TPDA in the next [1]), with the question: *what can a focus on theory bring to research on TPDA?*

THEORIES, PERSPECTIVES, PHILOSOPHY

The concepts in ME research are embedded in flourishing general or specific frames: Activity theories (Vygotski 1978, Leontev 1984…), Cultural-Historical Activity Theory (Engeström 2001), PCK (Shulman 1986), Balacheff's cKc (1995), Instrumental Approach (IA) (for a recent overview, see Artigue 2020 in the ICMI Awardees MOOC *AMOR [2]*), Documentary Approach (Gueudet & Trouche 2009), Pedagogical Technology Knowledge (Thomas and Hong 2005), TPACK (Mishra & Koehler 2006), Structuring Features of Classroom Practice framework (Ruthven 2007), Theory of Communities of practice (Wenger 1998), Framework of Teacher-Curriculum relationship (Remillard 2005), Theory of Semiotic Mediation (Bartolini Bussi & Mariotti 2008)… This non-exhaustive list seeks to show the diversity in topics, scales and dates. Besides, some approaches derive from others: IA is based on Chevallard's Anthropological Theory (2006) and on that of Verillon & Rabardel

(1995) in psycho-ergonomics. This latter itself is partly based on Activity theories… We could continue to "climb" back over these frames to reach main perspectives dealing with more general ideas of education, learning, cognition: Lakoff & Nunes (2000) Embodied mathematics, Piaget's (1980) constructivist perspective on child development, socio-constructivist theory (Vygotski 1978), Bloom's (1956) psychological taxonomy, Skinner's (1953) application of behaviorist perspective… The step further reaches the underlying philosophical foundations on what knowledge ultimately is, how it is acquired, transmitted, with for instance the innate/acquired debates... The dialectic materialism, a philosophical background of Radford's (2019) Theory of Objectivation or Bachelard's (1938) concept of epistemological obstacle in philosophy of science, used in Brousseau's theory foundations, are two examples.
The level that interests us here is the first one: what issues does the focus on this different theories panorama bring up for research on TPDA? To answer, we can examine: 1. the theories used in the research within the CERME TWG related to TPDA? 2. the papers dealing with TPDA within the TWG on theories (TWG17).

LOOKING AT THE THEORIES IN THE TWGS RELATED TO TPDA

From a methodological viewpoint, the idea would be to see which theories are jointly used, how and why. Focussing for instance on the 2 last CERME (2017, 2019), we can list the theories used in the TWGs related to TPDA i.e. **TWG15** *Teaching mathematics with resources and technology* **TWG18** *Mathematics teacher education and PD*, **TWG19** Mathematics teachers and classroom practice*s*, and **TWG20** Mathematics teacher knowledge, beliefs, and identity, but also **TWG16** *Learning mathematics with technology and other resources*, and complete with papers that review CERMEs' groups. The work is quite large. I have carried it out in detail for TWG15, and in a more global view for the other TWGs 2017 and 2019. From this, raises first a landscape of theories that we can cluster following the purpose for which the theory is invoked. Extending the categories mentioned in the MEDA2 announcement, we thus describe the clusters that we have obtained by distinguishing:

Theories that are used for research on technology use in mathematics education:

- o theories to design, prescriptive theories (offering design directions, investigative strategies)
- o theories addressing the link research and practice
- o theories addressing the transfer of research learning design to usual classroom conditions, or more generally the transfer of learning arrangements to other learning conditions
- o theories to understand, describe and model practices

Theories that are used to address collaborative dimension, identify conditions for productive dialogue between:

- o actors (e.g. developing collaborative research with educationalists, including teachers and educational technologists)
- o research fields (theorists within mathematics education and beyond).

Beyond this classification, to advance in our question, it would be interesting to revisit these theories by the characteristics/dimensions they focus on. For instance, which theories foreground the institutional dimension so important in questions of technological integration? The next section presents the investigation of TWG 15-

2017. The analyses lead to some reflections and questions for research on the TPDA.

Analyses of the theories used in CERME 20017 TWG 15

The Table 1 synthesizes my review: TWG15 included 26 contributions: the introduction of the group, 19 papers, 6 posters (mentioned below with the letter "P"). The numbers are those of the proceedings [3]. The panorama obtained shows we are moving towards a more coherent, yet not unified, theoretical backdrop, with a limited set of specific theories frequently used, sometimes completed by concepts or theories less frequent in the field. Have-we fulfilled the request made in the CERME 4 technology group (Barzel et al., 2005, p. 929) for a more systematic approach "*which combines various theories focusing on each of these subsystems (didactics, instrumental approach, situated and distributed cognition, community of practice)*"?

	Alone	With others
TPACK (Koehler & Mishra, 2005) (based on Shulman 1986 PCK)	**03** (with notion of "attitude")	**04** (+ Situated Abstraction, Noss & Hoyle 1996) **09** (+ Valsiner's three zones (1997)
I.A (Artigue, 2002, ; Guin & Trouche, 2002, Lagrange, 2002…) (based on ATD and cognitive ergonomics Rabardel 2002)	**13**	**01** (+ Double Approach, (Robert & Rogalski, 2005) **19** (+MTD (Meta Didactical Transposition) Arzarello et al. 2014 + Connectivism, Siemens 2004; Downes 2012) **21P** (+ ATD)
Documentational Approach (Gueudet, Trouche, 2009) (based on I.A.)	**25P**	**10** (+ Teaching Triad, Jaworski, 1994) **13** (+Social Creativity, Boundary Crossing) **26** (+ MTD (Arzarello et al., 2014)+communities of practice, Wenger, 1998)
Structuring Features of Classroom Practice (Ruthven, 2009)	**06** **22P** **24P**	
ATD (Chevallard, 1985)	**02** (extended with in/ outsourcing) **18** (with references to didactics of algebra)	
Others	**07** ACOT steps (Dwyer et al., 1994) **11** **20P** teachers' professionalism (Dale 2003) and models for action research (Asiale et al., 1996, Borba & Skovsmose, 2004) **23P** references to programs dealing with automatic theorem proving (geometry)	**05** various references to analyze tasks **08** several references to barriers of teachers' technology integration (+ TPACK to design PD not to analyse) **12** (Half-baked microworlds, Kynigos 2007) + Social Creativity and Communities of Interest (Fischer 2005; 2014) **14** (assessment) **16** (flipped classroom, Abeysekera and Dawson, 2015) **17** Semiotic representations (Duval, Janvier), semiotic bundle (Arzarello and Robutti, 2004)

Table 1. theories in the TWG 15 of CERME 2017 [4]

The last line "Others" reveals there is still a certain fragmentation. This is reinforced by the other lines if we look them more finely, not only quantitatively (how much are used?) but qualitatively: *how/ why* are they used? This second stage overview (Table 2) shows that theories are at times used as they are, or extended, or still associated. Besides, they are used for objectives of different nature. The landscape then seems to go all directions, even more if we add to this overview the reflexions cited in the introductive paper of the group (Clark-Wilson et al. 2017). The issues discussed among the members overflow, raising a huge variety of topics, from the acknowledgement on the need of multi-perspectives understanding, to the attention on digital assessment in mathematics. Certainly, among the various topics addressed, that of the technology integration comes more frequently. Yet, this latter is dealt so differently according to researchers that it still not represents a point of regularity. The nature of theories used, the ways they are, and the reasons why are different.

The TPACK (Koehler & Mishra 2005) frame is used for **analyzing** large-scale professional development (PD), but also for **designing** PD courses, not for **analyzing** data. Thus, the same theory has somewhat a different status there, it is a support for the design of the experiment.
We observe that it is also combined with the frame of *Valsiner's three zones* (Valsiner, 1997) to investigate how a tool (GeoGebra) is introduced in various mathematics tasks.
The *Instrumental Approach* (Artigue 2002; Guin & Trouche 2002) is used along with the *Documentational Approach* (Gueudet et al., 2012) to describe the teachers' **collective processes** in the use of a platform to plan their lessons.
Another paper also uses this IA and DA combination but adds a third frame: the *Teaching triad* (Jaworski, 1994), for the **collection** and **analysis** of data on **teachers' considerations** when implementing tasks in mathematics lessons.
ATD (Chevallard 1985) is used to value if the technological tool is applied in a way that is consistent with **an epistemological analysis of the top**ics.
In another paper, it is used with the suggested addition of the concepts of *out/in-sourcing*, used as metaphors within the dialectics of tool and content in the planning of teaching, to support teachers' reflection on crucial **choices between instrumented and non-instrumented praxeologie**s when planning their use of technology in mathematics lessons.
Some researchers **extend** the Structuring Features of Classroom Practice framework (Ruthven, 2009), with the addition of a new (sixth) structuring feature to capture teachers' **knowledge related to their students' attitudes and behaviors** with technology. Other use it as it is, to analyze **teachers' rationales for technology integration** in the mathematics classroom.
To analyze the **integration of technolog**y in teachers' practices, others call upon the *Ergonomic theoretical approach* (Robert & Rogalski, 2005).

Table 2. A qualitative overview of the theory use in the TWG 15 of CERME 2017

What reflections and questions does this work bring for future research on TPDA? To answer, this only opening work should be followed by a similar review of the other CERMEs' groups related to TPDA: **16**, **18**, **19** and **20.** This was beyond the scope of the request, yet, the study of the TWG15 already raises some reflections and questions. Below, I present them, taking a perspective broader than CERME.

Some reflections and questions from this review of the technology group

From the reviewing work described above, we draw several reflections and questions: that we grouped under 3 topics: **1.** The importance of the "Networking issue", that offers the possibility for researchers to share theoretical constructs. **2.** The question of "Societal dynamics", that relates to the future innovations (new tools, new interaction forms, new types of resources, artificial intelligence, big data…) but more of all, to the constant moving character of our society, and to the fastness of these moves; and **3.** The issue of the "Theory-practices links", that addresses some challenges and questions raised by the field of the PD. Due to the space constraints, these 3 topics and the questions raised are detailed in the *Annex* 2.

Some methodological perspectives to continue

To answer the question *what can a theory focus bring to TPDA*?, the study initiated above could be furthered for a better view of the state of the art, by a similar methodology applied to the study of the TWG related to TPDA (15 to 20) of the two (or more) last CERME, then by extending it to a more systematic literature review. This can be based on the following questions: What do the theories chosen for this theme tell us about this theme? What do the very choices of theories, the theoretical constructions themselves tell us about this theme? Three axes could be questioned: **1.** On a epistemological but also cultural axis: why these theories? what models are made of teaching/ (ou pedagogical?) practices? what aspects are explored? How is considered the specificity of the "digital" context? **2.** On a dynamic or "developmental" axis of the theories: how do they evolve? On which dimensions are they enriched and on which dimensions do they encounter obstacles? Which constructs are forsaken and why? **3.** On a "networking of theories" axis: how do these theoretical frames articulate, complete or oppose, contrast each other?

LOOKING AT RESEARCH ON TPDA THROUGH THE LENSE OF THE THEORIES AND THEIR NETWORKING

What are combinations, filiations, complementarity or on the contrary oppositions between the theories seen above? The need of networking theories emerged at CERME 4 in 2005 and was explored in TWG17 of the ensuing CERME conferences. The questions that multiplicity of theories arises, addressed in the "networking" field, apply well to the TPDA theme here: why so multiple theoretical developments? Is it due to communication strains among various native languages? (see Bikner-Ahsbahs & Prediger 2014 or the TWG17 also showing examples of the vocabulary barrier [5]), or cultural aspects? (the various educational cultures within countries may explain theoretical fragmentation and be an obstacle to connections (Kynigos & Psycharis 2009); the cultural obstacles may hinder 2 types of transfer, *from* foreign cultures and *towards* different educational contexts (Bikner-Ahsbahs et al. 2017 [6])).
The TPDA theme addresses two networking “sets”: among theories directly focused on TPDA, and between general studies on teaching and those more specific to teaching with technologies. Despite the language and cultural difficulties, many researchers have networked, cross-analysed theories within these two sets. Relevant papers can be found in TWG17 group [7] but not only. A broad literature review is therefore interesting.

A review of networking theories papers related to TPDA

In 2010, Drijvers et al. provide a state of the art of the theories that significatively address the technological integration in teaching practices. Through this historical overview, they claim for “*integrative theoretical frameworks that allow for the articulation of different theoretical perspectives*.” (Drijvers at al., 2010). Ten years later the work is still going on, even if several hybridizations have clearly developed over time, with a greater or lesser influence from one field to another, according to the authors. Today, numerous ME papers technology-centred put different theories or constructs in perspective, to compare, contrast or look for filiations between them.

Some deal with more than 2 theories: Ruthven (2014) explores commonalities, complementarities, and contrasts between TPACK (Koehler and Mishra 2009); Instrumental Orchestration (Trouche 2005); and Structuring Features of Classroom

Practice (Ruthven 2009). Drijvers (2011) find share points between the Realistic Mathematics Education view, the IA and the Embodied cognition. Instead of looking for unification, some researchers contribute to developing strategies to cope with the theoretical diversity. This is the case of Maracci et al. (2013), who *cross-analyse* the Theory of Didactical Situations and the Theory of Semiotic Mediation. Networking activity provides not only theoretical results but also concrete applications. An example is the Tabach & Trgalová's research. They first achieve (2017, 2018) relevant connections between IA and TPACK through the theoretical construct of double instrumental genesis (Haspekian 2011). In 2019, they add to the previous connections a more general discussion, comparing and contrasting with the Thomas and Hong's PTK(2005). Then, using the Mathematics Knowledge for Teaching framework (Ball et al., 2008), they gain insight in the research field of the PD (Fig.1) by defining several concrete PD stages, where personal instrumental genesis precedes professional genesis. Sacristan's introductive chapter in the same book (2019) discusses this position asking for more flexible implementation of PD programs. Thus, opening discussions in the ME research community, Tabach and Trgalova progress both at theoretical level and in the results of research (better understanding and defining the specific knowledge to be developed at each stage).

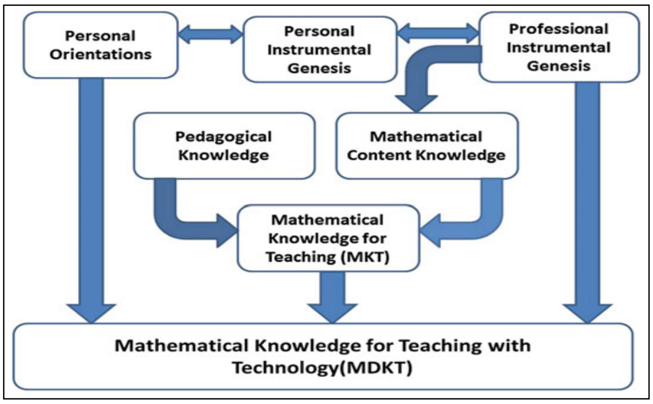

Fig 1. (MDKT) framework (Tabach & Trgalovà, 2019, p. 201)

Much more has been done. In order to derive new perspectives from this focus, we could further the list, and characterize more finely each of these networking cases, which are not of the same nature regarding the networking degrees (Bikner-Ahsbahs & Prediger 2008). Due to space restriction, I limit myself to these examples and present below some methodological perspectives to further the networking dialogue.

Methodological proposal to advance TPDA networking: cross-domain research?

Trgalova et al. (2018) report a discussion on how to organize the technology group in next CERMEs. Since CERME9, the thematic is split in two groups, respectively foregrounding teachers and students issues, which does not afford space for research addressing both. They note that another division, such as educational phase, still does not satisfy. It look like any fixed topic division would not meet the need of overlapping areas. Yet, "trans-TWG" sessions devoted to specific mutualized work on multi-perspective issues are needed. For example, "*the topic of teaching practices with* *technology raises the "need [for teachers] to develop new knowledge to design relevant technology-enhanced tasks*" (Tabach & Trgalova 2019). This competency addresses the *PD*. Researchers in this field (Hegedus et al. 2017) point a disappointment of the PD outcomes, explained by discrepancies between teachers' needs and PD program (Emprin, 2010). But reducing this discrepancy needs to better grasp both *standard practices* and perturbances caused by the *technology*. This multi-perspective raised the issue of "ICT competency standards" (Tabach & Trgalova 2019) to make PD and teacher educators more efficient. Referring to Trgalova et al. (2018) tetrahedron, there is a dialogue between 3 "faces" here: (teacher–technology–maths); (PD–technology–maths) and (teacher–PD–maths). Making these 3 areas dialogue appeals then to the whole tetrahedron, which is thus no more operational to describe the situation if a new summit is needed in the dialogue (for instance the "theory" issue dealing with this whole). In this example, the "knowledge" summit is mathematics for all, so not an actual dimension to play on (it would be such in studies dealing with added domains as in Lagrange & Laval 2019). Taking it as an common element already present frees a summit making the tetrahedron operational for new organizations: I thus suggest creating discussion times addressing a face of the new tetrahedron formed by a new foregrounded topic (Fig.2). This could be "Theory", "Representation", "A given device as Scratch"… Unlike in the initial tetrahedron, it's not fixed but has to be flexible for organizing "turning" mutual session times. It could be defined not upstream but after the submissions, according to the needs emerging from these. For example, choosing "Learners" we can benefit from de Freitas et al. (2019) work. They used cognitive psychology theories in ME to renew the role of affect at a collective level on students' side. This can be explored on teacher side, where affect, sympathy, play as well important roles not only at an individual scale (many research already explored it with the role of affect, beliefs on ICT integration) but on a cooperative scale. There, a dialogue with Sensevy's (2012) Joint Action theory could be used, teaching/learning being a joint activity (also in Radford 2019).

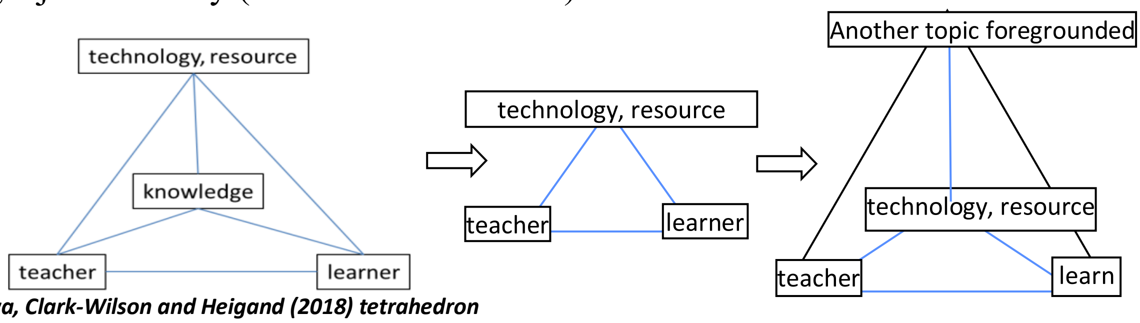

Fig 2. New forms of organisations for researchers' dialog

I did not elaborated further these reflections in concrete organization but the idea of "time modalities" with turning sessions may help organize dialogue in order to advance research on technology-enhanced teaching and learning, by adding another lever on which to play, so that. The idea is to **network topics** (and find methodologies for that) in addition to network theories on a given topic.

CONCLUSION

We made a journey among the theories in ME with a particular concern on *teaching practices in digital age*, with the broad question: *what a focus on theory can bring to TPDA research?* More specially, can the specific prism of theories *on TPDA* advance research results on this theme? (in general, can studying theories on a topic advance the research results on that very topic?) We proceeded along two directions. In the first one, we examined the theories in the CERME TWG related to TPDA. In the second, we examined research on TPDA through the lens of theories and networking, within and beyond the TWG17. Some reflections, emergent research issues and questions resulted. For example, the research of Tabach & Trgalova (2018, 2019) described above illustrated a networking case that brings both theoretical and "action" research results. Yet, my journey in both parts has only been initiated and would benefit of being furthered. For both perspectives I made methodological suggestions to continue the work. The qualitative state of the art could be combined with quantitative ones as the new Drijvers et al (2020)'s methodology mentioned above, which advanced on a theoretical concept using a bibliometric study. Yet, a more systematic literature review can help but would not be sufficient. The second part above explored the networking dimension, which is crucial for advancing on TPDA. To further it, it is necessary to find ways for researchers to dialogue.

Regarding this journey, to advance research on TPDA seems urgent as for the "*constant technological flux [which] makes it difficult to develop proper teacher training programs.*" (Sacristan 2019, p. 173). Gaining robust theoretical frames and tool that resist this flux is needed. Networking may undoubtedly help and the TPDA research field is fairly mature for this!

NOTES

1. Note that theories can hardly be disconnected from methodologies as the teacher issue can hardly be disconnected from learners' one. Operating a focus only puts one element on the scene front. On this topic, TWG17(2019) provides an interesting emphasis on the theories/methodologies interplay.
2. Awardees Multimedia Online Resources Project
3. https://hal.archives-ouvertes.fr/CERME10-TWG15/ (the n°15 being the introduction of the group)
4. The distinction "Alone/With others" is not strict but only a subjective appreciation: all the papers mention more than one theoretical reference, but these are more or less used by the authors.
5. French milieu, German Grundvorstellung have no English translation (Bikner-Ahsbahs et al 2017)
6. "theoretical tools (…) borrowed from other fields must either be adapted to mathematics education (…) or complemented with content-related theoretical tools" (Bikner-Ahsbahs et al. 2017)
7. Two recent CERME examples: Kuzniak et al. (2017), who illustrate the plasticity of their model by connecting it to several theories, Lagrange & Laval (2019), with working spaces in algorithmics.

REFERENCES, ANNEXES, FIGURES

Due to space limitation, the supplementary material connected to this text (references and annexes) can be found outsourced here:

https://www.researchgate.net/publication/344042875_MEDA_2_-2020_Plenary_Teaching_practices_in_digital_era_some_theoretical_and_methodological_perspectives

Assessment of mathematics in the digital age: The case of university mathematics

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In this paper I will reflect on the experience of TWG21: Assessment in Mathematics Education at CERME10 and CERME11, with focus on contributions linked to the use of digital technologies. I will then compare research concerning Computer Assessment Systems (CAS) at university level to the research in the general literature on assessment to find common themes, omissions and themes that are germane to the digital nature of this assessment method and to the mathematics. I conclude with some suggestions for future research as they apply to CAS in university mathematics, but that are relevant to assessment of mathematics in the digital age.

Keywords: computer assessment systems, university mathematics, assessment validity, formative and summative assessment.

INTRODUCTION

It is impossible to overestimate the impact that digital technologies have had and continue to have on the assessment of mathematics. A quick search on Google Scholar for the terms 'assessment mathematics digital technology' yields in excess of 94 thousand results since 2016, with entries concerning the assessment of mathematics at any level of instruction (from kindergarten to university and beyond), the potential of digital technologies for formative and summative assessment, the investigation of what can and cannot be assessed by digital technologies, and much more. In this paper I will first summarise the CERME experience of TWG21: Assessment in Mathematics Education to illustrate the breadth of this topic and some of the direction that the research has taken. I will then briefly discuss what are the main areas of research in assessment and I will map those onto the case study of the use of Computer Assessment Systems (CAS [1]) for mathematics at university. I will then show where this CAS research aligns with general assessment research, what is omitted and what are examples of research questions that are germane to the mathematics and to the use of technology. I start from the position that research on assessment in higher education is a rich field of enquiry and that mathematics education assessment research needs to confront its thematic against the thematic of this larger body of research, highlighting how findings transfer to the specific case of mathematics. I will conclude with some reflection of the role of digital assessment in the teaching and learning cycle of mathematics and what could be important areas for future research.

THE CERME EXPERIENCE

(Summative) assessment of mathematics at university level is one of my research interests and I was surprised to realise that there had not been a TWG on assessment for many years at CERME, despite assessment being a very important part of mathematics teaching and learning at any stage. Therefore in 2017, together with Michal Ayalon (Israel), Jeremy Hodgen (UK) and Michiel Veldhuis (Nederlands), I started TWG21 at CERME10. The group has now met twice and has received 42 submissions altogether, of which 14 concern assessment involving digital technology. These 14 papers clearly demonstrated the variety of research on digital assessment. Some papers discuss new assessment methods which just would not be available without the aid of technology such as comparative judgment (Davies, 2017), or the creation of a complex formative assessment tool in a blended modality for university mathematics (Barana & Marchisio, 2019; Cusi & Telloni, 2019). Other investigate the implications of transferring a task from pen and paper to a computer assessment system (Lemmo & Mariotti, 2017); report on the use of digital assessment to facilitate self-assessment (Hasa, Rämö & Virtanen, 2019); or disseminate findings of large projects investigating the design of digital activities that provide rich feedback to school students (Cusi, Morselli & Sabena, 2017a, 2017b). Some of the papers discuss the types of mathematical reasoning that CAS can test (Sangwin, 2019) and how CAS can be an effective tool for providing students with rich feedback (Beck, 2017). Finally, a good number of papers address the affordability that a large database of students' answer created through computer assessment systems can offer to researchers (Garuti et al. 2017; Ferretti & Gambini, 2017; Garuti & Martignone, 2019; Lasorsa et al., 2019; Bolondi et al., 2019) allowing them, for example, to classify students' difficulties with basic concepts like operations between exponentials. This variety of submission reflects only a fraction of the variety of research strands related to assessment of mathematics in the digital age. This research cannot however be carried out in a vacuum and needs to relate to the general research on assessment in education.

THEMES IN ASSESSMENT

If I were to name the four most important areas of research related to assessment these would be *reliability, validity, feedback* and *fairness*. In a naïve way *reliability* concerns the outcomes of assessment in terms of grading. An assessment is highly reliable if two distinct markers of the same paper return the same (or very close) results by using the same assessment scheme. *Validity* has recently developed into a complex concept and encompasses various aspects of the impact that assessment has on the teaching/learning cycle. Validity at a basic level concerns what is assessed and the aims of the assessment. An assessment method is valid if it assesses what it was supposed to assess. A mathematics exam in French administered to English students would not be a valid assessment of mathematics as it would also (and possibly mostly) be an assessment of the French language knowledge that the students have. A more realistic example is that of an assessment which asks pupils to reproduce seen computational techniques. This would probably not be a valid assessment of conceptual understanding (although it may be a valid assessment of procedural fluency). Messik (1995) breaks up the concept of validity into four dimensions: construct validity (the theoretical basis of the construct being assessed), criterion validity (the relation of that assessment item to other assessment measures), content validity (expert judgment on the content matching the construct subject of the assessment) and consequential validity (the impact that the assessment has on the participants to the teaching/learning cycle). The latter aspect of validity has been one focus of my recent research on assessment and its importance is highlighted by the work by Entwistle and Entwistle (1991). These authors describe how assessment is amongst the main factors that impact on students' approaches to learning, as the students' perceptions of what the assessment requires to be successful influence strongly the way in which they engage with the subject and the teaching of that subject. I have added *feedback* separately to my list as this is a much-debated aspect of assessment and feedback implementations, timing and effects are much studied in the education community. Finally, *fairness* deals with issues of inclusion and equity across the implementation of the assessment (e.g. are there any participants to the assessment who are excluded from it? Is the assessment fair across the body of students to whom it is relevant?). I will discuss below how existing research on CAS at university level (which I choose as a rather narrow case study part of the large body of research on assessment in the digital age) maps onto these aspects of assessment research.

CAS AND UNIVERSITY MATHEMATICS

CAS has become very popular in university mathematics assessment, at least in the UK. One reason is that mathematicians find very welcome the time saving coming from the electronic marking that these systems afford, but other advantages of these systems are also becoming clear. Before describing the match between research on CAS and the general assessment research it is important to note that reliability of assessment, which is of great importance when discussing human-marked work, it is far less important when discussing CAS systems as the marking process, once the marking grid has been established by those who have designed the assessment, will be automated. This is a big advantage that CASs provide both to the markers, and to researchers.

As an interesting exercise for this paper I have reviewed the literature on CAS, and I have grouped the papers found in some broad themes. I have mention one paper next top each theme as a representative, but the body of literature in most of the themes is extensive. The themes are:

1. What mathematical competencies can be assessed by CAS, including papers that addresses specific topics such as linear algebra (e.g. Sangwin, 2019);

2. Lecturers' perspective of the use of CAS (e.g. Marshal et al., 2012);

3. Students' perspective of the use of CAS (e.g. Rønning, 2017);

4. Potential of CAS as a source of rich feedback, and effectiveness of such feedback (e.g. Attali, & van der Kleij, 2017);

5. Effectiveness of CAS as a tool to catalogue students' misconceptions (e.g. Walker et al., 2015).

In this list there are some important themes that can be matched to the current research in assessment, some omissions, and some research issues which are germane to the presence of the digital technology. I will discuss those in turn below.

Matches: Many of the research areas listed above for CAS are related to validity in a wider sense. Research on what mathematics competencies can be assessed by CAS is paramount to construct a valid assessment of some given capabilities. While there is widespread agreement that CAS is very suitable to assess procedures and procedural fluency, it is still a matter for debate if it is suitable also to assess complex responses such as proof. This is also related to construct validity in that it relates to the distinction, in mathematics, between procedural and conceptual understanding. The difficulties with assessing complex processes like proof is, at least in part, that traditionally this work is submitted by the student in a free written form which is not compatible with automated marking. However, work is currently in progress (see Bikerton & Sangwin, 2020, for a report of some promising developments) and new questions are being written in order to compose proof comprehension tests that can be assessed by CAS. Research on stakeholders' perceptions and perspective is related to consequential validity. Not only there is an established link between students' approaches to learning and their perceptions of assessment, but teachers' perceptions of the assessment they adopt are bound to influence its implementation and ultimately its success. From the students' perspective, Rønning (2017) reports that one of the drawbacks of using CAS for assessment was that students stopped paying attention to the process of obtaining an answer as only the final answer can be input in CAS for grading. This is an approach to learning akin to procedural understanding and it is not desirable when assessing mathematics. From the mathematicians' perspective, Marshal et al. (2012) report that without full integration of CAS in the teaching of mathematics and without the creation of a shared forum for discussing its use (i.e. a community of practice), the use of CAS at university will not be sustained beyond the novelty trial stage. However, despite the presence of some study involving the CAS stakeholders, this area remains under-researched. Investigation of the effectiveness of feedback most associated to CAS is quickly becoming a significant research area. This area too can be related to consequential validity and, given that systems like STACK or NUMBAS can offer individualised feedback to students, it is quickly attracting the attention of many researchers. The type of feedback given to students, its timing and the impact of feedback on exam outcomes have all been studied in relation to CAS. As an example, Attali and van der Kleij (2017) report that although there is still no conclusive answer as to when it is best to administer feedback, if immediate or delayed, it seems that students' previous knowledge is still the dominant factor in deciding about effectiveness of feedback. Lastly, I would like to mention the potential of CAS to easily create a bank of data that researchers can analyse in order, for example, to build a comprehensive list of students' misconceptions at university level with a given topic, or to map how these misconceptions disappear or persist across educational levels. Although this is not strictly related to assessment, it is nevertheless an important by-product of the use of CAS, as also Bolondi et al., (2019) have discussed.

Omissions: Thinking back to the assessment research, the one big omission in CAS research is the investigation of whether there are stakeholders excluded from this type of assessment, and how can assessment include them. While it is true that CAS can help including university students who have to study remotely by offering flexible and effective assessment for online courses, the experience during the COVID-19 pandemic has highlighted just how wrong it is to assume that all students have easy access to digital tools and the conditions to focus on studying away from their university. This of course it is not the only section of the student body excluded from this assessment, those who do not conform to an ableist view of what a student can and cannot do are also excluded, and more effort and research should be dedicated to develop the right tools so that they too are included.

Questions related to the digital nature of CAS: There are also research questions that and are germane to the digital nature of the assessment and to the mathematics. Probably one of the most relevant concerns is the implications of the transfer between the pen and paper medium and the digital medium. This is particularly relevant for CAS assessment where the questions are often a 'translation' of questions that could be asked in pen and paper mode. There is an assumption in much of the literature that this transfer is immediate and without implications, although research in school settings shows that this is not always the case (Lemmo & Mariotti, 2017). Issues related to question design and implementations are also relevant to CAS. Answer to these questions will for example help deciding issues of validity in respect to what questions can be asked in a CAS environment.

SOME CONCLUDING REMARKS

I have described here the research on CAS as a case study of assessment in the digital age. The picture that emerges from this case study is that, although research around the implementation and impact of this assessment is growing, and this assessment is increasingly adopted in mathematics departments around the world, there are still important areas that demand attention. The main under-researched area is the impact of the assessment on the stakeholders, i.e. students and teachers. In my experience this is of paramount importance as this assessment can have unexpected consequences on students' engagement with mathematics, as the paper by Rønning (2017) shows. The CAS case study also shows one characteristics of assessment in the digital age: that it is thus far failing to realise its full potential and that usually it is designed in a conservative way. In a review of assessment carried out in Australia by Masters (2013) the author states:

> Most technology-based assessments to date have not capitalised on the potential of technology to transform assessment practice. In fact, most current computer-based assessment in school education is little more than paper and pen testing on a screen. (Masters, 2013, p. 27)

This is a position which is also to some extent reflected in the TWG21 papers I discussed earlier. Amongst all the submissions there has been only one assessment type that deviates significantly from a very traditional view of assessment. Comparative judgment for the assessment of mathematics (Bisson et al. 2016; Davies, 2017) is one example of assessment that is unlike anything else that has been used for assessment before and challenges the way in which we understand assessment. The idea beyond comparative judgment is that judging two items of work comparatively to find which one is better than the other is quicker and more accurate that judging a number of items against a marking scheme, i.e. adopting a criterium based judgment (Thurstone, 1927). For this method of assessment, a number of 'judges' assess pairs of student work and create a rank order which pools the 'collective knowledge' of the judges. The work by Bisson et al. (2016) shows how this method of assessment – which rely on the use of a computer interface for judgments and could not be implemented without this interface - is suitable to assess typically difficult mathematical competencies such as problem solving and conceptual understanding. There are also other applications of comparative judgment for peer assessment and for formative feedback, as the paper by Jones and Alcock (2014) shows.

If we want to exploit the full potential of digital technologies in assessment, we need to re-think the way in which we design and implement assessment and not just transfer uncritically the current assessment we use onto a digital format. As it turns out, even this un-critical transfer is not devoid of problems!

NOTES

1. Throughout the paper I refer to Computer Assessment Systems (CAS) to assessment whereby the students reply to mathematics questions administered via a computer system which are then marked by the system. STACK (https://stack.maths.ed.ac.uk) and NUMBAS (https://www.numbas.org.uk) are examples of CAS commonly used in the UK.

REFERENCES

Attali, Y., & van der Kleij, F. (2017). Effects of feedback elaboration and feedback timing during computer-based practice in mathematics problem solving. *Computers & Education*, *110*, 154-169.

Barana, A., & Marchisio, M. (2019). Strategies of formative assessment enacted through automatic assessment in blended modality. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4041-4048). Utrecht University, Utrecht, Netherlands.

Beck, J. (2017). Written documentations in final exams with CAS. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3427-3435). Institute of Education, Dublin City University, Ireland.

Bickerton, R. T., & Sangwin, C. (2020). Practical Online Assessment of Mathematical Proof. *arXiv preprint arXiv:2006.01581*.

Bisson, M. J., Gilmore, C., Inglis, M., & Jones, I. (2016). Measuring conceptual understanding using comparative judgement. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 141-164.

Bolondi, G., Ferretti, F., & Santi, G. (2019). National standardized tests database implemented as a research methodology in mathematics education. The case of algebraic powers. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4049-4056). Utrecht University, Utrecht, Netherlands.

Cusi, A., & Telloni, A. (2019). The role of formative assessment in fostering individualized teaching at university level. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4129-4136). Utrecht University, Utrecht, Netherlands.

Cusi, A., Morselli, F & Sabena, C. (2017a). Designing and analysing the role of digital resources in supporting formative assessment processes in the classroom: The helping worksheets. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3452-3459). Institute of Education, Dublin City University, Ireland.

Cusi, A., Morselli, F & Sabena, C. (2017b). Enhancing formative assessment in mathematical class discussion: a matter of feedback. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3460- 3467). Institute of Education, Dublin City University, Ireland.

Davies, B. (2017). A case for a new approach to establishing the validity of comparative judgement as an assessment tool for mathematics. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3476- 3483). Institute of Education, Dublin City University, Ireland.

Entwistle, N. J., & Entwistle, A. (1991). Contrasting forms of understanding for degree examinations: the student experience and its implications. *Higher education*, *22*(3), 205-227.

Ferretti, F & Gambini, A. (2017). A vertical analysis of difficulties in mathematics by secondary school to level: Some evidence stems from standardized assessment. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3493- 3499). Institute of Education, Dublin City University, Ireland.

Garuti, R., Lasorsa, C., & Pozio, S. (2017). The Italian national education assessment system: Building mathematics items. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3500- 3507). Institute of Education, Dublin City University, Ireland.

Garuti, R., & Martignone, F. (2019). Assessment and argumentation: an analysis of mathematics standardized items. In *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education*, 4075-4082.

Häsä, J., Rämö, J., & Virtanen, V. (2019). Evaluating students' self-assessment in large classes. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4083-4090). Utrecht University, Utrecht, Netherlands.

Jones, I., & Alcock, L. (2014). Peer assessment without assessment criteria. Studies in Higher Education, 39(10), 1774-1787.

Lasorsa, C., Garuti, R., Desimoni, M., Papa, D., Costanzo, A., & Ceravolo, R. (2019). Mathematics described proficiency levels: connecting psychometric and teaching analyses. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4115-4117). Utrecht University, Utrecht, Netherlands.

Lemmo, A., & Mariotti, M. (2017). From paper and pencil-to computer-based assessment: Some issues raised in the comparison. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of CERME10* (pp. 3540- 3547). Institute of Education, Dublin City University, Ireland.

Marshall, N., Buteau, C., Jarvis, D. H., & Lavicza, Z. (2012). Do mathematicians integrate computer algebra systems in university teaching? Comparing a literature review to an international survey study. *Computers & Education*, *58*(1), 423-434.

Messick, S. (1995). Validity of psychological assessment: Validation of inferences from persons' responses and performances as scientific inquiry into score meaning. *American Psychologist*, *50*(9), 741–749.

Masters, G. N. (2013). *Reforming Educational Assessment: Imperatives, principles and challenges—Australian education review; no. 57*. Camberwell: ACER.

Rønning, F. (2017). Influence of computer-aided assessment on ways of working with mathematics. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *36*(2), 94-107.

Sangwin, C. (2019). Developing and evaluating an online linear algebra examination for university mathematics. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.) *Proceedings of CERME11* (pp. 4121-4128). Utrecht University, Utrecht, Netherlands.

Thurstone, L. (1927). A law of Comparative Judgement. *Psychology Review*, *34*, 273–286.

Walker, P., Gwynllyw, D. R., & Henderson, K. L. (2015). Diagnosing student errors in e-assessment questions. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *34*(3), 160-170.

Quality of (digital) resources for curriculum innovation

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In this conceptual paper I re-conceptualize the notion of quality of digital (curriculum) resources in terms of a number of criteria: relevance, coherence, practicality, effectiveness, scalability, sustainability. I explain and illustrate them with two groups of studies in different context; both contexts could be characterized as oriented towards curriculum reform. It appears that in the context of reforms, the two criteria of 'relevance' and 'practicality' are important criteria for the quality of digital curriculum resources, whilst a thought-provoking distinction was made between static and dynamic quality.

Keywords: Digital curriculum resources, quality of curriculum resources, curriculum innovation.

INTRODUCTION

Whilst I am writing this article, the whole world is plagued by the COVID-19 pandemic, also changing policy makers' and teachers' perceptions of how education should be designed and provided. Moreover, it has changed students' perceptions of how to learn and study, and which resources to beneficially use for their learning. Certainly, since COVID-19, education has been expected to be provided mainly 'at the distance', via the internet, and technology will play a major role in how education will be provided in the future.

At the same time, internationally, teachers and students increasingly rely on digital resources (including open educational resources) to plan their lessons (for teachers), to solve their tasks and develop their learning trajectories (for students); in short to build their mathematics curriculum. Whilst there is an abundance of digital resources, both teachers and students often experience difficulties in choosing from the abundance of resources available (e.g. Siedel & Stylianides 2018), in evaluating their quality, and in integrating them in their instruction and learning in a systematic and meaningful manner. Curriculum resources, including digital materials, are known to be key tools for teachers (for preparing their teaching), and students heavily rely on them for their learning. Moreover, in many countries (e.g., The Netherlands, United Kingdom, United States), teachers are increasingly encouraged to (re-) design the curriculum in planning their instruction. In particular in higher education, mathematics students are stimulated to work on challenges and projects that need a high level of autonomous learning to solve the challenges, often auto-didactically acquired through/with digital (curriculum) resources. There is potential for these resources to provide stimulating and meaningful learning experiences for students, and motivating opportunities for teacher collaborative learning. One of the concerns arising is about the quality of the digital (curriculum) resources that teachers and students may use, and the coherence of their work with digital curriculum resources.

In terms of previous work that I build from in my analysis of quality of digital curriculum resources, I present three frameworks here. First, Choppin et al. (2014) created the Digital Typology framework. They outlined three categories to consider when analysing digital curriculum resources: students' learning experiences, curriculum use and adaptation, and assessment systems. Moreover, they conceptualised the learning space in terms of learning experiences, differentiation/individualization, social/ collective features. Second, in their second framework Choppin and Borys (2017) looked at digital curriculum resources with respect to four perspectives that inform the design, development, dissemination of curriculum resources: private sector perspective, designer perspective, policy perspective, and user perspective. They also explore how these perspectives lead to a foregrounding (or backgrounding) of the features described in the Choppin et al. (2014) framework. In particular, they explain that the four perspectives are often in tension with each other in terms of the purposes for design, the resources and capacity necessary to adopt digital programmes, and the potential to develop teacher (design) capacity. In the third framework, Pepin et al. (2016), distinguish between three types of e-textbooks (according to their model of development and their functionality): integrative e-text, evolving or 'living' e-textbook, and the interactive e-textbook (see below).

If we define 'curriculum' as 'design for learning' (van den Akker & Nieveen 2020), then 'curriculum resources' can be those designs for learning (e.g. mathematical tasks, lesson plans), or the tools that help us to design (and evaluate) learning (e.g. design tools). The research question is the following:

What do we know about the *quality* of such curriculum resources, in particular if they are digital, and how can we (re-)conceptualize the 'quality' of (digital) curriculum resources, in particular in times of curriculum renewal?

THEORETICAL FRAMES

In this section I explain and define the concepts of resources, e-textbooks and digital curriculum resources.

Several studies lean on the notion of *resource* to study what kinds of resources and materials teachers and students have access to, use, and orchestrate for their teaching and study of mathematics (e.g. Remillard 2005). To clarify the concept of curriculum resources, Pepin and Gueudet (2018) referred to mathematics curriculum resources as

> "all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom." (p. 1)

It is important to add that we have distinguished the term curriculum resources from, for example, social resources (e.g. web-based conversations with colleagues), and/or cognitive resources in mathematics education (e.g., frames used in professional development sessions to develop particular competences). Curriculum resources would thus include (1) text resources (e.g., textbooks, teacher guides, worksheets, tests); (2) other material resources (e.g., calculators or manipulatives used for a particular part of the curriculum); and digital curriculum resources (e.g., interactive e-textbooks).

In an earlier handbook chapter (Pepin, et al. 2016), we have defined e-textbooks. For the purpose of this paper I have slightly amended that definition, to become the following:

> E-textbooks can be defined as an evolving structured set of digital resources, dedicated to teaching (and learning), initially designed by different types of authors, but open for re-design by teachers (or students), both individually and collectively. (p.644)

They identified three kinds of e-textbooks:

> 1- the integrative e-textbook refers to an 'adds-on' type model where the digital version of a (traditional) textbook is connected to other learning objects [..];
>
> 2- the evolving or 'living' e-textbook refers to an accumulative/developing type model, authored where a core community (e.g. of teachers, IT specialists) has authored a digital textbook, which is permanently developing due to the input of other practicing members/teachers [..];
>
> 3- the interactive e-textbook refers to a 'toolkit' model where the e-textbook (authored to function only as an interactive textbook) is based upon a set of learning objects: tasks and interactives (diagrams and tools) that can be linked and combined. (p. 640).

In terms of differences between digital curriculum resources and digital (educational) technologies, we have proposed (Pepin, et al. 2017) to see the main differences as being the particular attention that digital curriculum resources pay to:

- The aims and content of teaching and learning mathematics;
- The teacher's role in the instructional design process (i.e., how teachers select, revise, and appropriate curriculum materials);
- Students' interactions with digital curriculum resources in terms of how they navigate learning experiences within a digital environment;
- The impact of digital curriculum resources in terms of how the scope and sequence of mathematical topics are navigated by teachers and students;
- The educative potential of digital curriculum resources in terms of how teachers develop capacity to design pedagogic activities. (p. 647)

Thus, we regard as *curriculum* resources (e.g. textbooks) those materials that are related to the (mathematics) curriculum, whether it is a one-off worksheet, or a test to assess a particular part of the curriculum (in terms of topic or grade), or a full-blown curriculum program.

> "It is the attention to sequencing—of grade-, or age-level learning topics, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates digital curriculum resources from other types of digital instructional tools or educational software programs." (p. 647)

THE STUDIES

In this section I explain and discuss two groups of studies: (1) one on (the design and evaluation of) mathematics (e-)textbooks (mainly) in the French context (e.g. Gueudet, Pepin, & Trouche 2013; Pepin, Gueudet, Yerushalmy, Trouche, & Chazan 2016); and (2) one on the design of students' actual study/learning paths with digital (curriculum) resources in the Dutch context (e.g. Pepin & Kock 2019).

Study group 1 – Design and evaluation of mathematics e-textbooks

In an earlier study (Gueudet, Pepin, & Trouche 2013) we reported on the comparison of the design and conceptualization of two very different French lower secondary mathematics textbooks: one which was developed, as it is 'traditionally' done, by 'experts' (teacher educators and researchers); and one which was developed, innovatively, by teachers using a digital platform. These different designs and conceptualizations had implications on the content, structure, potential and intended use of the books (which we investigated on the basis of specially designed questionnaires to the two groups of textbook authors). Our results pointed to a re-conceptualization of the notions of 'quality' (and 'coherence') of curriculum resources, such as (e-)textbooks. In terms of quality, we claimed that one of the textbooks (PDF version of paper book, with digital resources attached) was of high didactical albeit *static quality*: it offered many rich tasks, organized according to a carefully considered and complex structure. The second e-textbook appeared to be, in its initial version, of a lower intrinsic quality: it offered less problems and less rich tasks. In terms of structure it simply followed the structure of the official French National Curriculum. However, the 'digital additions' and possibilities of the e-textbook prompted us to re-consider the notions of 'quality'. The online version of e-textbook had already been modified several times, to take account of 'user comments', i.e. users' experiences and needs. The digital means offered possibilities for modifications, and these were integrated by the e-textbook in the process of re-design. This was perceived by the authors (a mathematics teacher association) as a necessity for meeting users'/teachers' needs in order to ensure the quality of the textbook- we called this *dynamic quality*. Only this e-textbook supported user adaptations and drew on user contributions.

In our quest for identifying the 'quality' of e-textbooks and digital curriculum resources, we recognized the notion of connectivity as an important issue (Pepin et al., 2016). Hence, in that handbook chapter we stated that the quality of an e-textbook depends on the nature and number of connections it makes. In particular, we identified connections at two levels:

(a) *External connectivity* refers to the potential of linking to and between subjects/users and resources/tools outside the textbook. It includes the potential to create virtual communities, connecting users with users (both teachers and students), as well as users and designers, and the textbook's interaction with other resources, via web links, or on platforms, for example. More generally, we have argued (ibid) that this *external connectivity* could include the following criteria:

> Connections to the national curriculum;
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> Connections across grades;
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> Connections with other disciplines (e.g., physics);
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> Connection to the assessment system;
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> Connections to other resources (files to download or websites of different kinds)
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> Connections between the textbook and teacher resource systems (for synergetic effects)
>
> Connections between teacher and students;
>
> Connections in terms of teacher collective work;
>
> Connections between teachers using the textbook and the author/s of the textbook.
>
> (p. 651)

(b) *Internal connectivity* refers to connections made inside the e-textbook. It concerns the specific mathematical content, i.e., that the e-textbook offers different kinds of combined materials (which can be definitions, properties, exercises but also, in the case of an e-textbook, software files, videos, etc.) and specific didactics (i.e., differentiation). *Internal connectivity* could include the following criteria:

> Connections between different topic areas;
>
> Connections between different semiotic representations (e.g. text, figures, static, and dynamic);
>
> Connections between different software/s for carrying out a particular task
>
> Connecting different concepts;
>
> Connecting different strategies for problem solving—this is linked to the issue of procedural vs problem-solving tasks as proposed by the textbook;
>
> Connecting different moments of appropriating a given concept (e.g. spiral progression, progressively deepening a concept instead of proposing a complete presentation of it in the same chapter) (p. 651/652)

From this group of studies, we retain first, that there are different forms of quality (i.e. static and living quality), and second, that the quality of an e-textbook depends on the number and nature of the connections it makes, at different levels.

Study group 2 – Students' use of digital resources for their actual student study paths

In a second kind of study we actually asked undergraduate mathematics students about their use of (digital) resources, in three different courses (Calculus, Linear Algebra, Bachelor End project based on challenged-based learning approach). In these studies (Pepin & Kock, 2019) we used a case study approach to investigate what kinds of resources were selected by students working on their two mathematics courses (Calculus, Linear Algebra) and the challenge-based projects they did as bachelor end projects. We were particularly interested how they used and orchestrated their chosen resources. Results showed that the students working on Challenge-Based-Learning projects used more resources outside the realm of curriculum resources (offered to them in traditional courses), and the teacher became the main 'resource' for monitoring and stimulating progress. Students' Actual Student Study Paths were iterative/cyclical. This was in contrast to the 'linear' study paths found e.g. in traditionally taught Linear Algebra courses. In the blended learning course (Calculus) students had an abundance of resources (most of them digital) to choose from, and they felt lost in this environment. The kinds of resources students used most were the following:

to their peers/colleagues, and to the lecturer/tutor;

to the textbook/reader (provided by the university);

to the resources provided for the course (by the course leader);

to resources outside the university (e.g. Khan academy);

to their family members (e.g. asking for help with tasks);

to the stakeholders of the (e.g. in case of the challenge-based bachelor end projects).

From this group of studies, we retain that students evaluate digital resources according to their 'usability' for a particular project or mathematical task, and their practicality (e.g. easy downloadable).

RESULTS

From the studies I used to illustrate particular aspects of quality, I take particular note of the following: First, it appears that the potential of a resource (for a particular purpose, in a particular context) leads to an exploration of its quality. Moreover, it can be said that whilst the mathematics education literature claims an intrinsic quality of a resource (e.g. didactical quality), this has to be seen in connection with (and distinguished from) its suitability with respect to a particular context (in which it is used) and users' goals and expectations. This rings true with the works of Trgalová and Jahn (2013) who suggest that the quality of a teaching resource depends on the user/s, the users' working context/s, and their design objectives. It can be said that these three aspects are connected (like in a web), and if only one is considered and addressed, it might bring the system out of balance.

Second, it appears that the notions of *quality* (of digital curriculum resources) and *coherence* (see Gueudet, Pepin, & Trouche 2013) go hand-in-hand. Whilst in one type of textbook, quality was afforded by rich didactical considerations (of the authors), it was stayed *static* by nature. In the other e-textbook, a different quality emerged, *dynamic* quality: users could change the content and so the textbook could evolve according to users' comments and proposed adaptations.

Third, in our study on students' use of digital resources, we identified other aspects of quality: for students the resource (e.g. previous maths course/video on the web) was of quality, if they saw its relevance (in a particular learning situation) and if it was easy to use, if it was practical. Digital software tools such as Matlab were important resources to shape the mathematical practice of the students and to help develop a solution to the challenge, based on the mathematical concepts involved.

Leaning on the curriculum innovation literature (e.g., van den Akker & Nieveen 2020), we can summarise the aspects of quality of digital (curriculum) resources under the following headings (see below), whilst noting that in particular situations students and teachers pay particular attention to some and less to others:

- relevance: both teachers and students only choose a digital resource if they see its relevance for their purpose and in the situation.
- coherence: the coherence of a digital resource is typically evaluated by the number and nature of its links to e.g. the curriculum, or fittingness within the lesson series.
- practicality: this refers to easiness of use and practicality in terms of feasible within a certain situation (e.g. usable with 30 students).
- effectiveness: this refers to whether the resource is 'doing' what it claims to 'do', e.g. does it help to understand a certain mathematical concept better.
- scalability: can this resource be used by only a small number of students, or by the whole school population?
- sustainability in context: can this resource be sustained over a prolonged period of time?

It appears that in the context of reforms, the two criteria of 'relevance' (there is a need for this resource, linked with the new curriculum) and 'practicality' (the resource is useful/useable in practice) are important criteria for quality of digital curriculum resources, whilst a thought-provoking distinction was made between static and dynamic quality. However, we have to realize that quality is especially depending on the user/audience and his/her purpose.

REFERENCES

Choppin, J., Carson, C., Borys, Z., Cerosaletti, C., & Gillis, R. (2014). A typology for analyzing digital curricula in mathematics education. *International Journal of Education in Mathematics, Science, and Technology, 2*(1), 11–25.

Choppin, J., & Borys, Z. (2017). Trends in the design, development, and use of digital curriculum materials. *ZDM Mathematics Education*. doi:10.1007/s11858-017-0860-x.

Gueudet, G., Pepin, B., Sabra, H., & Trouche, L. (2016). Collective design of an e-textbook: Teachers' collective documentation. *Journal of Mathematics Teacher Education, 19*(2–3), 187–203.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.). (2012). *Mathematics curriculum material and teacher development: From text to 'lived' resources*. Dordrecht: Springer.

Gueudet, G., Pepin, B., & Trouche, L. (2013). Textbooks' design and digital resources. In C. Margolinas (Ed.), *Task design in math- ematics education* (pp. 327–337). Oxford: ICMI Study 22.

Pepin, B., & Gueudet, G. (2018). Curriculum Resources and Textbooks in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 1-5). Cham: Springer International Publishing.

Pepin, B. & Kock, Z.J. (2019). Towards a better understanding of engineering students' use and orchestration of resources: Actual Student Study Paths. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education.* Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. (2016). E-textbooks in/for teaching and learning mathematics: A potentially transformative educational technology. In L. Eng- lish & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 636–661). New York: Taylor & Francis.

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246.

Siedel H., Stylianides A.J. (2018) Teachers' Selection of Resources in an Era of Plenty: An Interview Study with Secondary Mathematics Teachers in England. In: Fan, L., Trouche, L., Qi, C., Rezat, S., Visnovska, J. (eds.) *Research on Mathematics Textbooks and Teachers' Resources*. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-73253-4_6

Trgalová, J., & Jahn, A.-P. (2013). Quality issue in the design and use of resources by mathematics teachers. ZDM—The International Journal on Mathematics Education, 45(7) (this issue). doi:10. 1007/s11858-013-0525-3.

Van den Akker, J. & Nieveen, N. (in press) Combing curriculum and teacher development through design research.

Theme 1

Mathematics Teacher Education and Professional Development in the Digital Age

Papers

Liceo Matematico in Catania: a first analysis on teachers' professional development

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The Liceo Matematico (LM) is an Italian project in which in high school curricular hours extra hours are added where mathematical contents and interdisciplinary activities are proposed. The additional activities, always of laboratory nature, are aimed at broadening students' cultural education and developing their critical skills. The LM is characterized by a strong collaboration between school and university through teacher training activities during the school year. In this paper, we make an initial analysis of the impact that this training experience has on the teachers who followed the LM in Catania for three years. From data analysis, obtained from the administration of a questionnaire, professional development and collaboration arise in this community of teachers that is emerging.

Keywords: Liceo Matematico, mathematics machines, professional development, teachers' collaboration, communities of practice.

INTRODUCTION

The Liceo Matematico (LM) is a research project started at the University of Salerno in 2014. Main pilasters of the LM are: interdisciplinarity, laboratory teaching, elaboration of didactic proposals dealing with mathematical topics that are not in the curriculum (Capone et al., 2017). The LM project was joined by several Universities in Italy. Afterward, classes called "LM classes" started in several Italian high-schools (from grade 9 to 13). In these classes, in addition to the curriculum activities, one or two hours per week are dedicated to laboratory activities in which mathematics acts as a glue between different disciplines (literature, philosophy, chemistry, biology, computer science, robotics, art) (Capone et al., 2017). Schools hosting LM classes are called LM high schools (www.liceomatematico.it). There is a strong collaboration between university researchers and school teachers of LM high schools, in both cases not only in mathematics. The collaboration is expressed through the organization of periodic meetings aimed at designing and discussing laboratory activities to be tested and implemented in the classes. Annually, a National LM conference is offered were teachers and professors can share the various experiences. The activities are generally proposed by the University Poles and each Pole proposes training activities that respond to the skills of researchers, always having in mind the above pilasters. For example, the module that will be analysed in this work was designed within the Catania Pole for the LMs of Eastern Sicily.

To date, in Italy, there are 12 University Poles, to which about 100 secondary schools refer to. The LM project proposed by the University of Catania, the city where the authors of this paper work, was joined by about 10 schools in Eastern Sicily and 13 LM classes started in the school year 2017/2018. To date 13 schools have started the LM in Eastern Sicily for a total of about 300 students, per year. The educational project of the University of Catania is in progress (to date covers grades 9 to 11). The school teachers involved in LM activities, periodically, more or less every two weeks, between October and April, meet together with the university researchers and examine the proposals that the researchers elaborated. In fact, the university researchers do not go to LM schools, but the teachers who have followed the training course at the University will implement the activities in their classes. In this paper, we consider the mathematics teachers who have been following the LM work for these first 3 years. The following research questions guide our study: i) What impact does such a training experience have on their professional development? ii) If so, what kind of collaboration does take place between these teachers?

In particular, we will illustrate the training activity on the use of Virtual Mathematical Machines (VMM). The teacher training was conducted by one of the authors; the construction of the questionnaire and the analysis of the collected data were carried out by all the authors, putting together their mathematical and psycho-pedagogical skills.

THEORETICAL FRAMEWORK

The ICME-13 survey on teachers working and learning through collaboration (Robutti et al., 2016) examines different research studies, offering a common interpretative frame, in which placing and interpreting experiences of teachers working together. It is based on three themes: i) contexts and features of mathematics teachers working collaboratively; ii) theories and methodologies; and iii) outcomes. The first theme is particularly useful in framing educational initiatives, because it is spread out in different dimensions: 1) The initiation, foci and aims of collaborations; 2) The scale of collaborations (numbers of teachers and time-line); 3) The composition of collaborative groups and the roles of the participants; 4) Collaborative ways of working and their conception. In what follows, we present the LM experience in Catania according to this frame, in order to contextualize it in a general perspective. As for the second theme, it seems appropriate to recall the following theoretical lenses, that we will use as well.

The meaning of teachers collaborating and the meaning of community

"Collaboration implies co-working (working together) and can also imply co-learning (learning together). It involves teachers in joint activity, common purpose, critical dialogue and inquiry, and mutual support in addressing issues that challenge them professionally. It helps them in reflecting on their role in school and in society." (Robutti et al., 2016, p. 652). Community is used colloquially to mean groups of people who engage together socially, professionally, corporately, or officially. However, the community is usually seen to have some joint purpose and some stability over time. Wenger (1998) defines communities of practice as groups of people engaged with each other focused on a joint enterprise and creating a shared repertoire - a set of resources which support their engagement in relation to the joint enterprise. “Tools and resources are important for collaborative professional work and learning among teachers” (Brodie, 2020, p. 37).

CONTENT AND TEACHER TRAINING

Every year (starting from the school year 2017/2018), LM school teachers attend training meetings at the University of Catania on educational activities (modules) that they can bring to class. Each teacher decides which modules propose to his/her students, depending on his/her aptitudes and on the class. In any case, in the classroom, they work through laboratory teaching (Anichini et al., 2004). The activities for grade 9-10 students of LM classes are well structured, with a well-defined path designed by the researchers, and worksheets already tested in the classes. This school year (2019/2020) training for teachers with grade 11 students has started with a module on VMM. It was decided to actively involve the teachers in the design of the activities. The school teachers, in fact, have already been following LM activities for two years and have probably understood the *LM approach*. In the following, we will focus on the VMM module.

1) The initiation, foci and aims of collaborations: A mathematical machine (in a geometric context) is a tool that forces a point to follow a trajectory or to be transformed according to a given law (Bartolini Bussi & Maschietto, 2006). In the VMM module, mathematical machines related to conics are examined. The researcher proposed real mathematical machines and built their virtual representation with the GeoGebra software (or vice versa), highlighting the problems that arise in the design phase of each machine. In fact, the virtual construction of a machine and its physical construction present very different difficulties. The aim of the training was to involve teachers in the design of activities on mathematical machines that they would have to replicate in their classes. Observe that the passage from physical to virtual is not easy: in the activity, students are guided towards this passage by activities that underline the difference in the use of some GeoGebra tools. For example, the difference between *Segment* and *Segment with given length*. When drawing machine bars, we have to use the latter.

2) The scale of collaborations (numbers of teachers and time-line): The training involved 15 teachers, in three meetings of 3.5 hours each, between November and December 2019.

3) The composition of collaborative groups and the roles of the participants: Participants were high-school in-service mathematics teachers, teaching in LM classes. At the first meeting, the researcher proposed the topic, mathematical machines, and presented, in general, machines producing conics and machines producing geometrical transformations [1]. The whole group decided to work on machines producing conics. In fact, conics are part of grade 11 curriculum. Then, the teachers and researcher decided to deal with the antiparallelogram (that draws an ellipse), with two machines,

similar to each other, that draw an ellipse and a hyperbola (based on the use of a circle), and with a machine drawing a parabola.

4) Collaborative ways of working and their conception: At each meeting, the researcher was introducing the topic (a mathematical machine) and afterward teachers and researcher worked on the school activity, thinking on how to build the machines, which material they could use or "correcting" [2] the worksheets that the researcher prepared. Between meetings, teachers would discuss in small groups, comparing ideas. The groups were not predefined, they were born spontaneously during the meetings, even sometimes they were made up of the teachers sitting "close by". The groups discussed the proposals made and discussed the activities proposed in class. Everything was then shared with the rest of the colleagues.

The approach used to introduce the machines in class was different from time to time, shared and proposed by the participants. The approach to the first machine was proposed by the researcher: it was decided to start from the physical machine, the antiparallelogram, and then move on to the virtual one. The path proposed to the school students foresees the physical construction of the machine with wooden sticks, balsa cutter and paper fasteners. The teachers, however, at the next meeting also proposed other materials for the construction of machines (wood, plastic materials ...). In the case of the two other machines, the ones that draw an ellipse and a hyperbola (based on the use of a circle), it was chosen to start from the virtual machine and then move on to the physical one.

For the last machine, the one that draws parabolas, it was chosen to introduce the machine only through a verbal description (Figure 1): "Let us consider an articulated system of bars representing a rhombus ABCD and two perforated bars, *a* and *d*. The bar *a* slides along two guides, represented in the figure by the straight lines r and s, perpendicular to it, and has one end at point C, while the bar *d* contains points B and D".

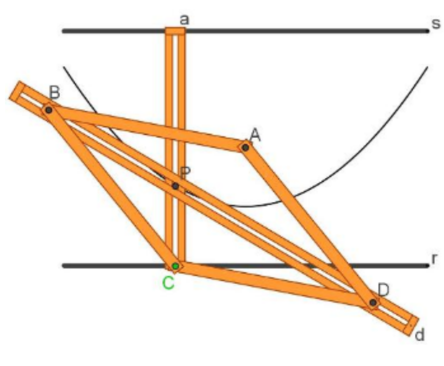

Figure 1: Parabolograph (taken from [3])

For each machine, worksheets of the activity have been produced by the researcher and discussed with the teachers. The worksheets have the task of guiding the school student in the analysis of the machine by identifying its characteristics and highlighting why that machine produced that conic. Worksheets also highlight the constraints to be imposed in the design of the virtual machine and those to be imposed in the design of the physical machine. Therefore, in this type of activity mathematical and engineering aspects interact with each other.

QUESTIONNAIRE AND DATA ANALYSIS

At the end of the training meetings, in December 2019, an anonymous questionnaire was administered. It was produced using Google Forms, an open source application for online surveys, and sent to teachers via email. It is divided into two sections: one related to LM in general and the second more specific to the course VMM. It contains multiple choice questions, Likert scale questions - from 1 (absolutely false) to 4 (absolutely true) - and some open-ended questions to motivate some answers. The analysis of responses was performed with Excel.

All 15 teachers who attended the VMM module answered the questionnaire. Among them, 3 followed the LM activities for the first time, 1 is in the second year and the other 11 are in the third year of the training. Our analyses focus only on these 11 teachers, 7 women and 4 men, all mathematics teachers in service in Eastern Sicily, with teaching experience ranging from 10 to 40 years. The following analyses, unless otherwise indicated, refer to questions formulated on Likert scale from 1 to 4. Examples of teachers' answers to open-ended questions will be shown in italics.

Let us start by considering the data referring to the first part of the questionnaire. As mentioned, the methodology that the LM prefers is laboratory teaching. We asked the teachers how familiar they were with this methodology before starting the LM and now, three years later. In the following, we consider 3+4 together in the Likert scale. 73% was already familiar with this methodology before the LM, but only 45% used to practice it in their lessons. Now, three years after the start of the LM, 73% use this methodology, even in activities outside the LM. In addition, this same percentage is now using this methodology with more awareness. The first part of the questionnaire ends with a self-reflection part. We asked the teachers to reflect on whether, before the LM, they expected any repercussions on their teaching professionalism and their students. For both questions, 73% (3+4 together in the Likert scale) answered that they had such expectations. The same questions were asked at the present time. For both questions, 74% (3+4 together in the Likert scale) answered that they perceived such effects. An open question asked to describe the possible effects they perceived. We received only 6 answers. They all agree that a new way of teaching mathematics is being experimented and that this has a positive influence on students:

> I had the opportunity to pose some topics differently and the students have partly understood that mathematics can be done differently;

> Students are more motivated and are more focused, interested and enthusiastic in math lessons.

Let us now consider the part of the questionnaire related to the VMM module. We were interested in understanding, whether during the hours of training at the university, discussions/sharing of ideas were made between the teachers on the contents that were dealt with. 55% (3+4 together in the Likert scale) did this with teachers from the same school [4]. 54% (3+4 together in the Likert scale) did this with all the teachers present.

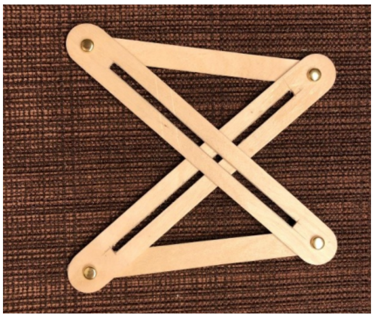

Figure 2: Antiparallelogram

In addition to discussions about the contents of the module, the teachers produced some physical models of mathematical machines with the help of simple materials (wooden sticks, paper fasteners...). An example of a model realized by the teachers is shown in Figure 2. It is a mathematical machine that allows to draw an ellipse. The model has not been 'invented', but rather the teachers have made an ingenious choice to choose simple materials to use for its construction and how to cut and assemble all the pieces.

The design of the models was carried out by 4 teachers. Although not everyone made proposals on how the models could be built, 64% (3+4 together in the Likert scale) said they felt involved in the design phases of VMM module. This type of collaboration, both discursive (discussion, reflection) and design (construction of physical models), which is also based on the fact that the teachers know each other and have been working together for three years now, has led us to believe that these teachers form a community of practice. Although this assumption can be further demonstrated by finer analyses, which we want to conduct in the near future, in the questionnaire we were interested in understanding what degree of perception of belonging to a community the teachers had. We then asked "With regard to the group of teachers with whom you followed the VMM module, how much do you feel part of a community?". 55% feel absolutely part of a community, 27% more yes than no, 18% more no than yes (nobody answered absolutely no).

> I have been collaborating with the family of the Liceo Matematico for three years now and I find that they have been three intense years full of experience and collaboration;
>
> Suggestions, advice, opinions continue even outside the university classroom;
>
> Everything that in so many years of experience you have imagined, materializes when you compare yourself with others.

Finally, we asked: "How much do you agree that this training experience has affected your way of teaching mathematics to your students?". 18% absolutely agree, 73% more yes than no and 9% more no than yes. So the majority (91% if we consider 3+4 together on Likert scale) believe that having designed the activities together with the researcher, with the possibility to intervene with their ideas, had an impact on their professionalism.

> It made me reflect on my way of teaching;
>
> It encouraged me to use new approaches closer to the students' interests;
>
> [I give] more importance to the sense of discovery;
>
> I deal with topics more casually in ways that students sometimes don't expect.

DISCUSSION AND CONCLUSION

In light of the analysed data, we can answer the research questions. The first part of the questionnaire, although at a general level, allows us to answer question i). The effects on the professional development of the teachers who have been participating for three years in the LM of the University of Catania are a confirmation of the expectations they had before starting the LM. Most of the teachers had expectations of change both on their professionalism and on their students' attitude towards mathematics. These expectations are becoming a reality over the years of the LM. Most have stated that they perceive a change in the way they teach mathematics and that this is spilling over to their students. In particular, changes in one's professionalism concern the laboratory teaching. Prior to the LM, it was known to the majority, however, they did not make much use of it in class. Since attending the LM, the majority not only use it in activities outside the LM, but also use it with more awareness. With the second part of the questionnaire, which focuses on VMM module, we can answer more precisely the research questions. In fact, within this module, there has been collaboration both between the researcher and the participating teachers and among the participating teachers themselves. Working groups were created in the presence and the discussion took place in a double direction. On the one hand, there have been discussions about the content of the course. These discussions took place both among colleagues from the same school and among colleagues from other schools. On the other hand, there have been discussions on how to make physical models of mathematical machines. The most proactive teachers were 4. However, it was the majority who stated that they felt involved in the whole VMM module. In particular, the majority stated that the experience of designing together with the researcher and colleagues had a positive impact on their professionalism in terms of self-reflection and their way of teaching. We can therefore observe that co-working involved co-learning. We need finer analyses to be able to assert this with more certainty, but we can begin to say that for these teachers the collaboration results in the creation of a community of practice. They have a domain of common interest: they joined the LM and meet periodically in university to receive training on the mathematics topics proposed by the researchers and that they then bring back to class to their students. They, therefore, engage in joint discussions and activities, learning from each other, as has been the case with the design of physical models of mathematical machines with simple materials. Over the course of three years, they have developed a shared repertoire of experiences and tools. They are therefore practitioners. Most of them feel that they belong to a community. Some speak of "*family* of the LM".

This is a first analysis after three years of LM activity made only on teachers who have been following the LM for 3 years. It is our intention to do interviews, case studies, to study in more depth the professional development from which these teachers are benefiting.

NOTES

1. For more information, see: http://www.mmlab.unimore.it/site/home/laboratorio-visite-mostre/la-collezione-di-macchine-matematiche.html

2. In the sense of suggesting changes, make easier some points of the mathematics proofs, etc.

3.http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=216&Itemid=298&lang=it

4. These 11 teachers come from 6 schools. In particular, 3 are from 3 different schools, 2 from one school and 6 in two groups of 3, from two schools.

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REFERENCES

Anichini, G., Arzarello, F., Ciarrapico, L. & Robutti, O. (2004). *Matematica 2003*. Lucca: Matteoni stampatore.

Bartolini Bussi, M. G., & Maschietto, M. (2006). *Macchine matematiche: dalla storia alla scuola. Collana UMI Convergenze*. Milano: Springer.

Brodie, K. (2020). Resources for and collaboration: a conceptual framework. In Borko, H. & Potari, D. (Eds.), *Proceedings of the e Twenty-Fifth ICMI Study* (pp. 37–48). Lisbon, Portugal.

Capone, R., Rogora, E., & Tortoriello, S. (2017). La matematica come collante culturale nell'insegnamento. *Matematica, cultura e Società – Rivista dell'Unione Matematica Italiana*. Serie 1, Vol. 2(3), 293-304.

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration: June 2016. *ZDM – The International Journal on Mathematics Education*, *48*(5), 651-690.

Wenger, E. (1998). *Communities of practice: learning, meaning, and identity*. New York: Cambridge University Press.

New teaching techniques aiming to connect school and university mathematics in geometry

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This paper covers the conceptualisation of an innovative teaching format that helps students build bridges between university geometry and school geometry to counteract the effect of Klein's (1924/2016) "double discontinuity". After a theoretical discussion of the used frameworks, we present the design-based research process for the teaching format and results of the accompanying evaluation using the framework of concept image (Tall & Vinner, 1981). Finally, we present the constructed e-learning tool and future research plans.

Keywords: design-based research, transition, concept image, mathematical maps.

INTRODUCTION

"Teachers matter" is the statement and title of an OECD (2005) paper presenting a study conducted in 25 countries. They agree that "demands on schools and teachers are becoming more complex" and teachers are "the most significant resource in schools" to reach high-quality education (OECD, 2005, p. 7). Since quality of future teachers depends heavily on university training (cf. Venter, 2017), it is necessary to improve university teaching if high standards in schools are to be maintained. 100 years ago, Felix Klein (1924/2016) described a common issue in teacher education, the "double discontinuity", which concerns problems of future teachers when transitioning from school to university and back to school. Winsløw and Grønbæk (2013) showed that this problem is still relevant today, especially when it comes to autonomous work. Klein stated that the reason for this discontinuity, the lack of connections between mathematical contents at school and university, results in teachers falling back on traditional teaching culture after graduating from university and eventually in low quality of teaching. Additionally, subjectively perceived lack of meaning of university contents (in the sense that students think, they won't need them for their future work) is one reason for students' poor academic performances in mathematics teacher education (cf. Cooney & Wiegel, 2003). However, the important connections between school and university math contents do not emerge incidentally (e.g. Bauer & Partheil, 2008; Winsløw & Grønbæk, 2013).

THEORETICAL BACKGROUND

In order to counteract the problem of the double discontinuity and its consequences, we designed and evaluated an innovative teaching format based on the theory of praxeology (Chevallard, 2006). A praxeology consists of a praxis block and a logos block. The praxis block is made of types of tasks or problems (T) and techniques (τ) to solve them. The logos block is comprised of a technology (θ), i.e. description and justification of the technique, and the theory (Θ), which is a broader discourse justifying the technology. Using the notion of praxeology, the problem under consideration is to counteract Klein's "double discontinuity" by showing connections between school and university geometry (T) and the technique to do this is the presented innovative teaching format (τ). Specifically, the teaching format consists of a course (**τ_1**), which should illustrate the interdependencies of mathematics taught at school and mathematics taught at university. In order to illustrate them more clearly, an interactive "mathematical map" (**τ_2**) (Brandl, 2009) as a digital learning tool is developed and used. It is intended to "offer the student an optimal solution for establishing successful learning processes" (ibid., p. 106) by integrating the historical origin of mathematical concepts as well as interdependencies between them. The "mathematical map" should combine these two characteristics in one three-dimensional representation, a kind of graph or tree. One dimension represents time, while the other two represent inner-mathematical dependencies. More details can be displayed by using added functionalities, where only one characteristic is considered (cf. present status below). Overall, the praxis block of the teaching format can be identified with the tuple [T, τ_1, τ_2]. In the notion of praxeology, the evaluation of the teaching format can be seen as part of the technology (θ), as it justifies the techniques [τ_1, τ_2] to address the problem (T). The research question is if the techniques [τ_1, τ_2] are able to build bridges from the "conceptual-embodied" or "perceptual-symbolic world" of school into the "axiomatic-formal world" of university (Tall, 2008) and therefore smoothen the first discontinuity.

DESIGN-BASED RESEARCH PROCESS FOR [τ_1, τ_2] AND EVALUATION

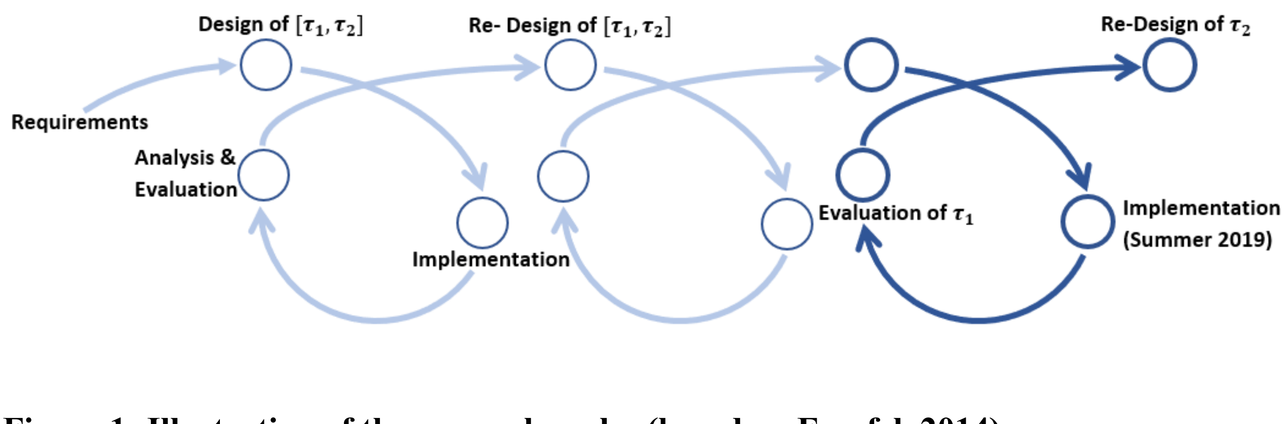

Figure 1: Illustration of the research cycles (based on Fraefel, 2014)

For the design and evaluation of [τ_1, τ_2], we decided to use a design-based research (DBR) approach (cf. Anderson & Shattuck, 2012), which is "a practical research methodology that could effectively bridge the chasm between research and practice in formal education" (ibid., p. 16). To support students' autonomous learning processes, a blended learning design was implemented via an e-learning platform. The use of e-learning has many advantages, for example, time and location independence and the possibility of self-paced learning (Elkins & Pinder, 2015).

As an e-learning management system, we used ILIAS, which is provided by the University of Passau and has the same functionalities as Moodle. The presented ILIAS course contains multiple learning modules for the different topics discussed during the semester. Each learning module consists of essential definitions, theorems, and examples from university textbooks and school textbooks as well as related tasks for the students to work on in groups. So far, the teaching format passed three cycles (see Figure 1) with slight changes to the design in every cycle.

In the following, DBR process and adaptions made to the teaching format [τ_1, τ_2] during all three cycles are addressed briefly. Afterwards, the **Evaluation of** the last implementation in Summer 2019 is described in detail. Hereby, the focus is on results for the course $\boldsymbol{\tau_1}$. Finally, we present the **Re-Design of** the interactive mathematical map $\boldsymbol{\tau_2}$, whose didactical benefit will be evaluated in detail in a future research cycle.

The design-based research process for [τ_1, τ_2] and adaptions

Before starting the conceptualisation of [τ_1, τ_2], a preliminary questionnaire on school geometry knowledge of first semester students was conducted with 136 participants to identify knowledge gaps. Additionally, scripts of university geometry lectures were compared with the school curriculum for geometry to identify relevant topics for the teaching format, such as axiomatic structure of geometry, motions, congruence, Pythagoras theorem and absolute geometry. For $\boldsymbol{\tau_2}$, information from relevant literature about the historical developments of geometry was collected. A first version of [τ_1, τ_2] was piloted in the winter semester 2017/18. Analysis of this pilot led to reducing mathematical rigor in favor of vividness of concepts. Therefore, GeoGebra applets were embedded in ILIAS learning modules to add visualisations. Due to technical difficulties with the implementation of the three-dimensional map, it could not be used in the first cycle. For the second version of [τ_1, τ_2] in the winter semester 2018/19, the theoretical model of concept image and concept definition (Tall & Vinner, 1981) was used to grasp the concept of connections between university and school geometry together with guided interviews as a research instrument. Observations during the semester and the analysis of the guided interviews suggested some adjustments, which led to increased use of visualisations in the third cycle (Datzmann & Brandl, 2019). To solve the technical difficulties with $\boldsymbol{\tau_2}$, the original concept was split into two separate parts, a timeline, which shows the historical development, and a two-dimensional map, which shows similarities of the geometrical contents (cf. Datzmann & Brandl, 2018). These were used to categorise geometric developments in terms of their content and the time they emerged.

Making the teaching format credible for the mandatory study plan in the summer semester 2019 led to an increase in the course size to 11 students. The two separate parts of $\boldsymbol{\tau_2}$ were connected by introducing links leading from the two-dimensional map to the respective content in the timeline (e. g. theorem of Pythagoras). The accompanying evaluation again consisted of a guided interview as a pre-post-test to compile the concept image of geometric concepts and is now described in detail.

Evaluation of τ_1 in the third research cycle (Summer 2019)

As mentioned above, we used Tall and Vinner's (1981) framework of concept image and concept definition for the evaluation accompanying the design-based research process in the last two cycles. The concept image is "the total cognitive structure that is associated with the concept, which includes all mental pictures and properties" (p. 2). The concept definition splits into the formal concept definition, like the one in a textbook, and the personal concept definition, "which is the personal reconstruction by a student of a definition" (p. 2). The personal concept definition is often attributed to the concept image and we follow this habit. At different times different portions of the concept image may be activated, these are then called the evoked concept image.

We were interested in how the concept image of the students changed because of the teaching format. The survey focused on the concepts of line, circle, congruence, and the sum of interior angles since all of them are covered in school and university.

For these concepts, the most relevant mental pictures, properties and related mathematics concepts were identified based on schoolbooks and university scripts. The aspects found were then divided into four subcategories as shown in Table 1.

Concepts	Line	Circle	Congruence	Sum of interior angles
Subcategories	Conceptions of a line Properties of lines Lines in different geometries Connections to related concepts	Conceptions of a circle Properties of a circle Special circles Connections to related concepts	Conceptions of congruence Classification in mathematics Connections to related concepts	Derivation of the sum of interior angles Sum of interior angles in different geometries Extension to n-gons Connections to related concepts

Table 1: The four concepts and their subcategories

As a research instrument, a guided interview was conducted with nine students, who completed the teaching format in the summer semester 2019, before the first and after the last session. The guided interview was used to allow the students to talk about everything in their mind regarding a concept and the interviewer could then use questions to evoke subcategories of the concept image, which were not mentioned. The statements of the students were assigned to aspects of the respective subcategory (i.e. aspects for "Conceptions of a line": infinite straight dash; set of points; line equation; vectors; determined by two points; axiomatic) and assessed regarding their correctness via a five-level Likert scale from mainly incorrect to mainly correct. Aspects that were not mentioned were marked as not existent. A sample of the assessments (20%) was countercoded by a second researcher with a Cohens Kappa of $\kappa = 0.772$.

The coded statements of the students were then compared between the pre- and post-interview. The concept images in the post-interview contained more aspects that are allocated in the "axiomatic-formal world" of university geometry than in the pre-interview, whereas aspects that refer to the "perceptual-symbolic world" of school geometry were still present. Klein described the issue when a student is "confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly" (1924/2016, p. 1). However, results show that the concept images have mostly been extended by abstract university aspects and that concepts from school have not been replaced by them. This suggests that most students were able to create links between what they knew from school and what they have learned at university. Therefore, the developed teaching format was to some extent able to counteract the first discontinuity.

RE-DESIGN OF τ_2, THE INTERACTIVE MATHEMATICAL MAP

Subsequent to the last research cycle, it was possible to redesign the existing prototype of the "mathematical map" in accordance with the original concept in order to show similarities of concepts and connections more clearly (see Figure 2 left; Brandl, 2009).

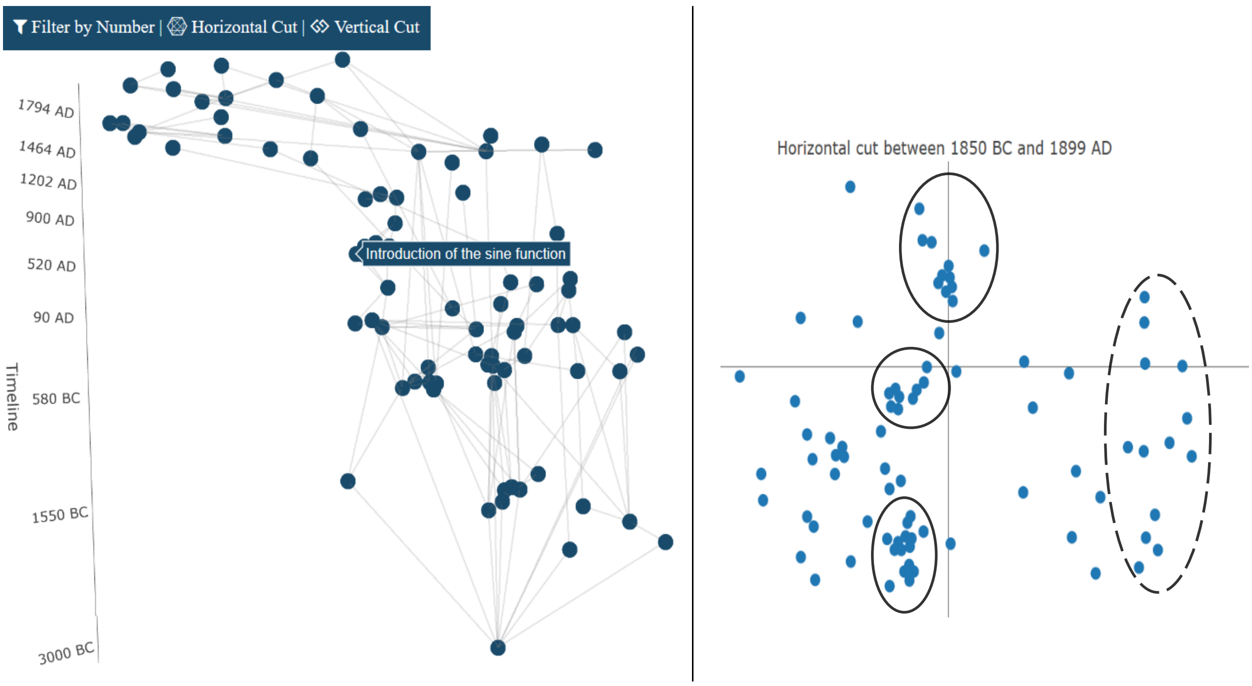

Figure 2: Screenshots of the mathematical map (front and top view; Status 09.06.2020)

To translate the thematic relatedness into the Euclidean distance, we use a force-directed method out of graph theory (technical details can be seen in Przybilla et al., in press). In the top view (see Figure 2, right), the thematically related contents are displayed close together in clusters. By clicking on a point, the corresponding content opens on a timeline, where milestones of geometry are shown and material such as files, videos, or interactive media can be accessed. An example of such a node on the timeline is shown in figure 3. In order to visualise genesis and similarities of contents, some functionalities for students to use were added.

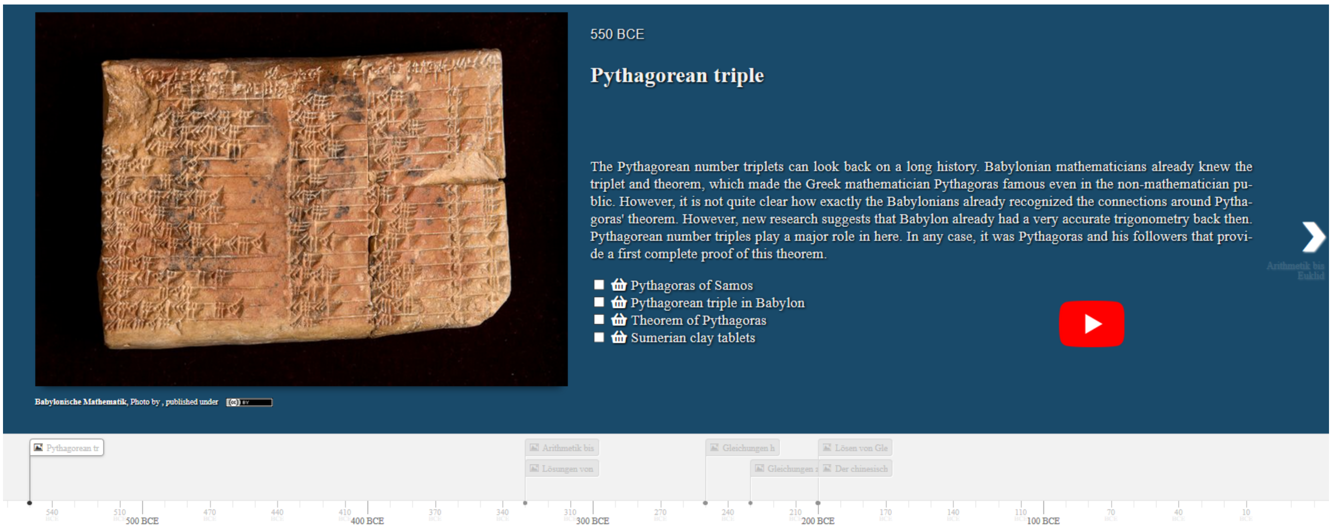

Figure 3: Screenshot of the node *Pythagorean triple* on the timeline (Status 09.06.2020)

Historical Genesis of Mathematical Contents and Vertical Cuts

To overcome Klein´s "double discontinuity" (1924/2016), teachers have to point out connections between the "axiomatic-formal" university contents and the "perceptual-symbolic" school contents (Tall, 2008). One possibility to create such links is by recognizing and emphasizing mathematics as an emerging science. Klein describes this type of learning as "intuitive and genetic, i.e., the entire structure is gradually erected on the basis of familiar, concrete things" (1924/2016, p. 9), Tall would say, on contents out of the "perceptual-symbolic world". Historically, university contents are often built upon school contents. For example, the curriculum for geometry in Germany covers mostly the Greek Euclidean geometry. Only in the last years, analytical geometry, founded in the 17th century by René Descartes, is discussed in part. Whereas the focus at the university lies on Hilbert's axiomatic geometry, which had developed at the transition to the 20th century. In addition, the non-Euclidean geometries, studied in the 19th century, are treated. Therefore, the functionality "Vertical Cut" (Brandl, 2009) visualises all development steps, which led to a chosen content. In figure 4, the Vertical Cut of the node *Spherical geometry* is depicted.

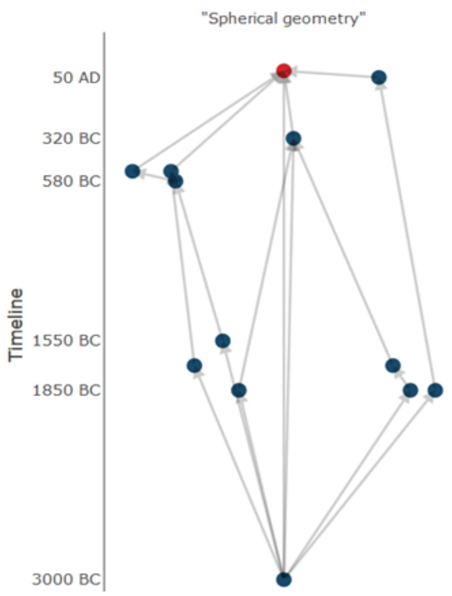

Figure 4: Screenshot of a Vertical Cut (Status

Inner-Mathematical Similarities and Horizontal Cuts

In order to facilitate the recognition of similar contents, the system offers the functionality "Horizontal Cut" (Brandl, 2009). After projecting all nodes to one level, so that the thematic proximity becomes visible, the user can filter them related to time or thematically. Looking at the horizontal cut from 1850 BC to 1899 AD (figure 2 right), similar contents are clustered (manually highlighted by ellipses). The lower ellipse contains all *Trigonometry* contents in this period. Additionally, the increasing differentiation in mathematics in the 18th and 19th centuries becomes visible through the contents of non-Euclidean geometry cluster and included in the dashed ellipse.

UPCOMING RESEARCH CYCLE

A further research cycle will be implemented to investigate the didactic benefit and several possible applications of the mathematical map ($\boldsymbol{\tau}_2$) in different universities. First, benefits of the usage as an interactive data structure will be tested. Students should make use of the functionalities to become aware of the connectedness of mathematical concepts and the historical origins of concepts. On the other hand, it is planned that students create some contents for existing nodes themselves. In this way, they shall learn from the historical development and the research process.

ACKNOWLEDGMENTS

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REFERENCES

Anderson, T., & Shattuck, J. (2012). Design-based research: A decade of progress in education research? *Educational Researcher*, 41 (1), 16–25.

Bauer, T., & Partheil, U. (2008). Schnittstellenmodule in der Lehramtsausbildung im Fach Mathematik. *Mathematische Semesterberichte*, 56 (1), 85–103.

Brandl, M. (2009). The vibrating string – an initial problem for modern mathematics; historical and didactical aspects. In: Witzke, I. (Ed.), *18th Novembertagung on the History, Philosophy & Didactics of Mathematics*, 95–114. Berlin: Logos.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In: Bosch, M. (Ed.), *European Research in Mathematics Education IV: Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education (CERME 4, February 17–21, 2005),* 21–30. Sant Feliu de Guíxols: FUNDEMI IQS – Univeritat Ramon Llull and ERME.

Cooney, T. J., & Wiegel, H-G. (2003). Examining the Mathematics in Mathematics Teacher Education. In: Bishop, A. J. et al. (Eds.), *Second International Handbook of Mathematics Education*, 795–828. Dordrecht: Kluwer.

Datzmann, A., & Brandl, M. (2018). A new defragmenting teaching format for teacher education using mathematical maps. In: Weigand, H-G. et al. (Eds.), *Proceedings of the Fifth ERME TOPIC CONFERENCE (ETC 5) on Mathematics Education in the Digital Age,* 305–306. Copenhagen: University of Copenhagen.

Datzmann, A., & Brandl, M. (2019). Evaluation of a connecting teaching format in teacher education. In: Jankvist, U. T. et al. (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME 11, February 06–10, 2019)*, 2462–2463. Utrecht: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Elkins, D., & Pinder, D. (2015). *E-Learning Fundamentals: A Practical Guide*. Alexandria: ADT Press.

Fraefel, U. (2014). *Professionalization of pre-service teachers through university-school partnerships.* Partner schools for Professional Development: development, implementation and evaluation of cooperative learning in schools and classes. Edinburgh: WERA Focal Meeting.

Klein, F. (2016). *Elementary Mathematics from higher standpoint. Volume I: Arithmetik Algebra Analysis*. (G. Schubring, Trans.) Berlin, Heidelberg: Springer. (Original work published 1924).

OECD (2005). *Teachers Matter: Attracting, Developing and Retaining Effective Teachers, Education and Training Policy*. Paris: OECD Publishing.

Przybilla, J., Brandl, M., Vinerean, M., & Liljekvist, Y. (in press). Interactive Mathematical Maps – A contextualized way of meaningful Learning. In Nortvedt, G. A. et al. (Eds.), *Papers from NORMA20. Proceedings of the Ninth Nordic Conference on Mathematics Education*, Oslo, June 1–4, 2021. (postponed)

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12 (2), 151–169.

Tall, D. (2008). The Transition to Formal Thinking in Mathematics. *Mathematics Education Research Journal*, 20 (2), 5–24.

Venter, E. (2017). Good teacher education – an exploratory study. *The Independent Journal of Teaching and Learning*, 12 (1), 63–73.

Winsløw, C., & Grønbæk, N. (2013). *Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus*. Retrieved from: https://www.researchgate.net/publication/243963587

Scratch programming and student's explanations

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Programming is being included in many educational policies, also in Norway. A study involving first-year pre-service teachers and year four students is undertaken to address the increased emphasis on programming. The focus is on links between ScratchJr functions and students' mathematical explanations and justifications. The results indicate that some functions in ScratchJr have the potential to foster such mathematical argumentation, but it requires appropriate mathematical tasks and teacher awareness about how to support the students' work.

Keywords: programming, mathematical explanations and justifications, task design.

INTRODUCTION: BACKGROUND, RATIONALE, RESEARCH QUESTION

ICT literacy is the ability to use technology to develop 21st-century knowledge and skills such as critical thinking, communication, and collaboration (Dede, 2010). In recent years, this focus on ICT literacy has re-orientated to include programming. According to Balanskat and Engelhardt (2015), programming has been included in more than 20 national curriculums. This is likely to increase with the EU's policy document *Digital agenda for Europa* (http://ec.europa.eu/digital-agenda). In their research, Bocconi, Chioccariello, and Earp (2018) found that programming as an important 21st-century skill was becoming more evident in national policy documents. In the future, Balanskat and Engelhardt (2015) argue that many of today's students will be involved in developing technology.

In Norway, there has been an emphasis in the national curriculum on digital skills and students' abilities to express themselves orally and in writing, and these focuses are maintained in the revised curriculum that will take effect during autumn 2020 (Norwegian Directorate for Education and Training, 2020). As well, there is an increased focus on students' argumentation, their explanations and justifications. However, the most debated change is the prominent role given to programming. In the new curriculum, programming is implicitly included in the competence aims in mathematics from year two and explicitly included from year five and onwards.

Programming in school is not new. In the 1970-80s, the use of programming languages like BASIC and LOGO faced challenges such as adding additional demands on students, little transfer value, and effects on students' mathematical understanding took time to emerge (Hoyles & Noss, 1987). In two recent reviews, Forsström and Kaufmann (2018) and Popat and Starkey (2019) have found that there remains a lack of convincing evidence for the educational potential of programming in mathematics education. Thus, there appears to be a gap between education policies and education research. Investigating the relationship of programming and mathematical explanations and justifications can address this gap. There are, however, studies such as Kaufmann and Stenseth (2020) that document links between programming and mathematical argumentation. According to Mariotti (2012), particular tools can be resources that support teachers' didactical actions, and in this study, we investigate if and how programming with ScratchJr can be a tool that facilitates student's argumentation. The question we pose is therefore: *what makes grade four students engage or not in (multimodal) mathematical explanations and justifications when programming in pairs with ScratchJr?* Several factors might play a role, such as the program functions of ScratchJr, the teacher's introductions and follow-ups when helping the students, and the kinds of tasks given to the students. We emphasise which program functions were used and if and how they fostered or hindered students' mathematical explanations and justifications.

ANALYTICAL FRAMEWORK

ScratchJr programming provides opportunities to communicate mathematics with multiple modalities in addition to language like gestures and screen elements (code blocks and animations). Morgan and Alshwaikh (2012) argued for the importance of considering the contribution of different modalities and how they can be related. We investigate, inspired by Albano, Iaconor, and Mariotti's (2017) scripting approach, how ScratchJr functionalities can mediate students' explanations and justifications.

A communicative function of an explanation is to provide an answer to an explicitly- or implicitly-posed question. According to Donaldson (1986), the question type defines a *mode of explanation*: empirical (what has happened to cause …?), intentional (for what purpose …?), deductive (how do you know that …?), or procedural (how do you DO …?). By studying explanations and justifications in a mathematics class, Yackel (2001) differentiated between mathematical explanations and justification as social *constructs* based on the communicative function they serve. Mathematical explanations have a main communicational function of clarifying aspects of mathematical thinking that may not be completely clear to others (Yackel, 2001), while the function of mathematical justification is to respond "to challenges to apparent violations of normative mathematical activity" (Yackel, 2001, p. 13).

From the perspective of Donaldson's explanation modes, mathematical explanations seem equivalent to the empirical mode and mathematical justifications seem similar to the deductive mode. Although both empirical and procedural modes include a directional indicator through the inclusion of words such as "because" or "so", empirical explanations use it to show that something is caused by something else, while procedural explanations use it to emphasize the temporal order. The procedural explanation mode does therefore not necessarily provide information about mathematical thinking. "The role of causal connectives in deductive sentences is to make explicit the links in the deductive process, rather than causal relations between events" (Donaldson, 1986, p. 104). Thus, a deductive mode of explanation uses evidence through logical reasoning to support why something is the case.

METHODOLOGY

This paper concerns year four students' exploration of programming with ScratchJr in two mathematics lessons led by a first-year pre-service teacher (PT). The students worked in pairs and shared a tablet. The Jr.-version of Scratch was used because it works well on touch-based tablets and its iconic representation for the function blocks makes it user-friendly for beginners. In this paper, the functions' names are italicised and correspond with the block descriptions in the ScratchJr user manual.

The study is part of the first loop of a design-based research project (e.g. Sandoval & Bell, 2004), supported by the Norwegian Research Council, called *Learning about teaching argumentation for critical mathematics education* (LATACME) in multilingual classrooms. The data comes from a partnership school during the PT's second practicum period. Two audio and video recordings of the PT's introductions and five audio and video recordings of the students' work were collected across two days. The screen recordings and the students' discussions were combined in picture-in-picture movies and transcribed. Here we focus on one of the pairs, Per and Tor.

The students' task was to use multiplication tables (corresponding number sequences) to navigate characters when creating an animation in ScratchJr. The assumption was that this open task would make students engage in mathematical explanations and justifications. However, neither the PT nor the students had much experience with ScratchJr. The PT had little time to prepare the lessons and did not have the opportunity to discuss the plans with his teacher educators. Therefore, we report on what Skovsmose and Borba (2000) refer to as "the current situation" which forms the base stage from which a design-based research project can be developed.

When analysing the data, we first identified the excerpts containing mathematical explanations and justifications by searching for explanation modes as well as the ScratchJr functions the students used. Then, to address what made the students engage or not in explanations and justification, we focused on the use of three different modalities: the spoken language, the codes, and the animations. Other factors such as the PT's follow-ups were also considered.

FINDINGS AND DISCUSSION

ScratchJr has some constraints that influence how a task can be solved. For example, the grid in ScratchJr is fixed to 20x15 squares and the movement is circular. This means that a character can move a maximum of 20 steps before it disappears from the screen and reappears on the other side. The requirement to use multiplication tables to generate a sequence of moves makes it easy to exceed 20 even with the multiplication table for 2. Per and Tor experience the issue of characters disappearing on both days, but only in some cases are mathematical explanations and justifications provided. In this section, we chronologically present analyses of three excerpts illustrating different explanation modes from both days accompanied by a brief description of the lessons.

On the first day, the students get the task to "create an animation that uses the multiplication tables for 2 and 3". In the introduction, the PT explains the use of *motion* functions and asks the students to use multiplication tables to navigate characters, otherwise no specific instructions are given. Per and Tor use some time to explore the program before they start programming a crab moving on a beach background.

Excerpt 1. Tor finishes the program for the crab (Figure 1), and Per starts the animation. The crab leaves the screen and then reappears on the left side to proceed almost to the centre where it stops. Tor says, "No! We have to take eeh … I guess we have to take away …" Tor drags the last two blocks aside and, while holding them, asks, "How do we delete these?" Per moves the blocks back to the menu and they disappear. Tor then runs the remaining program (Figure 2) and concludes that the crab stays on the screen.

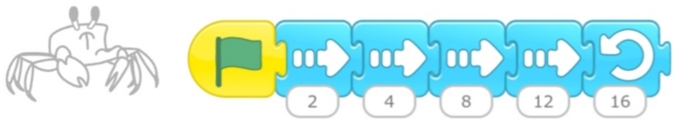

Figure 1: Code when crab leaves screen

Figure 2: Code when crab stays on screen

When the code makes the crab leave the screen and reappear on the other side, the students says "No!" and start immediately to revise the code. The animation mediates an important idea of measurement, namely the indirect comparison of length. The code can mediate the idea of a function for the total path length expressed as the number of steps per block as it consists of four *move right* blocks grouped according to the multiplication tables for 2 and 4. It does not though mediate the idea of measurement as clear as the animation because there is no limitation to the numbers to be used in the code due to the circular movement. Combined, the code and the animation help them to see how to keep the crab's movement on the screen. Tor says "I guess we have to take away …" and he disconnects the two last code blocks. This can be regarded as a procedural mathematical explanation because it, although it requires knowledge about the solution, mainly concerns how to solve the problem. Per deletes the two disconnected blocks by moving them back to the menu, and this manipulation of the blocks is also a procedural explanation, though non-mathematical. When they had done the code adjustments, they checked if they had solved the problem by running the code again.

In this excerpt, the differences between the ideas mediated by the code and the animation invite the students to solve a mathematical problem and provide explanations. ScratchJr provides visual modalities both to understand the problem and to communicate the solution. After 19 minutes, a teacher educator turns on the *grid* function to help the students program a boat approaching the beach, so that it includes perspective effects. They try out the *enlarge* and *normal size* functions from the *control menu*, but no use of the *grid* is evident.

During the introduction on the second day, the PT asks some of the students to explain the functions in the different menus of ScratchJr. He comments and adds several examples when the *say command* (from the *looks* functions) and the *sound* and *motion* functions are discussed. Then Per and Tor get the task to create an animation using the multiplication tables for 6 and 8. They choose a birthday party as a topic. After two minutes, Tor turns on the *grid* function.

Excerpt 2. During the first nine minutes, Per and Tor program the pink alien to move on the scene, add sound at several steps, and review the result several times. The PT approaches the students, looks at the screen, checks his notes, and decides to take part in their discussion:

PT:	Now you actually have to start including bigger numbers [points at the screen]. Is it big multiplication tables, the multiplication table for 6 and … [checks his notes] 8?
Tor:	[Looks at PT] Yes, we have 12 [points at the rotation block] (see Figure 3).
PT:	But 12 in itself is not how the multiplication table for 8 is built up.
Tor:	But it is very difficult with eight because then we have to go so far, then we have to do something else.

Figure 3: Code for Pink alien

Figure 4: Code for Blue alien

The students seem to try to avoid the problem of the limited width of the screen by only using the number 2 on their *motion* blocks (Figure 3). This is not in line with the task of using the multiplication tables for 6 and 8. When the PT points this out, Tor argues that they cannot use larger numbers from the multiplication table for 8 because it will move the alien too far. He answers the PT's why-question by explaining their decision not to use the multiplication table for 8, and this can therefore be considered as an example of an intentional explanation mode. The PT's involvement makes the students look for how actions, other than movements, can be used in the animation in order to follow the pattern of the required multiplication tables.

Excerpt 3. After the PT leaves, Per and Tor add a new page with the blue alien as the character. They disagree whether they should use the multiplication table for 6 or 8, but then Per makes a program (see Figure 4) and the blue alien stops having just a few more steps left before it might disappear from the screen:

Tor:	We have to think about how many steps it is.
Per:	Six.
Tor:	One, two, three, four, forward [points at the four squares of the grid ahead of the alien]. So, we have to take four [types 4]. We have to take four. Look, it will be as here. And start!

While Per makes the code, Tor counts and presents an argument for how many steps the alien has left before disappearing from the screen. The mathematical problem is similar to the one in the Excerpt 1. However, this time the *grid* function has been turned on, and this adds an iconic representation for the number of steps in the animation. Together, they mediate the idea of the measurement by counting. This supports Tor's argument by providing evidence for his point of view. The combination of gesture (counting by pointing at the grid) and language can be considered a mathematical explanation. Although it has a clear procedural goal (how to program the alien to move to the end of the scene), it is a deductive explanation, because Tor first counts the number of squares between the alien and the goal, and then concludes that they have to write four in the program.

On the second day, Per and Tor use the *looks* (only *say*), *sound* and *motion* functions (except *go back*). They use *reset characters* several times, but do not use the *control* functions as they did the first day. Although the *grid* function is turned on the first day as well, it is not used to solve the mathematical problem. On both days, the functions the PT emphasises in his presentations are the ones most frequently used by the students. The *motion*, *grid*, *sound*, and *looks* functions seem to foster explanations and justifications. Procedural and intentional explanation' modes were identified both days, often related to ScratchJr functionality, but the deductive mode of explanation occurred only on the second day. The students concentrate mostly on forming meaningful storylines, but they adhere more closely to the assigned multiplication tables on the first day. The communication moves from focusing on the exploration of ScratchJr functions during the first day to become more mathematical and result-oriented on the second day as they get more familiar with the program. This is in line with how the PT organized the two introductions, from brief comments to discussion of the functions.

CONCLUSION

What makes grade four students engage or not in mathematical explanations and justifications when programming in pairs with ScratchJr? The focus in this paper has been on which program functions were used and if and how they fostered or hindered such argumentation. Summarized, the study provides results showing that students' mathematical argumentation can be mediated (cf. Albano et al., 2017) by: conflicting differences between modalities in line with Morgan and Alshwaikh (2012) (cf. excerpt 1); thinking on the solution of the task mediated by the program or one of the modalities (cf. excerpt 3), and, in some cases, negotiating the task when challenged by the PT about how well they addressed it. ScratchJr mediates explanations and justifications by making several representations of mathematical ideas simultaneously available through different modalities.

The study contributes, like Kaufmann and Stenseth (2020), to documenting an educational potential of programming in mathematics education. The affordances and constraints of ScratchJr have some didactical implications for mathematics teachers who plan to use it. For example, ScratchJr allows sending messages and wait a certain time for an action, but it lacks conditional blocks. The animation with the *grid* visualises well that one step forward increases the x-coordinate by one. The circular nature of the *grid* surprises the students and can trigger explanations and justifications. Combining the *motion* and *control* functions can create a perspective effect for a moving character, while the rotation of a character is limited and only works for some of them. These aspects should be taken into consideration during task design and other didactic actions aiming to facilitate student's argumentation.

Although using multiplication tables when making animations might not seem challenging enough mathematically for year four students, in a programming context with ScratchJr, it provides opportunities to investigate measurement ideas. A more explicit focus in the task formulation on number sequences in the multiplication tables could invite students to focus more on functional reasoning. Discussing the mathematical ideas students experience while programming and how the program can be made to do what it does, can facilitate students' mathematical argumentation.

Teachers' preparation, and the presentations and follow-ups in class, influence to what extent students explain and justify with mathematics, but there are also other aspects that deserve further research. Investigating the role of students and teachers' gestures, how teachers can develop and use knowledge about ScratchJr and task design to facilitate mathematical argumentations, and how teacher educators can help PTs gain sufficient knowledge about how to include programming in their mathematics teaching, are just some examples.

REFERENCES

Albano, G., Iacono, U. D., & Mariotti, M. A. (2017). A computer-based collaboration script to mediate verbal argumentation in mathematics. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 2507–2514). Dublin, Ireland: DCU Institute of Education & ERME.

Balanskat, A., & Engelhardt, K. (2015). *Computing our future: Computer programming and coding - Priorities, school curricula and initiatives across Europe*. Brussels, Belgium: European Schoolnet.

Bocconi, S., Chioccariello, A., & Earp, J. (2018). *The Nordic approach to introducing computational thinking and programming in compulsory education. Report prepared for the Nordic@BETT2018 Steering Group*. https://doi.org/10.17471/54007

Dede, C. (2010). Comparing frameworks for 21st century skills? In J. Bellenca & R. Brandt (Eds.), *21st century skills. Rethinking how students learn* (pp. 51–75). Bloomington, IN: Solution Tree Press.

Donaldson, M. L. (1986). *Children's explanations.* Cambridge: Cambridge University Press.

Forsström, S., & Kaufmann, O. T. (2018). A literature review exploring the use of programming in mathematics education. *International Journal of Learning, Teaching, and Educational Research*, *17*(12), 18–32.

Hoyles, C., & Noss, R. (1987) Synthesizing mathematical conceptions and their formalization through the construction of a Logo-based school mathematics curriculum. *International Journal of Mathematical Education in Science and Technology*, *18*(4), 581–595, DOI: 10.1080/0020739870180411

Kaufmann, O. T., & Stenseth, B. (2020). Programming in mathematics education. *International Journal of Mathematical Education in Science and Technology*, DOI:10.1080/0020739X.2020.1736349

Mariotti, M. A. (2012). ICT as opportunities for teaching-learning in a mathematics classroom: The semiotic potential of artefacts. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 25–40). Taipei, Taiwan: PME.

Morgan, C., & Alshwaikh, J. (2012). Communicating experience of 3D space: Mathematical and everyday discourse. *Mathematical Thinking and Learning*, *14*(3), 199–225.

Norwegian Directorate for Education and Training (2020). *The mathematics curricula year 1–10*. Retrieved from https://www.udir.no/lk20/mat01-05

Popat, S. & Starkey, L. (2019). Learning to code or coding to learn? A systematic review. *Computers & Education*, *128*, 365–376.

Sandoval, W. A. & Bell, P. (2004). Design-based research methods for studying learning in context: Introduction. *Educational Psychologist*, *39*(4), 199–201.

Skovsmose, O., & Borba, M. (2000). *Research methodology and critical mathematics education.* Aalborg: Centre for Research in Learning Mathematics, Aalborg University, Roskilde University, Royal Danish School of educational Studies.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 9–25). Norwich: PME.

Examining educational staff's expansive learning process, to understand the use of digital manipulative artefacts to support the students' computational thinking and mathematical understanding

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This paper discusses the need for a professional development process if the educational staff were to use digital manipulative artefacts in primary mathematics education. The project is based on how to develop a teaching sequence to use robots as digitally manipulative artefacts to support students' mathematical understanding and computational thinking. This paper is a part of an ongoing project and will demonstrate how educational staff through an expansive learning process, develop a teaching sequence to support the students' mathematical and computational understanding by working with robots as a digital manipulative artefact.

Keywords: mathematics education, primary level, computational thinking, digital manipulative artefacts, work development research

INTRODUCTION AND RELEVANCE

With this paper, I would like to shed light on the possibility of incorporating technology into mathematics in the form of robots already by the early years, in order to support work with the students' understanding of mathematical concepts supported by computational thinking. With the rise of the digital age, more and more digital manipulatives have become available, such as robots that can be used in teaching contexts. According to Nugent, Barker, Grandgenett, and Adamchuk (2009) the digital manipulatives embed computational capabilities, and can be seen as catalysts that can help youth in problem-solving approaches using the digital manipulatives as a tool to help them in development of their thinking.

In an article published in 2006, Jeanette Wing described computational thinking (CT) as a way of "solving problems, designing systems, and understanding human behaviour by drawing on the concepts fundamental to computer science" (p.33). CT is a problem-solving strategy in which a complex problem can be solved either by drawing on known strategies or by means of decomposition and abstraction. According to Grover and Pea (2018), the core elements are logic, algorithms, abstraction, pattern recognition, evaluation and automation. They also define computational practices such as decomposing, the creation of computational artefacts, testing, and debugging and iterative processes. In this project, problem-solving should be seen as a context in which the students can experience and develop their mathematical understanding using the concepts from CT, and it considers how the educational staff can help to support these processes.

According to Lee et al. (2011) there is a lack of opportunities for teachers to learn CT as a part of their professional development, and the teachers do not have the necessary access to materials and technology to use in their teaching. It is not just important that the teachers are offering professional development courses, but also, as reported by Black et al. (2013), there is a need to establish communities of practice, to provide ongoing support and sharing of resources. However, there is an increased need to focus on how to establish professional development in order to support the educational staff (the participating classes' maths teachers and teacher assistants) to embed the students' CT in mathematics. There is therefore a need to explore how the educational staff can be supported in using robots as a manipulative artefact in mathematics and how the built-in computational thinking in the robot can help to support the students' mathematical understanding.

THEORETICAL BACKGROUND

In adopting the term 'learning environment', I consider the teaching and learning situation as a whole (Bottino & Chiappini, 2002). This means that I am interested in analysing teaching and learning processes to use in the educational staff's professional development. As a part of this, robots have an important role as a manipulative artefact in mediating teaching and learning processes.

The expansive learning process (Engeström, 2001) gives me a framework that is useful for describing and developing the educational staff's collective professional development process. The expansive learning process is concerned with the historical and cultural development of activity, and I am specifically interested in the mediation role of the digital manipulative artefact, and how the educational staff are using it to develop the students' mathematical understanding. For Engeström (2001), it is important to learn from new forms of activity which are not yet available. In this case, the knowledge and skills are learned as they are being created by the educational staff. "An expansive learning activity will produce culturally new patterns of activity, and expansive learning at work will produce new forms of work activity at the workplace" (Engeström, 2001 p. 139). The typical sequence of learning action in an expansive cycle is the following: *Questioning* is the first action, where a question will be asked, criticising or rejecting some of the already accepted practices and existing knowledge. *Analysing* is the second action. This analysis involves a mental, discursive, or practical transformation of the situation to discover reasons or explanatory mechanisms. Analysis evokes the "why" questions and exposes the principles. *Modelling* is the third action. Here, the participants construct an explicit and simplified model of the new idea that explains and offers a new resolution to the problematic situation. *Examining* the model is the fourth action. Here, the participants are running, operating, and experimenting with the new model in order to understand its dynamics, potentials, and limitations. *Implementing* the new model is the fifth action. Here, the participants are using and testing the new model. *Reflecting* is the sixth action. Here, the participants are evaluating the new model and perhaps adjusting it. *Consolidating* the model and its outcome into a new stable form of practice is the final action (Engeström, 2001). These seven steps for increased understanding should be seen as an outwardly expanding cycle but with many different kinds of action that can take place at any time. Mapping the educational staff’s learning action can provide a collective mirror for the educational staff and help them to identify problems in the activity. The seven steps also allow the researcher to identify and analyse which types of learning action is most dominant in a particular period of time.

In order to support the educational staff’s expansive learning processes, a teaching sequence, centred on the use of robots as a manipulative artefact in geometry, was analysed together with the educational staff. I then focused on the moment of the teaching sequence where the use of the robots was expected to unfold mathematical meanings. The teaching sequence was considered using three second-grade classes. I will present the analysis of selected episodes drawn from the teaching sequences that have been used in the educational staff’s professional development to support their expansive learning processes. The concept of teaching sequences emerged from the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), and offers a framework to design teaching sequences embedding digital manipulative artefacts, and to analyse the collected data in order to gain insight into the use of robots to support students’ CT and mathematical understanding. The structure of a teaching sequence can be defined as a didactical cycle that consists of an iteration where different typology of activities consist such as; activities with an artefact, individual/small group production of signs, collective production of signs, and mathematical discussion (Bartolini Bussi & Mariotti, 2008).

The teaching sequences have been developed by the educational staff, and the tasks and activities were the same for all three classes. This allowed me to look at the same teaching sequence concerning geometry in three different classes, and explore the potential for using robots to gain a deeper understanding of geometry for the students. This leads to the following research question: How can the use of a teaching sequences support the educational staff’s expansive learning process in using robots as a digital manipulative artefact to aid the students’ computational thinking and mathematical understanding?

METHOD

The study has been conducted with the participation of three second-grade classes (A, B and C), and the classes’ educational staff. The teaching sequences were developed throughout two lessons, each of them lasting one and a half hours in each class.

Ethnographic data in the form of observations and video have been used to analyse the teaching sequences, and unfold the students’ use of the robots to gain a better understanding of how the educational staff can create tasks and activities that support the students’ CT and mathematical understanding. Through professional development sessions, the educational staff was presented with quotes from the teaching sequences, and were then guided by the researcher to analyse and gain a better understanding of the tasks and activities. This analysis was used to gain new insight, and to develop the teaching sequence further. There have been two professional development sessions with the educational staff, one after the second didactical cycle, and one after the last didactical cycle. These sessions were videotaped, which allowed for mapping and analysing the educational staff's expansive learning process (Engeström, 2001).

Overview of the teaching sequence

The teaching sequences that the educational staff has formulated for the students were structured in order to support the students' geometric thinking. According to Goldenberg, Dougherty, Zbiek and Clements (2014), is it important to help students develop a precise language about geometric figures, to give them words and language as well as the ability to participate in discussions about the categorisation of the figures. Attention will therefore be paid to tasks and activities that encourage the students with help from the manipulative artefact to categorise the geometric figures, and to use the appropriate words. As outlined, there was a need for the educational staff to develop and model new solutions such as tasks and materials that were to be used in connection with the robot and the geometric field of study.

As stated above, the teaching sequences are followed by an iterative process of didactic cycles.

The first didactic cycle involved the robots. The task was that the students should become familiar with the robot, and figure out how it works.

The second didactical cycle focused on how to get the robot to make a square, and then they had to investigate how small and how large a square the robot could make. According to CT, the students were asked to create an algorithm for making the square.

The third didactical cycle; here the students were asked to work with the geometric properties of polygons. The task (Fig. 1) was to make the robot land on, for example, all squares, triangles, etc., and describe the characteristics of the individual polygon.

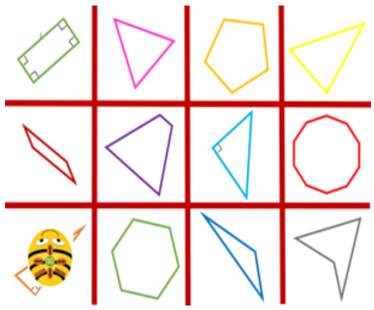

Figure 1. Task geometric properties

The fourth didactical cycle had a problem-solving approach, and the students had to figure out which type of polygon the robot could make. The students were required to investigate the type – from one-sided to ten-sided polygon – the robot could make, and see if they could make any generalisations from it. In the next section, I will focus on the first and the third didactical cycles to unfold how the teaching sequences were used as a part of the educational staff's expansive learning process.

Focus on the first didactical cycle

The students worked together in pairs, and each pair had their own robot. During the first task, the students were required to become familiar with the robot. The teacher provided a short overview of the different buttons on the robots, and after that the students was asked to figured out what the different buttons were for. Subsequently, the teacher followed-up on it in the class discussion. The following quote demonstrates how the students are trying to understand how the robots work. The quote is taken from class A, and the discussion section at the didactical cycle.

Student 1:	We clicked a whole lot and but it didn't do what we wanted it to.
Student 2:	It was because we forgot to press delete. Then we clicked two forwards and two to the side, two backwards and two to the side.
Teacher:	What did you think it was doing? A square?
Student 1:	Yes, but it didn't. It just started driving around and going backwards (the student moves his body to show what the robot had done). Because we forgot to press delete.
Teacher:	If you had clicked on delete, would it then have made a square? Did you try it?
Student:	You must press two forward, one left, two forward, one left, two forward, one left, two forward, one left.
Teacher:	This is something we should actually try out in a moment. I actually saw it when I was with you when you said that you were going to try to make it do a square and it just didn't.
Student 1:	(The student turns around to demonstrate the ability to make a square.)

This was a good example of what the student was struggling with in the first task. Many had trouble remembering to clear the robot after ending an activity, and thereby trouble with creating a new activity. Student 1 also used his body to demonstrate the movement of the robot, which helped him to make the movement more understandable. The fact that the students were initially given time to investigate the robot made them more focused on the mathematics in the later exercises.

Professional development

After the second didactical cycle, the educational staff were given a training session to help them develop the teaching sequences further. The quote above was shown to them to let them know how the students had worked with the robots. The quotes also showed that, even when the task did not include mathematics, the group attempted to get the robot to make a square. The quote showed the potential for using the robots as a digital manipulative in mathematics, but also the need to develop tasks and activities to support this. Here, the educational staff started to ask questions on the previous practices, as part of their expansive learning processes. They became aware of the fact that there was a need to design tasks and additional material that could be combined with the robot if it were to be used to support the students' mathematical concepts of understanding. It appeared that the robot could not be regarded as a manipulative artefact for mathematics itself, but that it was the didactical framework and the task which created the possibilities for supporting the mathematical concept of understanding. For this reason, the educational staff had to act to ensure that the developed materials could be used in connection with the robot if it were to support the geometric subject area.

Focus on the third didactical cycle

In the third didactical cycle, the students worked on two tasks. In the first, they were asked to work with the geometric properties of polygons, see Fig. 1. The students had to make the robot land on, for example, all squares, triangles, etc., and describe the characteristics of the individual polygons. When the students sorted the polygons, it helped to initiate a process in which they focused their attention on the characteristics of the polygon. This would allow them to increase their knowledge of the individual polygons. The students must both relate to the robot through CT, where they must first get an overview of, for example, the triangles on the worksheet, and then create an algorithm that moves the robot from one triangle to another. The students use the robots as a manipulative object by describing the characteristics that lie behind the polygon the robot lands on.

In the second task, the students had to categorise different polygons on the basis of different criteria such as a right angle, acute angle, etc. This helped to support the students' study of various properties such as a 'right angle' and 'equally long sides'. Categorising the polygons from Fig. 1 on the basis of new criteria helps to support students' reasoning at a higher level of abstraction (Goldenberg et al., 2014). This also gave the students the opportunity to distinguish between polygons which resembled each other and to become aware of the common characteristics of polygons which did not appear to have the same characteristics. Through their work on categorising the polygons, the students developed an understanding of the fact that the different polygons could have the same characteristics. When the students worked with the robots as a manipulative artefact, they are working with CT when they are programming the robots. Through CT, the students work to get the robots to move in different sequences, for example when the robot has to move around all the triangles (Fig. 1). During the task, they were continually debugging and correcting their codes if the robot did not land on the desired polygon. In this way, the students were trained in their CT when they introduced what they had worked with in the classroom as well as the way in which they had solved the rewarding task.

Professional development

After the fourth didactical cycle, the educational staff were given another session. The session focused on the third and fourth didactical cycles, and the teaching sequences as a whole. The educational staff was asked to mention which part of CT the students had been working with in the two types of tasks in the third didactical cycle. However, this proved difficult for them: “I actually don’t know. I know that we were working computational with the robot, but which part of CT we used, is hard for me to say” (Teacher, Class C). This showed that the educational staff were using the digital manipulative artefacts as a tool to let the students solve the task. It is thereby suggested that the educational staff gain a better understanding of CT, so they can support and distinguish signs of CT that can support the students’ mathematical understanding. Together with the researcher the educational staff were questioning and analysing the teaching sequence to gain a better understanding of the possibility to work with the robots as a digital manipulative tool in mathematics. Through the teaching sequence the educational staff modelled the didactical cycles, and examined the teaching sequence by using the robots to develop the students’ CT and mathematical understanding. “This form of development has given me the courage to do more, and use the technology in small steps” (Teacher, class A). The educational staff stated that working with the digital manipulative artefacts through a collective professional development process had an impact on them, based on the fact that they had to develop new teaching sequences together. In other words, it has given the educational staff the courage to work with technologies in teaching. The educational staff felt that they were gradually being supported in their own expansive learning process and were getting help to change their activities through joint collective actions that helped to develop their practice regarding the use of digital manipulatives to support of the students’ mathematical and computational understanding.

CONCLUSION

Expansive learning should be seen as an iterative process, where the educational staff together with the researcher examine the current practices, along with the teaching sequences. Considering the educational staff’s expansive learning processes, they are still in the beginning of the processes. By analysing the didactical cycles, it helps the educational staff to gain a greater understanding of how the robots could be used as a manipulative artefact to support the students’ mathematical understanding.

From an educational perspective, working with a robot as a digitally manipulative artefact helps the students to reason, problem-solve, generalise, and predict, which may lead to a deeper mathematical understanding. The possibilities for supporting the students' mathematical learning are present with digital manipulative artefacts under the right pedagogical and didactical prerequisites. During the study, it was found that the robot itself could not be regarded as a mathematical manipulative artefact, but that the didactical cycles and the work on the tasks, which meant that the robot, and the built-in CT helped to support the students' development of their mathematical and computational understanding.

REFERENCES

Bartolini Bussi, M. G. &Mariotti, M. A. (2008): Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In: *Handbook of international research in mathematics education*, New York. 746-783.

Back, J., Brodie, J., Curzon, P., Myketiak, C., McOwan, P.W., & Meagher, L.R. (2013). Making computing interesting to school students: teachers' perspectives. *ITiCSE* '13.

Bottino & Chiappini (2002): Advanced Technology and Learning Environments: Their Relationships Within the Arithmetic Problem-Solving Domain, in *Handbook of International Research in Mathematics Education,* red. English, L.D. chapter 29.

Engeström, Y. (2001): Expansive Learning at Work: Toward an activity theoretical reconceptualization, *Journal of Education and Work*, 14:1, 133-156

Goldenberg, P. Dougherty, B., Zbiek, R.M. & Clements, D. (2014): *Developing Essential Understanding of Geometry and Measurement for Teaching Mathematics* in Pre-K-Grade 2, National Council of Teachers

Grover, S. & Pea, R. (2018). Computational Thinking: A competency whose time has come. In *Computer Science Education: Perspectives on teaching and learning*, Sentance, S., Carsten, S., & Barendsen, E. (Eds). Bloomsbury.

Lee, I., Martin, F., Denner, J., Coulter, B., Allan, W., Erickson, J., Malyn-Smith, J., & Werner, L.L. (2011). Computational thinking for youth in practice. *Inroads*, 2, 32-37.

Nugent, G., Barker, B.T., Grandgenett, N., & Adamchuk, V. (2009). The use of digital manipulatives in k-12: robotics, GPS/GIS and programming. 2009 39th *IEEE Frontiers in Education Conference,* 1-6.

Wing, J. M. (2006): Computational thinking, in *Communications of the ACM*, 49(3), p. 33–35.

Facilitating the design and enactment of mathematics curricula through digital mapping

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In this paper, members of three research teams describe their digital mapping projects: Math-Mapper 6-8 (USA), the Dynamic Mathematics Curriculum Network (Canada), and the Cambridge Mathematics Framework (England). Each researcher shares early evidence of the ways their map is being used. This evidence illustrates the potential of these maps to enhance the design and enactment of mathematics curricula and to foster professional learning and knowledge sharing among groups engaged in mathematics education. The maps are likely to be increasingly meaningful for their intended audiences because they incorporate user feedback and new research. Other insights arising from dialogue among the researchers are shared.

Keywords: digital technology, curriculum design, mathematics professional learning.

A GENERATIVE DIALOGUE AMONG RESEARCH TEAMS

In this paper, we consider the potential impact of three projects focused on the digital mapping of school mathematics. Each project uses digital technology to make the many connections in school mathematics more visible. *Math-Mapper 6*-8 (MM6-8) is a learning map being developed in the United States; the *Dynamic Mathematics Curriculum (DMC) Network* is a digital network that emerged from a Canadian research project; and the *Cambridge Mathematics Framework* is a knowledge map being developed in England. Members of the research teams for these projects began meeting after the DMC Project (Koch, Suurtamm, Lazarus & Masterson, 2018) was presented at the *Fifth ERME Topic Conference on Mathematics Education in the Digital Era*.

Our initial discussions revealed that the connections as well as the underlying basis for the connections, were unique to each project. At the same time, as noted in Koch, Confrey, Clark-Wilson, Jameson and Suurtamm (in press), we discovered that each map illustrates how mathematics concepts relate to one another, includes connections from school mathematics to related research, and offers connections to instructional resources. Each project also uses digital tools to facilitate knowledge sharing among teachers, curriculum designers, teacher educators, and researchers. Our ongoing discussions led to insights into how each map functions as a dynamic, emergent space as summarized in Koch et al. (in press).

In this paper, we introduce the “spaces of representation” (Siegert, 2011) in each map by describing their respective purposes, intended audiences, underlying basis, and structure. Each researcher then describes some ways their map is being used to facilitate the design and enactment of mathematics curricula. Here, we use the term *curricula* to convey both the mandated curriculum standards that specify the concepts and processes students are expected to learn in each grade, as well as the curricular materials educators use to enact these standards. Each researcher also provides examples of changes made to their map in response to user feedback and considers the ways their map functions as a shared resource across professional communities.

We see these projects as situated at the intersection of two MEDA 2020 themes: mathematics teacher education and professional development in the digital age (Theme 1), and curriculum development and task design in the digital age (Theme 2). We look forward to critical engagement with conference participants on these topics.

THREE DIGITAL MAPPING PROJECTS

Math-Mapper 6-8 (Jere Confrey)

Math Mapper 6-8 (MM6-8) is one component of a digital learning system which covers the content of middle grades mathematics in the United States (Siemens & Confrey, 2015). The map can be accessed by registering an account at sudds.co. The purpose of the map is two-fold. Firstly, the map creates a visual representation of the relationships among the big ideas and sub-constructs within middle school mathematics. We view big ideas as concepts that connect the content, processes, and forms of argumentation in mathematics. In doing so, big ideas can help avoid viewing mathematics as a set of fragmented topics and skills. Secondly, the map provides teachers with direct access to empirically-based learning trajectories (LTs) (Clements & Sarama, 2004; Confrey, Toutkoushian, & Shah, 2019) which can guide learner-centred instruction and ground the map's related diagnostic assessments. Confrey (2019), wrote a synthesis of research on mathematics learning trajectories which summarizes the map's theoretical foundation. MM6-8 was built and refined in a partnership among learning scientists and psychometricians in a "trading zone" that allows revision and refinement of the map (Confrey, 2019).

Figure 1: Components of Math-Mapper 6-8

The principal audience for the MM6-8 map is both students and teachers. The map replaces the linearity of a book's table of contents in favour of multiple levels of visual illustration. MM6-8 uses a non-linear, hierarchical structure which includes nine big ideas, 25 relational learning clusters (RLCs), and 62 constructs, each of which is associated with a LT. Students who use the map can see how what they are learning connects to a small but powerful set of big ideas. Teachers who use the map gain access to empirically established ideas about learning using LTs. In addition, every level of LTs in MM6-8 has a related set of assessment tests and practice accompanied by intuitive student reports to guide diagnostically-valid instructional moves. Thus, teachers can use MM6-8 to re-examine instructional materials and curriculum standards and diagnostically assess student progress along LTs (Confrey, Gianopulos, McGowan, Shah, & Belcher, 2017). Connections to other resources are also offered including the Common Core State Standards - Mathematics (CCS-M) and access to illustrative resources from the “Resource library”.

Early users of MM6-8 include partnerships with six middle schools with varied demographics. Over the last four years, students have taken over 75000 MM6-8 assessments enabling the research team to use item-response theory (IRT) to conduct on-going validation of those assessments. Annual interviews with teachers have led to modifications to the map based on the use of the diagnostics resulting in more explicit delineation of misconceptions in mathematics, revisions to the map, and shorter, more focused assessments. Student data show positive correlations of increased use with improved end-of-year growth on MM6-8 tests (Confrey, Toutkoushian & Shah, 2019). Data also indicate that it takes time for teachers to learn to trust the learning trajectories and to see their relationship to instructional practices.

Dynamic Mathematics Curriculum Network (Martha Koch)

The DMC Network [1] is the result of a research project to represent the connections within school mathematics as perceived by individuals who are engaged in mathematics education (Koch et al., 2018). Theoretically rooted in complexity thinking (Davis & Simmt, 2003; Doll, 2008), the DMC Network was derived from analysis of the concepts, connections and related resources suggested by K-12 teachers, school division mathematics curriculum leaders, teacher educators, researchers, and graduate students from across Canada. In the first phase of data collection, participants engaged in video-recorded collaborative problem-solving sessions and created physical models of connections they perceived as they worked on the task. The task they were given is one that often prompts algebraic thinking. In subsequent phases, we invited mathematics educators to view the digital version of the DMC Network that we had developed through analysis of the models from the first phase, and to contribute their ideas through an online portal.

In the DMC Network, mathematics concepts and processes are represented as nodes connected to one another with curved lines. Clicking a node or connection line reveals definitions, explanations, examples, and links to research-informed resources. The position of any node is determined solely by connections between that node and other nodes. Readers are invited to view these features at dynamicmathcurriculum.ca. Based on our analysis of input from participants, some nodes connect to many others (e.g. “Algebraic expressions” currently connects to 12 nodes) while other nodes have fewer connections (e.g. “Proportionality” currently connects to 4 nodes). Educators can create many paths through the concepts and processes that are shown. A central feature of the DMC Network is the “Add to the Network” tab which invites any user to suggest new nodes or connections or recommend related resources. Contributors are asked to explain their thinking to assist the research team with evaluating their suggestions and deciding which changes should be made in the DMC Network.

We think of teachers as the main audience for this resource. The DMC Network can help teachers deepen their understanding of mathematics concepts, plan a sequence of lessons, discover a way to support a student struggling to understand a concept, or share ways of teaching mathematics that they have found effective. Teacher educators may use the DMC Network to help teachers become more aware of the connections that are called for but typically not made explicit in curriculum documents.

Participants who contributed to the first iteration of the DMC Network found the process of articulating and representing the connections they perceived as they engaged in collaborative problem solving to be challenging yet generative. Many noted that these activities deepened their understanding of mathematics concepts and processes in relation to the curriculum standards in their jurisdiction. Most saw mathematics as much more interconnected than they had realized and many sought to represent the connections they perceived by adding iterative elements to their models. Those who contributed their ideas through the "Add to the Network" feature more often suggested new connections rather than new nodes. The first iteration of the DMC Network had 10 nodes with 31 connections while the next iteration had 13 nodes and 59 connections. Here again, we noted the tendency for participants to see mathematics as deeply interconnected. A few contributors suggested changes that reflect initiatives in their context such as one teacher educator who recommended including Indigenous views of mathematics in the DMC Network. In the most recent phase of the project, high school and college educators have been invited to envision a three-step path they might take within the DMC Network and to provide feedback on the nodes and connections that might be added.

Cambridge Mathematics Framework (Ellen Jameson)

In the Cambridge Mathematics Framework (CM) project a team of designers, teachers, and researchers are developing a tool to enable the dynamic generation of maps which highlight and describe connections between ideas and experiences in school mathematics (www.cambridgemaths.org). The maps, and associated content, are representations of knowledge about mathematics learning interpreted from reports of research and practice according to our design methodology (Jameson, McClure & Gould, 2018). The purpose of the CM Framework is to support coherence in mathematics education by facilitating shared understanding of connections in mathematics learning within and between communities involved in curriculum design and enactment such as curriculum and resource designers, teachers, and teacher educators. Our purpose, theoretical influences and design methods are more fully elaborated in recent papers (Jameson, McClure & Gould, 2018; Jameson, 2019).

In order for the CM Framework to serve as a shared frame of reference, these mathematical ideas are not curriculum-specific but can be mapped to various sets of standards. Likewise, these ideas and relationships are expressed in ways which are recognisable and useful to audiences in multiple communities. Some people may be looking for a ‘way down’ to get a more detailed perspective, while others may need a ‘way up’ to see a bigger picture. Some might be looking back to see what ideas students may need to be working with at a particular point, while others may look forward to see what ideas students will need to be able to work with later on. Some may be working at a time scale of a few weeks, while others may be designing for learning over a few months, a few years, or a decade.

Mathematical content is expressed in the CM Framework maps as *waypoints*. Each waypoint contains a summary of the mathematical idea (the ‘what’) and its part in the wider narrative (the ‘why’), and lists examples of ‘student actions’ that would provide opportunities to experience the mathematics in meaningful ways. Waypoints are related to one another by *themes*. A theme is a way in which an idea develops into or is used when working with another idea. The CM Framework also includes *Research Summaries* which are documents that tell the story of a group of waypoints and themes. They include a literature review, an interactive map of the waypoints and themes, and a section which describes how research has influenced the structure and content of the map. An example of a Research Summary is available on our website (Jameson et al., 2019). Connections to other resources are also managed within this layered structure. These resources might be for designers (such as curriculum statements for curriculum comparison or revision), for teachers and teacher educators (such as professional development activities), or for both (such as glossary definitions of mathematical terms).

External reviewers evaluate our research summaries, and we conducted a Delphi study to evaluate our structure and theoretical influences (Jameson and McClure, 2020). We are piloting the use of the CM Framework for curriculum and resource design to identify core actions for the key uses of the CM Framework and to develop features, interfaces and training support. In one case, we used the CM Framework in the design of the UNICEF Learning Passport for Children on the Move (LPCM) mathematics curriculum framework, through which we were able to develop new tools and processes for mapping, analysing and revising curriculum statements, and for documenting the content and connections underlying the revised curriculum in order to provide a narrative for those who need to work with it (Jameson & Horsman, 2020). We are also currently running a survey, *CM DefineIt*, to collect data on preferences and critiques of published definitions of mathematical terms relative to teachers’ contexts (Majewska, 2019). Our next step will be to trial the features and interfaces we are developing.

INSIGHTS FROM ACROSS THE PROJECTS

Facilitating the design and enactment of curricula

Each of these digital maps is beginning to impact the design or enactment of mathematics curricula in distinct ways. Confrey identifies as a first order of impact of MM6-8, an increased awareness among teachers of their students’ thinking and a movement toward learner-centred approaches as teachers enact the curriculum standards in their setting. Koch notes that many participants who contributed to the DMC Network became more aware of the deeply connected nature of mathematics as they articulated their ideas. Insights from educators who describe their experiences planning a three-step sequence using the DMC Network will be shared at MEDA 2020. In the CM Framework, Jameson describes the integration of important conceptual connections, highlighted by evidence from research and practice, into curriculum analysis and design in ways that improve coherence.

Fostering professional learning and knowledge sharing

Feedback from early users of each project illustrates the ways these maps foster professional learning and knowledge sharing within and across the groups engaged in mathematics education. For example, Confrey characterizes MM6-8 as a "trading zone" (Confrey, 2019) among different communities where practitioners, learning scientists, and measurement specialists can discuss the map as a shared resource. On a smaller scale, participants in the DMC Network project describe their learning in comments such as "I could see some connections quickly and then started to wonder about other nodes that might be appropriately connected . . . I wanted to break some nodes down into smaller chunks". Others noted the value of the DMC Network for facilitating collaboration such as one participants' comment "For its users and contributors it seems that this type of tool can emphasize the importance of researcher-practitioner relationships". Returning to knowledge sharing at a larger scale, in the CM Framework LPCM pilot project, a curriculum development team used the CM Framework as a shared frame of reference driving discussions around strategies and trade-offs in the design of a curriculum with unusual constraints.

Digital maps as inherently dynamic tools

Each project includes processes for responding to feedback and for reflecting new research. These processes are essential for ensuring each map continues to support effective mathematics teaching and learning. The nodes, connections, definitions and resources in the DMC Network can be revised as researchers review contributions from mathematics educators. In the MM6-8 project, feedback from teachers has resulted in changes to the map to clarify mathematics misconceptions and facilitate ongoing development of assessment tools. Newly developed learning trajectories can also be added. The team developing the CM Framework has created a flexible format and structure to which modifications can be made. The CM team is currently using this flexibility to expand the range of content and to respond to feedback from external reviewers. As the CM Framework reaches a broader audience, opportunities to incorporate feedback will expand and new research can be included.

Dynamic tools can be used in ways which are creative rather than prescriptive; they leave room for choice and decision-making. A map or network can contain many overlapping paths, allowing users to focus on one or more paths relevant to their context while not losing the implications of the others. For example, the CM

Framework contains more than a single curriculum could cover, but this is what gives it the power to explore the implications of choices for content selection and sequence when designing a curriculum or a textbook.

These projects provide evidence of the ways digital maps can foster vertical integration across K-12 mathematics curriculum standards, encourage informal collaboration or more formal partnerships between groups engaged in mathematics education, and lead to a better understanding of the processes and impact of instructional change on teachers' knowledge and classroom practices.

NOTES

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REFERENCES

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, 6(*2*), 81-89.

Confrey, J. (2019). *A synthesis of research on learning trajectories/progressions in mathematics.* Paper commissioned for the OECD 2030 Learning Framework, by OECD Mathematics Curriculum Document Analysis Project Workshop. www.oecd.org/education/2030.

Confrey, J., Gianopulos, G., McGowan, W., Shah, M., & Belcher, M. (2017). Scaffolding learner-centered curricular coherence using learning maps and diagnostic assessments designed around mathematics learning trajectories. *ZDM Mathematics Education*, 49(*5*), 717-734.

Confrey, J., Toutkoushian, E. P., & Shah, M. P. (2019). A validation argument from soup to nuts: Assessing progress on learning trajectories for middle school mathematics. *Applied Measurement in Education*, 32(1), 23-42.

Confrey, J., Toutkoushian, E. P., & Shah, M. P. (2019). Working at scale to initiate ongoing validation of learning trajectory-based classroom assessments for middle grade mathematics. *Journal of Mathematical Behaviour,*

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 14*(2), 137-167.

Doll, W. E. (2008). Complexity and the culture of curriculum. *Educational Philosophy and Theory, 40*(1), 190-212.

Jameson, E. (2019). *Methodology: Building the research base*. Cambridge Mathematics.https://www.cambridgemaths.org/research/framework-documentation/

Jameson, E., & Horsman, R. (2020). *Using the Cambridge Mathematics Framework to refine the UNICEF-Cambridge Curriculum Progression Framework*. Cambridge

Mathematics. https://www.cambridgemaths.org/Images/UNICEF-cambridge-curriculum-progression-framework.pdf

Jameson, E., & McClure, L. (2020, June). Agreeing to disagree for the sake of formative evaluation: A Delphi panel deliberates on mathematical thinking and doing. Proceedings of the International Conference of the Learning Sciences 2020, Nashville, KY.

Jameson, E., McClure, L., & Gould, T. (2018). Shared perspectives on research in curriculum reform: Designing the Cambridge Mathematics Framework. ICMI Study 24 Conference Proceedings, 8. Tsukuba, Japan: ICMI.

Koch, M.J., Confrey, J., Clark-Wilson, A., Jameson, E. & Suurtamm, C. (in press). Digital mapping of school mathematics: Three innovative approaches. In A. Clark-Wilson et al. (Eds.), *Mathematics education in the digital age: Learning, practice and theory*. London: Routledge.

Koch, M.J., Suurtamm, C. Lazarus, J. & Masterson, L. (2018). Making connections: Launching a co-created digital mathematics curriculum network. In H. Weigand et al. (Eds.) *Proceedings of 5th European Society for Research in Mathematics Education Topic Conference: Mathematics Education in the Digital Age* (pp. 147-154). University of Copenhagen.

Majewska, D. (2019). *Glossary app: The development and pilot phase of CM Define It.* Cambridge Mathematics. https://www.cambridgemaths.org/Images/glossary-app-the-development-and-pilot-phase-of-CM-define-it.pdf

Siegert, B. (2011). The map IS the territory. *Radical Philosophy* (*169*), 13-16.

Siemens, G. S., & Confrey, J. (2015). Some possible implications of data-intensive research in education—the value of learning maps and evidence-centered design of assessment to educational data mining. In C. Dede (Ed.), *Data-intensive research in education: Current work and next steps. Report on two National Science Foundation-sponsored computing research association workshops* (pp. 79-87). Washington, DC: Computing Research Association.

Developing a digital tool for vignette-based professional development of mathematics teachers – the potential of different vignette formats

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Using vignettes for the professional learning of Mathematics teachers is considered to have a high potential for their growth related to mathematics education: vignettes are representations of practice which afford linking theory with practice contexts both in learning opportunities and in assessment instruments. The digital age offers various possibilities of creating and implementing vignettes in teacher education and professional development. However, evidence about best-practice use of different vignette formats, such as video, cartoon, or text vignettes, is relatively scarce. Responding to this research need, relevant existing findings are reviewed, and implications are drawn for the design of a digital tool for representing classroom situations in the framework of the European project coReflect@maths.

Keywords: Vignettes, representations of practice, teacher education, professional development, digital learning.

INTRODUCTION

Representations of professional practice contexts in vignettes can help to connect specific practical requirements of the mathematics teacher profession with theoretical contents in mathematics education (e.g., Buchbinder & Kuntze, 2018). This can be expected to have advantages for pre- and in-service teachers' professional learning: For instance, specific classroom situations can be analysed with the help of criterion knowledge – highlighting the relevance of corresponding professional knowledge for such analysis – and cooperative reflection can be encouraged, given the mostly obvious need to deal with such classroom situations. Beyond providing teachers with learning opportunities, vignettes can also be used in assessment instruments.

However, relatively little is known from empirical research about best-practice ways of using vignettes in professional learning opportunities and in assessment. The development of digital technologies offers a large variety of possibilities of representing practice contexts such as classroom situations in vignettes: One of the most recent developments, for instance, are 3D representations of classrooms with student (and teacher) avatars (e.g. Blume et al., 2018; Richter et al., 2019), which can be viewed by pre- or in-service teachers with virtual reality headsets. Vignettes using this technology may aim at placing teachers in the perception of an artificially designed representation of a situation, which can be standardised or – depending on the design and goals of the vignette use – be modified in controlled ways. More established is the use of video technology in mathematics teacher education and professional development. Video vignettes are expected to contain a high amount of context information of a classroom situation and are, therefore, considered as particularly close to the classroom situation they represent and to its professional requirements (e.g. Richter et al., 2019; Petko et al., 2003).

Still, there is a need of empirical evidence about the specific strengths of video vignettes in comparison with other vignette formats, such as text vignettes, cartoons, or animations. For the design of vignette-based digital learning or assessment environments, empirical evidence is, however, needed in the search for effective ways of technology use. Consequently, this paper addresses this research need and collects available findings with relevance for identifying potentials of different vignette formats. Implications will be drawn for the development of a digital tool for representing classroom situations in the European project coReflect@maths.

The paper will first outline a theoretical background related to vignette-based professional learning and assessment of pre- and in-service teachers. Based on the research interest derived from this background, we will then present and examine available findings related to different vignette formats and to their respective potential. Against these findings, conclusions will be discussed for the design of a digital tool for representing classroom situations in the project coReflect@maths, and an outlook on further related project goals and activities will be given.

THEORETICAL BACKGROUND

Vignette-based professional learning in mathematics education aims at fostering a broad range of aspects of professional expertise. In recent approaches of describing aspects of mathematics teachers' expertise, several interrelated and overlapping terms are used, such as "awareness" (Mason, 2002), "noticing" in the sense of "selective attention" (e.g., Seidel, Blomberg, & Renkl, 2013) or "knowledge-based reasoning" (Sherin, Jacobs, & Philipp, 2011), "professional vision" (Sherin & van Es, 2009), or "usable knowledge" (Kersting et al., 2012). All these terms have in common that they relate aspects of teachers' expertise to profession-related situation contexts, such as classroom situations (Kuntze & Friesen, 2018), so that in these approaches, profession-related situation contexts are particularly relevant for assessing teachers' expertise and for teachers' professional learning.

However, classroom situations are very complex (*Petko et al., 2003):* in classroom situations, many processes take place simultaneously and some of them are not directly visible or hearable. Moreover, an individual situation in the classroom happens only once in the same way and cannot be reproduced exactly as it had originally taken place. Even if a classroom situation is recorded on video, the video can capture only specific perspectives and only visual and auditive information can be stored, so that multitude of perspectives and thoughts of all the actors of the situation, for instance, cannot be mapped. Therefore, even if videos may cover many aspects of a classroom situation in an information-rich way, it is not possible to cover *all* aspects of it. Classroom videos thus should not be confounded with the situation they show, they are only *representations* of them (Buchbinder & Kuntze, 2018).

In general, classroom vignettes can be understood as *representations of practice*. According to Buchbinder and Kuntze (2018), a representation of practice (e.g. a classroom situation) is something that stands for this classroom situation (e.g. a cartoon showing a conversation between two students, or a transcript of a dialog) and represents some, but not all aspects of the classroom situation it refers to.

This reduction of information entails some key benefits of vignettes: Vignettes can be viewed (or read, etc.) repeatedly and independently from a specific classroom situation; they can be stored, collected and provided to other persons. Vignettes, therefore, afford an easy access to analysing elements of classroom situations by teachers (Kuntze, 2018), which makes vignettes a useful instrument in mathematics teacher education in many ways (e.g. Skilling & Stylianides, 2019). In particular, vignettes can help to bridge the gap between theoretical contents and goals in mathematics education on the one hand, and requirements of specific classroom situations on the other hand.

Vignettes can use different *formats,* which can be expected to have different specific advantages, but also challenges. While video vignettes, for instance, may cover a relatively high amount of information from the classroom situation they represent, creating such a vignette requires effort, and data protection issues requires care and may restrict their use. Compared to cartoons, for example, video vignettes can hardly be constructed quickly or modified easily so as to contain specific elements intended for reflection. Cartoon or text vignettes contain less information, but are more flexible in use and can be constructed relatively easily. Especially without the help of a digital tool, the design of cartoon vignettes requires creating and dealing with graphical elements, which may represent a difficulty compared with text vignettes. A reduced complexity also can be a benefit as it may support the access for learners and reduce cognitive load (e.g. Syring et al., 2015).

RESEARCH AIM

When it comes to developing vignette-based learning opportunities or vignette-based assessment instruments, it has to be decided which vignette format to choose and how to frame the vignette-based work. For these design decisions, empirical evidence about best-practice ways of using vignettes is needed, as different vignette formats might impact differently on teachers' analysis outcomes, for instance. A video vignette might, for example, better support teachers in engaging with the represented classroom situation compared to a text vignette, but the large amount of potentially irrelevant context information of the video vignette might also be an obstacle for successful noticing or criteria-based analysis. Empirical research related to the potential of different vignette formats is particularly essential for the design of digital tools and environments which aim at facilitating vignette-based work by supporting their design and their implementation as learning opportunities. In particular, the following questions are in the center: *Are there differences between vignette formats, as far as* *characteristics of their effectiveness for professional learning and assessment (such as noticing, analysis, perceptions of authenticity, etc.) are concerned? Which vignette formats can be considered as particularly effective according to available empirical findings? What implications can be drawn for the design of a digital tool facilitating the creation of vignette-based learning opportunities?*

Responding to this research need, we will in the following collect empirical findings concerning comparisons of different vignette formats. Based on this, we will discuss implications for the development of a digital tool, which aims at supporting the design of vignette-based learning opportunities.

EMPIRICAL FINDINGS ON DIFFERENT VIGNETTE FORMATS

In a study with N = 298 pre-service and in-service teachers (Friesen, 2017; Friesen & Kuntze, 2018), three different vignette formats (text, video, cartoon) were compared in terms of the extent of engagement perceived by the participants when analysing the vignettes; engagement can be seen as important prerequisite for sufficiently representing classroom situations by means of vignettes (ibid.). For evaluating to what extent teachers were engaged with the vignettes, they were asked by means of Likert scales about the extent of their (1) motivation to analyse the vignettes, (2) regarding perceived authenticity of the vignette, and to what extent they perceived (3) immersion and (4) resonance when analysing the vignettes. *Immersion* means that participants felt "put in" the situation; resonance refers to whether one thinks about the own professional practice when working with the vignettes (cf. Seidel et al., 2011). Moreover, it was investigated whether the participants' analyses varied among the different vignette formats, which would indicate that the vignette format can have an impact on teachers' analysis of the represented classroom situations.

For the study, sets of three vignettes in different formats (text, cartoon, and video) were designed representing the same classroom situation. In order to validate the accordance of the vignettes to the respective classroom situations, the sets of vignettes were subjected to an expert rating by teacher educators. Based on that rating, six sets of vignettes rated as highly authentic and representative were chosen. The vignettes together with Likert scales for measuring how teachers assess the authenticity of each vignette and their perceived engagement were administered in a multiple matrix design to the participants (Friesen, 2017; Friesen & Kuntze, 2018).

In the analysis of the teachers' responses it was found that the perceived motivation, immersion, and resonance was on a similar and relatively high level for all three vignette formats. The authenticity of the video-based vignettes was rated by most of the participants on average as less authentic than the text and cartoon vignettes; only the sub-sample of n = 22 in-service teachers rated the authenticity of video vignettes similarly as the authenticity of text and cartoon vignettes. Based on teachers' analyses of the vignettes, a Rasch analysis was conducted. No evidence was found implying multiple dimensions, e.g. according to different vignette formats. There were also no implications for interrelatedness of vignette formats and teachers' analysis of the represented classroom situations, which would have become apparent in divergent item difficulties. In conclusion, the results did not reveal relevant differences according to different vignette formats in terms of engagement, and there were no hints that the vignette format influenced teachers' analysis.

A study with N = 61 teacher candidates by Herbst, Aaron, and Erickson (2013) provided similar results. In this study, it was investigated whether there are differences of using animations (animated cartoons with audio track) compared with videos of classroom situations in the categories *genuineness*, *projectiveness*, *mathematics*, *noticing*, *reflectiveness*, and *alternativity*, which are described in detail in Herbst et al. (2013). The results show that participants only rated authenticity of the video vignettes higher; no other significant differences were found. Similar to Friesen's (2017) study, the study did not reveal any hints that the participants' analyses of the represented classroom situations were influenced by different vignette formats. The authors, therefore, conclude that videos and animations "can be comparably effective" (ibid, p. 11). In a further qualitative study (Herbst & Kosko, 2014), there were also no relevant differences between video vignettes and vignettes with animations. The authors, therefore, concluded, that "animations are just as useful as videos" (p. 515) for investigating teachers' professional knowledge.

In conclusion, the reported empirical findings indicate that the different vignette formats appear to be similarly effective to make classroom situations accessible for teachers' analysis.

DEVELOPING A DIGITAL TOOL FOR VIGNETTE-BASED LEARNING

A central aim of the European project coReflect@maths is to develop a digital tool (DIVER – Designing and Investigating Vignettes for Education and Research Tool), which supports pre- and in-service teachers as well as teacher students in creating, sharing, and collaboratively reflecting on vignettes representing classroom situations. Existing online tools are mostly limited to one vignette format (e.g. cartoons or videos), can only be used by speakers of one language (e.g. English), and do often not sufficiently solve issues with data protection, which is a major concern in educational contexts. The DIVER tool will be programmed as a plugin for the learning platform Moodle. Moodle is used by many European universities and provides a secure, data-protected learning environment in different languages. Since it is open-source, external applications such as DIVER can be integrated to add specific functionalities. Another advantage of Moodle is its potential for remote teaching since it allows the implementation of online courses as well as blended learning scenarios.

As the empirical findings outlined above do not imply general advantages of video vignettes in comparison with the other formats, the development of the DIVER tool mainly concentrates on cartoon vignettes, as they can combine advantages of video-based and text-based vignettes without sharing most of their disadvantages. For instance, cartoon vignettes allow to sketch various classroom situations and to vary them systematically, which hardly can be done with video vignettes. In addition, as cartoons afford representing aspects of classroom situations visually, long descriptions, which could be needed when choosing text vignettes, are not necessary. A central aim of the tool, therefore, is to support the easy creation of cartoon-based vignettes. At the same time, the tool will also facilitate the integration of text and video vignettes within the same environment, as multiple vignette formats might support specific goals of professional learning.

For developing vignettes, the tool is intended to have an easy access graphical interface containing several graphical elements (different student and teacher characters, classroom environments, classroom material, etc.) that can easily be arranged in order to create classroom scenarios. There will be possibilities to add speech bubbles, to edit students' notebooks, and to add writings on the board, so that there are possibilities of creating cartoon vignettes on the base of video material which cannot be published as a consequence of data protection limitations. The creation of cartoon-based vignettes in DIVER is supposed to support creating classroom scenarios in a systematic way, highlighting, e.g., certain quality aspects of teaching and learning in the mathematics classroom. It allows at the same time to create classroom situations based on personal teaching experience of pre-service and in-service teachers. By capturing that teaching experience, e.g. in the form of a cartoon-based vignette, it can be shared with colleagues and made accessible to collaborative analysis and reflection as well as to further improvement. For facilitating the sharing of vignettes and collaborative reflection on presented classroom practice, it is planned that the DIVER tool should also offer the opportunity to follow up on already designed cartoon-based or existing video-based vignettes that can then be analysed and commented on within the tool. This function of the tool can also be used for evaluating vignette-based university courses or PD programmes and holds, therefore, potential for corresponding evaluation research.

OUTLOOK

In coReflect@maths, we plan to implement the support of the languages English, Spanish, German, and Czech. The relatively quick possibility of translating speech bubbles in cartoon vignettes will enable a multinational use of some of the vignettes, highlighting a further advantage of cartoon vignettes. This possibility will also enable to detect, compare, and discuss cultural specificities of the mathematics classroom of the different member countries. Corresponding multi-lingual vignette-based material will be published on the project homepage (www.coreflect.eu) for public use by the end of the project.

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REFERENCES

Blume, F., Göllner, R., Möller, K., Dresler, T., Ehlis, A.-C., & Gawrilow, C. (2018). Do students learn better when seated close to the teacher? A virtual classroom study considering individual levels of inattention and hyperactivity-impulsivity. *Learning and Instruction*, *61*, 138 –147.

Buchbinder, O., & Kuntze, S. (2018). Representations of Practice in Teacher Education and Research – Spotlights on Different Approaches. In O. Buchbinder & S. Kuntze (Eds.), *Mathematics Teachers Engaging with Representations of Practice. A Dynamically Envolving Field* (pp. 1–8). Cham, Switzerland: Springer

Friesen, M. (2017). Teachers' competence of analysing the use of multiple representations in mathematics classroom situations and its assessment in a vignette-based test (Doctoral thesis). Retrieved from https://phbl-opus.phlb.de/frontdoor/deliver/index/docId/545/file/Dissertationsschrift_Friesen_Onlineversion.pdf.

Friesen, M. & Kuntze, S. (2018). Competence Assessment with Representations of Practice in Text, Comic and Video Format. In O. Buchbinder & S. Kuntze (Eds.), *Mathematics Teachers Engaging with Representations of Practice. A Dynamically Evolving Field* (pp. 113–130). Cham, Switzerland: Springer.

Herbst, P., Aaron, W., & Erickson, A. (2013). How Preservice Teachers Respond to Representations of Practice: A Comparison of Animations and Video. Paper presented at the 2013 Annual Meeting of the American Educational Research Association, San Francisco. Retrieved from http://hdl.handle.net/ 2027.42/97424.

Herbst, P. & Kosko, K.W. (2014). Using representations of practice to elicit mathematics teachers' tacit knowledge of practice: a comparison of responses to animations and videos. *Journal of Mathematics Teacher Educ., 17*(6), 515-537.

Kersting, N., Givvin, K., Thompson, B., Santagata, R, & Stigler, J. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *Am. Educ. Res. J.*, *49*(3), 568–589.

Kuntze, S. (2018). Representations of Practice in a Video-Based In-Service Teacher Professional Development Project and in Its Evaluation. In O. Buchbinder & S. Kuntze (Eds.), *Mathematics Teachers Engaging with Representations of Practice. A Dynamically Envolving Field* (pp. 47–70). Cham, Switzerland: Springer

Kuntze, S., & Friesen, M. (2018). The role of mathematics teachers' views for their competence of analysing classroom situations. In B. Rott, G. Törner, J. Peters-Dasdemir, A. Möller, Safrudiannur (Eds.), *Views and Beliefs in Mathematics*

Education. The Role of Beliefs in the Classroom (pp. 183–194). Cham, Switzerland: Springer.

Mason, J. (2002). *Researching your own practice. The discipline of noticing.* London: Routledge Falmer.

Petko, D., Waldis, M., Pauli, C., & Reusser, K. (2003). Methodologische Überlegungen zur videogestützten Forschung in der Mathematikdidaktik. *ZDM*, *35*(6), 265–280.

Richter, D., Wiepke, A., Richter, E., Huang, Y., & Zender, R. (2019). "Nutzung von Virtual Reality-Technologie zur Erfassung von Klassenführungskompetenzen in der Lehrerausbildung". Poster presented at the 7th annual conference of the Gesellschaft für Empirische Bildungsforschung (GEBF), Cologne.

Seidel, T., Blomberg, G., & Renkl, A. (2013). Instructional strategies for using video in teacher education. *Teaching and Teacher Education*, *34*, 56–65.

Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M., & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations: Does it make a difference whether teachers observe their own teaching or that of others? *Teaching and Teacher Education*, *27*(2), 259–267.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20–37.

Sherin, M., Jacobs, V., Philipp, R. (2011). *Mathematics teacher noticing. Seeing through teachers' eyes.* New York: Routledge.

Skilling, K. & Stylianides, G. (2019). Using vignettes in educational research: a framework for vignette construction. *Intern. Journal of Research and Method in Education*.

Syring, M., Kleinknecht, M., Bohl, T., Kuntze, S., Rehm, M., & Schneider, J. (2015). How problem-based or direct instructional case-based learning environments influence secondary school pre-service teachers' cognitive load, motivation and emotions: A quasi-experimental intervention study in teacher education. *Journal of Education and Human Development*, *4*(4), 115–129.

Student teachers’ geometric work and flexible use of digital tools

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In this paper, we surmise that a well-founded geometrical work of teachers is crucial for an adequate introduction of digital tools in mathematics education. Building on this assumption, we explore an approach to the teaching and learning of geometry based on the mathematical work that student teachers actually do. We begin by describing the forms of geometrical work performed by student teachers who solve an elementary geometric task in a paper-and-pencil environment. Next, we present a training process designed to help them better master their own work through the conjoint use of classical and digital tools in compliance with institutional expectations. The development and analysis of the research are supported on the theory of Mathematical Working Space.

Keywords: Geometrical work, Dynamic Geometry Software, Geometry teaching and learning, Teacher training.

INTRODUCTION

For more than 30 years now, mathematics instruction has benefited from the many digital tools developed by researchers and teachers in many countries. We have ourselves been involved in the development of teaching and learning situations based on Dynamic Geometry Software, inspired by Laborde’s (2001) or Mariotti’s (2000) contributions. Nevertheless, it must be acknowledged that this unprecedented mobilisation of new high-performance tools has not had the expected effects. According the PISA Study on Students, Computer and Learning (OECD, 2015), the substantial investments made might have a counterproductive effect. Indeed, and paradoxically, the authors of the report argue that the higher the investment in ICT, the lower the results. And, they conclude that there exists a negative relationship between computers use and performance in mathematics.

> Irrespective of the specific tasks involved, students who do not use computers in mathematics lessons perform better in mathematics assessments than students who do use computers in their mathematics lesson, after accounting for differences in socio-economic status (OECD, 2015, p. 158).

Moreover, in the case of France, the last twenty years have seen a dramatic decline in the level of students in absolute and relative terms since France. A study with primary school students (Chabanon & Pastor, 2019) show a dramatic decline in students' knowledge since in some mathematics areas, half of the present students belong to the last decile of the 1980s. Of course, reasons for this failure are complex and not all of them are related to the entry into the digital age and we continue to think that digital tools can positively transform the mathematical work of students. For the ICMI study on ICT (Hoyles & Lagrange, 2010), contributors were invited to consider the influence of teachers on this point. Most of the papers insisted on the conditions of use and appropriation of digital tools leaving aside possible problems related to initial teachers' knowledge of mathematical content. This issue is just evoked but not really addressed in the papers and surveys that we read. That is why we launched this joint research between our two teams (PUCP in Lima, Peru and LDAR in Paris, France), on the question of teachers' knowledge and training in this field. The present research aims to identify the real and effective mathematical work developed by teacher students and then to design a training process based on the mathematical work forms actually produced by the students. To be more precise, and working only in geometry, the research is grounded on the following teaching intents:

S1. To base the teaching and learning of geometry on existing forms of geometrical work among students in such a way that they can examine and question their mathematical knowledge, or be aware of their limits, conceptions, etc.

S2. To promote a flexible and non-compulsory use of digital tools seen as a set of resources from which students can draw according to their knowledge and taste.

The relevance and validity of these two teaching intents (S1 and S2) are studied through the lens of the theory of Mathematical Working Spaces (MWS) (Kuzniak, Tanguay & Elia, 2016) and geometrical paradigms (Kuzniak, 2013). A brief introduction to this theoretical framework is first given. Then, our report is divided in two parts. The first part is dedicated to the identification of the geometrical work actually performed by primary school teacher students when solving geometric problems in a paper-and-pencil environment. Next, we present the task, developed in a digital environment that is intended to provide students with the means to master mastering their own geometric work and help them overcome certain obstacles.

This contribution is part of the debate on the role and use of classical and digital tools in mathematics teacher education and professional development (topic 1) and in task design (topic 2).

THEORETICAL FRAMEWORK

Mathematical Working Spaces (MWS)

A Mathematical Working Space (Kuzniak, Tanguay & Elia, 2016), named MWS, is an abstract environment organized to enable the functioning and generation of mathematical work in a specific domain (geometry, probabilities, etc.). The space thus conceived is based on the articulation of epistemological and cognitive aspects represented by two planes in the MWS diagram (Fig1). The epistemological plane is directly associated to the mathematical content. This plane consists of three sets of components: a set of signs named *representamen*, a set of technological tools (*artefacts*), and a set of properties and theorems (*theoretical referential*). The second plane, cognitive plane, relates to the thinking of individuals solving mathematical tasks, and is the conjunction of three cognitive processes: visualization, construction and proving. The transition from one plane to another plane results from the three geneses connected with the model: semiotic, instrumental, and discursive (Fig. 1).

- a semiotic genesis that transforms the signs into operative mathematical objects;
- an instrumental genesis that makes the artefacts operative as construction instruments;
- a discursive genesis of proof that relies on properties and organizes them for producing a mathematical proof.

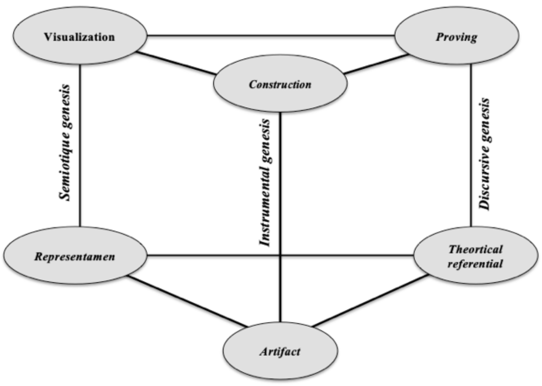

Figure 1: The MWS Diagram

The MWS diagram helps visualize the dynamics and evolution of the work through the different geneses and planes. To account for the various interactions necessary to generate this work, it is useful to consider three vertical planes in the diagram: semiotic-instrumental [Sem-Ins], instrumental-discursive [Ins-Dis], and semiotic-discursive [Sem-Dis]. A work will be considered as complete (Kuzniak & Nechache, 2016) when a circulation into the MWS diagram exists that mobilizes all these vertical planes.

Geometrical paradigms

In geometry education, three geometrical paradigms have been adapted from Kuhn's paradigms by Houdement and Kuzniak (Kuzniak, 2013) and help clarify and organize the various and conflicting points of view prevailing in education around geometry. The first paradigm called Geometry I is concerned by the world of practice with technology. In this geometry, valid assertions are generated using arguments based upon perception, experiment, and deduction. The paradigm called Geometry II, whose archetype is classic Euclidean geometry, is built on a model that approaches reality through a model. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. This geometry has an axiomatic horizon in relation to the modelling of the real world. To these two Geometries, it is necessary to add Geometry III, which is usually not present in compulsory schooling, but which is the implicit reference of mathematics teachers who are trained in advanced mathematics. In Geometry III, the system of axioms itself is disconnected from reality, but central.

Identification of geometrical paradigms and their interplay contributes to the understanding of what guides the work actually performed in a school setting and helps characterize the forms of this work. In the following, paradigms will help us clarify the extent to which the work performed complies with teaching expectations.

DIFFERENT FORMS OF GEOMETRIC WORK

First-year master teacher students were asked to perform a geometric task on estimating the surface area of a piece of land, "Alphonse's field".

> Alphonse has just returned from a trip to Périgord where he saw a quadrilateral-shaped field that his family was interested in. He would like to estimate its area. To do this, during his trip, he successively measured the four sides of the field and found, approximately, 300 m, 900 m, 610 m, 440 m. He's having a hard time finding the area. Can you help him by showing him the method to be followed?

Students were allowed to search for a solution and ask further information for ten minutes. Contrary to the initial teachers' and researchers' expectations, almost all students did not identify the need to have further conditions to fix the shape of the quadrilateral and determine its area. Indeed, they engaged in the search for the field area by spontaneously adding certain supplementary conditions (the quadrilateral had to be specific or all the quadrilaterals had the same area since they had the same perimeter). From the solutions that the students provided to the task, we have drawn the following characterization of their geometric work.

Forms of geometric work identified in a paper-and-pencil environment

The study (Kuzniak & Nechache, 2020), in Paris, was conducted over two years and engaged two cohorts of student's teachers (85 in all) and was completed and confirmed by the same experiment in Lima with 30 students this year. Using the MWS theory, five main forms of geometric work have been identified. In order to clarify these forms of work, we have observed in particular the place and role of the semiotic tools (figure and drawing), the artifacts (construction and length measurement tools) and the theoretical tools (formulas and properties). These forms of geometrical work also depend on their relation to Geometry I or Geometry II paradigm or an interplay between both geometrical paradigms.

1. Dissectors work is supported on a decomposition of the quadrilateral into sub-figures without any use of drawing tools. It is a form of work in conformity with the Geometry II paradigm with an exploratory proof work. This work did not produce the expected results and leads to a blockage. In the end, it is not complete because students do not engage in any exploration work based on figures and possible constructions.

2. Surveyors work is based on figure construction at scale with drawing tools. The constructed figure is then used as support for measuring, reasoning and proving. It is a

form of work that is in line with the expectations of the Geometry I paradigm. We consider it as complete, in the MWS context, because all the geneses of work are mobilized. However, the final outcome is not mathematically correct because students used a particular drawing and introduced sole additional data.

3. Explorers work refers to construction and exploration of different figures that satisfy the required conditions. Compliant to Geometry II paradigm, this work is not complete due to a lack of explanations and references to properties.

4. Constructors work is only dedicated to the sole construction of the quadrilateral with use of ruler and/or compass to transfer the lengths of the sides. This work is not complete and remains compliant to Geometry I in the sole plane [Sem-Ins].

5. Calculators work is guided by a calculation based on a formula specially invented for this purpose. It does not use any control, instrumental or theoretical, over the results. Calculators' work form poses the problem of the exact nature of the paradigms involved in work. It may be a kind of scholarly geometric paradigm which interferes with geometrical paradigms.

Lack of controls and false results

From this first report we were able to draw some conclusions about the students' lack of control over their work that leads them to mathematically false results. This is largely due to the fact that students have developed a cognitive repository (Kuzniak & Nechache, 2020) that contradicts the standard theoretical referential. This repository is based on a set of knowledge and assertions that are false or at least questionable. They introduce false theorems in action (equivalence of perimeter and area). They make systematic use of even imaginary formulas and finally they consider that the figures involved in a geometry problem are necessarily particular figures.

This cognitive repository comes from the students' previous practice of geometry and enables them, in the best of cases, to produce a geometric work in which processes and methods are in conformity with the dominant geometrical paradigm. But, the results they get are not correct due to a lack of control based on the theoretical referential. The working forms of Dissectors, Surveyors and Explorers are good candidates to develop a teaching and learning in the field of plane geometry adapted to the level of these students and their future schoolchildren. We suggest that it is possible to enrich these forms that are close to the way these students work and think in order to make them correct and conform to certain geometric paradigms (S1). According to our approach, this requires thinking about the design and implementation of didactic situations based on the use of various classical and digital tools (S2). We will detail this in the following part.

DEVELOPING A GEOMETRIC WORK BASED ON A FLEXIBLE USE OF DIGITAL TOOLS

Based on the geometric work forms previously identified in a paper-and-pencil environment, we decided to continue the study in Lima and Paris by developing the students' geometric work with use of digital tools. The development of the expected geometrical work is based on different types of controls in relation to each of the genesis of the MWS. The construction of figures with drawing and measurement tools is associated to approximation of the measurement and a set of procedures (triangulation, formulas and properties). In this way, we think the students' cognitive repository can be changed in order to adjust it to the epistemological referential expected at this level. In a way, the geometric paradigm involved is mainly related to both Geometry I and II paradigms with a strong emphasis on the discursive genesis associated with proof.

To challenge the work done previously, we chose to have the students explore the various configurations and formulas of possible areas by using a version of GeoGebra on a tablet. Moreover, this study is related to our theoretical framework (MWS theory) and that led us to consider and articulate the three MWS geneses in the project. The instrumental genesis supported by digital and classical tools is central and articulated with a semiotic genesis based on the use of different registers of representation (graph, table, numeric calculation) and with the proof discursive genesis generated by symbolic and analytic tools.

A three-stage project

The project is divided into three stages that will be illustrated and completed during the oral presentation at the conference.

First Step. Construction and exploration with GeoGebra.

The first is a classic step of construction and exploration in a digital environment. It aims to develop better controlled construction methods by integrating the circle entity with construction tools into the student's work, especially that of constructors. The exploration of the different quadrilaterals obtained must challenge student's theorems in action related to areas and perimeters and lead them to suggest the need of supplementary information, one of the quadrilateral diagonals, to construct the quadrilateral and solve completely the task.

This step has first been implemented in Lima with 30 student teachers who were working alone each with a computer and GeoGebra. We noted a significant difference between the students who were comfortable with the software, since they knew the tools needed to use it, and the others who had great difficulty building the figure. Among the students who had no difficulty using the software, there were two populations. The first constructed the quadrilateral by adjustment with the "given length segment" tool. Then with the "area measurement" tool, they estimated the area of the quadrilateral. Some of these students are then confident that they have found the right value for the area, while the majority of them have been able to see area variations and suggest other data to construct the quadrilateral (angle, particular figure...). For the second population, unfortunately, the construction they made turned out to be indecipherable and useless because of the remaining traces of all the circles necessary for the construction.

Second step. Towards an experimental proof of Heron's formula.

As mentioned earlier, many students are convinced that there is a formula for finding the quadrilateral area, and therefore the formulas issue is taken up for this second step. Using the triangulation method introduced by dissectors, the aim here is to explore the possibilities of obtaining a formula for the area of a triangle as a polynomial function P of the length of its three sides.

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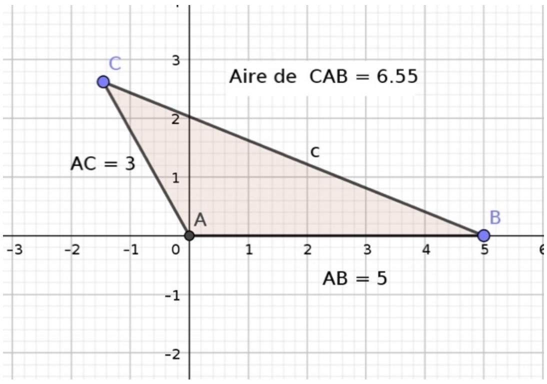
<td>
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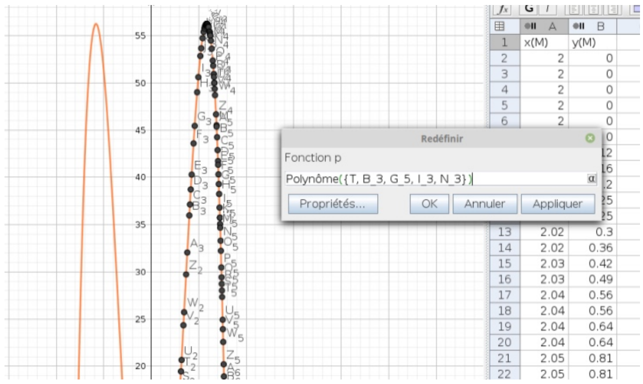
<td>
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<tr>
<td>Search for the area of a variable triangle having two fixed sides</td>
<td>Graph of the points (side, square of the area) to get the polynomial P.</td>
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Table 1. Towards an experimental proof of Heron's formula

Third step. Towards a discursive proof.

Finally, for the most advanced students, a search for proof is planned in connection with formal classical proof or with the possibilities proposed by GeoGebra around the automatic reasoning tools (ART). This approach uses advanced control instructions (Relation tools) and allows to verify certain demonstrations based on Pythagoras's theorem. It is based on an analytic expression of the conditions related to the construction of a triangle with the height relative to the diagonal.

CONCLUSION

This ongoing research focuses on a type of teacher training associated with the use of classical and numerical tools. The training is based on the students' geometric work forms (S1) that we have been able to describe through the use of MWS theory. We attempt to design tasks that allow the evolution of these forms of geometric work with the use of various digital or traditional tools (S2). We seek to develop students' exploratory work through the construction of figures in different environments. We also wish to make them use the semiotic registers related to calculation and drawing. Finally, our aim is to enable the students to elicit experimental and formal proofs. In this way, we think possible to make the students aware of their different forms of geometrical work in order to develop control means on their solutions to the task.

The first results of our experiment showed the difficulty for the students to use some tools, both classical and digital. However, we were able to observe that the joint activation of these different tools allowed the whole group to progress and overcome certain obstacles. This first experimentation also shows the interest of varying the tools used by the students. Indeed, some of them accept to use calculators, spreadsheets or CAS while rejecting GeoGebra. We also observed that some students were asking for experimental and others for formal proofs.

Our training method is based on the students' actual mathematical work, and combines the use of traditional and digital tools, around emblematic tasks. It seems to us therefore relevant to implement it not only in geometry but also in other fields of mathematics. But of course, it is still necessary to assess the extent to which this use of digital and traditional tools has really transformed their working form in relation to the institution's expectations.

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REFERENCES

Chabanon, L., & Pastor, J.M. (2019). L'évolution des performances en calcul des élèves de CM2 à trente ans d'intervalle (1987-2017). *Note de la DEPP, 19(08),* 1-4.

Kuzniak A. (2013). Teaching and Learning Geometry and Beyond... Plenary conference. Proceedings of Cerme8. Antalya Turquie

Hoyles, C., & Lagrange J-b, (eds.) (2010): *Mathematics Education and Technology—Rethinking the Terrain. The 17th ICMI Study*. New-York: Springer.

Kuzniak, A, Tanguay, D & Elia, I. (2016). Mathematical working spaces in schooling: An introduction. *ZDM-Mathematics Education, 48(6)*. 721-737.

Kuzniak, A. & Nechache, A. (2020). Développer un travail géométrique complet et conforme chez les étudiants de première année de master enseignement en France. *Revue de Mathématiques pour l'école. 233.* 92-103.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabrigeometry. *International Journal of Computers for Mathematical Learning, 6(3),* 283–317.

Mariotti, M.A (2000). Introduction to Proof: The Mediation of a Dynamic Software Environment. *Educational Studies in Mathematics.* 44, 25–53.

OECD (2015). S*tudents, Computers and Learning: Making the Connection.* PISA, OECD Publishing. http://dx.doi.org/10.1787/9789264239555-en

Math education master students focusing on teaching mathematics with digital resources

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A teaching activity on rotations has been used in a math education master course in order to provide students with an insight into mathematics teaching with digital resources. The design of the activity was framed by the Theory of Semiotic Mediation taking into account related research results concerning the synergy between manipulatives and digital artefacts at school level. The aim of the study described in this paper is to investigate the potentialities of the designed teaching activity in helping prospective teachers to reflect on the role and the use of digital resources in high-school mathematics teaching and learning.

Keywords: mathematics teaching with digital resources, Theory of Semiotic Mediation, prospective teacher education.

INTRODUCTION

Integration of digital resources in mathematics teaching and learning is one of the main research topics in mathematics education, at least for the last twenty years (Trgalová et al., 2018). The issue is addressed from different prospective such as: design and development of resources; mathematics curriculum development and task design; benefit for students' learning; and, more recently in particular, mathematics teacher education and professional development (Clark-Wilson et al., 2014). With respect to the latter, many research studies are being devoted to this area: to identify the specific knowledge and expertise that is required to efficiently/effectively teach mathematics using digital resources; and to design and evaluate teacher's education in mathematics and professional development programs, aiming to enhance this knowledge and expertise. Within this research field, this paper aims at contributing to investigate the prospective teachers' interaction with digital resources with a dual purpose: on the one hand we focus on their personal reflection on mathematical meanings through the accomplishment of a sequence of tasks involving different kind of resources; on the other, we pay attention to their professional development process in reflecting on the integration of digital resources in mathematics teaching.

To do this, we present a teaching activity and its implementation in a teaching experiment, involving master students in Mathematics (here conceived as prospective teachers). The activity, concerning rotation around a centre in the plane, is described within the framework of the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), highlighting the role of the synergic use of digital and non-digital resources. Some key episodes of the implementation of the teaching activity are described and analysed in order to answer a first research question: can the combined, intentional and controlled use of digital and non-digital artefacts within the activity enhance students' mathematical content knowledge? Moreover, we show some evidence supporting the hypothesis about the potentiality of the use of the activity to develop prospective teachers' professional knowledge and skills. This aims to answer the following research question: may the activity enhance the students' knowledge and expertise on the use of digital resources in mathematics teaching and learning?

THEORETICAL FRAMEWORK

The Theory of Semiotic Mediation (TSM) offers a theoretical framework suitable to design teaching sequences embedding digital resources and to analyse data in order to gain insight into the students' learning process. According to TSM, personal meanings, emerging from the activities carried out with an artefact, may evolve into mathematical meanings, which constitute the objective of the teaching intervention. Fostered by specific semiotic activities, the evolution can occur, in peer interaction during the accomplishment of the task and in collective discussions, orchestrated by the expert guidance of the teacher. Some research studies based on TSM (Mariotti, 2009) have pointed out the fundamental role of the teacher in fostering the construction of mathematical meanings throughout the students' learning process. Within TSM, the design of tasks develops on the base of a fine grain a priori analysis of the solution processes, and specifically on the identification of the schema of utilization that are expected. Moreover, Faggiano, Montone and Mariotti (2018), showed that a potential synergy may occur between the use of different digital and non-digital artefacts, providing a rich support to the development of mathematical meanings. For the purposes of the math education master course at stake, the TSM was chosen to offer to the students a framework to develop their reflection on the role and the use of digital resources in mathematics teaching and learning.

METHODOLOGY

Participants, procedure and data collection

The study was conducted with 12 master students in Mathematics, within a mathematics education course. They were familiar with the notion of rotation as an isometric transformation of the plane in itself, with one fixed point, called centre of the rotation. The teaching experiment was developed in three lessons of two hours each. In the first session students were asked to work in groups of four on a sequence of three tasks on rotation, involving different kind of resources. The second session was devoted to a collective discussion, conducted by the teacher, aimed at allowing personal meanings to emerge and evolve towards the meaning of rotation. The aim of the last session was to collectively discuss with the students the experience they had with the teaching sequence of tasks and to bring them to reflect on the way the activity, conveniently changed, can be developed in a high-school class in order to teach rotations through a synergic use of digital and non-digital resources.

Group work and discussions were videotaped, transcribed and analysed together with the students' protocols. In this paper we present some of the collected data with the aim to provide elements to answer to our research questions.

Overview of the teaching sequence

The aim of the teaching sequence was to focus on the properties of the rotation as a one-to-one correspondence between the points of the plane, described by the centre of rotation, the angle of rotation, and the direction of the turn, and which preserves the distances of each rotated point from the centre. The tasks have been designed according to TSM and combining the use of different artefacts in order to exploit the synergy among them.

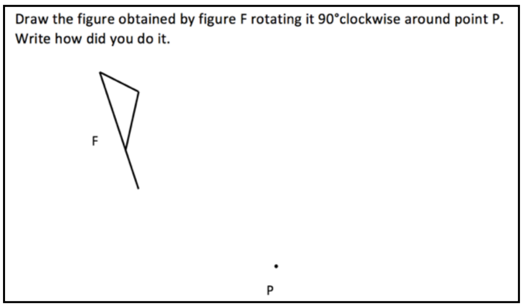

Figure 1a: Task 1

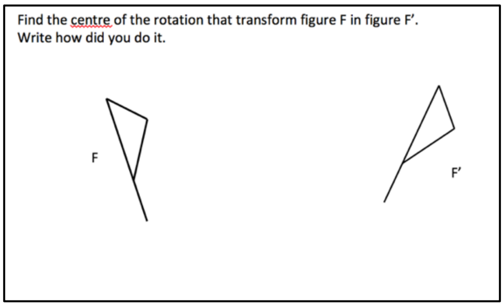

Figure 1b: Task 3

The first task (Fig. 1a) required students to draw the figure obtained by rotating 90° clockwise a given figure around a given (external) centre and to explain the way they did it. In order to carry out the request, the students were given tools such as the protractor, a ruler, set squares and a compass. The aim of this first task was to draw students' attention to the following aspects of the concept of rotation: the idea of *rotation as a circular rigid movement* of a figure around a point –called centre– by a certain angle, expressed by the use of the compass –to preserve the distances from the centre– and the use of the protractor –to rotate all the points by 90°; the idea of *rotation as a punctual correspondence*, expressed by the identification of the rotated points in the intersection points of the arches –drawn with the compass– with the rays –drawn from the centre creating 90° angles; the idea of *rotation as an isometric transformation* which, in particular, preserves the distances and the amplitudes of the angles and transforms segments into congruent segments, expressed by the process to join the obtained rotated points to draw the rotated figure.

In the second task students were required to use a Dynamic Geometry Environment (GeoGebra). They were asked: to construct the segment A'B' obtained by rotating 60° clockwise a given segment AB around a given point P, using the tool/button "Rotate around point"; to observe what moves and what doesn't move when dragging the extremes of the two segments or the point P and to explain the reasons why it happens. The aims of this second task were: to draw students' attention once more to the meaning of rotation as a punctual correspondence of the points of the plane and as an isometric transformation –by dragging the extremes of the segment AB and observing the resulting movement of the segment A'B'; to highlight the fundamental role played by the centre of rotation, during the construction of the required rotation –by dragging the point P and observing the resulting movement of the segment A'B'.

The third task (Fig. 1b) does not require the use of GeoGebra. Students were given two congruent figures drawn on a piece of paper and were asked to find the centre of the rotation that transforms one figure into the other. The aim of the third task was to draw students' attention to the centre of the rotation as the unique point at the same distance from each pair of corresponding points of the figures. That is, the centre of rotation can be found as the intersection point of the perpendicular bisectors of two segments joining a point of one figure to the corresponding point of the other figure.

RESULTS AND DISCUSSION

The transcripts and the protocols, presented and discussed in this section, were chosen for their features to give evidence of: the emergence and evolution of signs showing the students' enhancement of their mathematical knowledge; and the students' understanding of the role and the use of digital resources in high-school mathematics teaching and learning.

Students' accomplishment of the three tasks

During the first lesson students worked in groups of four, but each of them received his/her own sheet to work on. In order to obtain the 90° clockwise rotation of the given figure around the given point P, all of them immediately started using the compass, pointed in P, to draw four arches passing through the vertices of the given figure. Then they drew the four segments joining each of the vertices with the point P. Finally, as the required rotation was 90°, they drew, starting from P, the perpendicular lines to these four segments. To do that, some of them used the protractor and others the set squares. The rotated figure was identified joining the intersection points of the arches with the perpendicular lines (see. Fig. 2a).

A tablet was given to each of the groups in order to accomplish the second task. As requested, they: opened the given GeoGebra file; used the tool/button "Rotate around point" to create the 60° clockwise rotation of the given segment AB around the given point P; dragged A and/or B and then dragged P; discussed about what happened; wrote their observations concerning the dragging of the elements. The quote below is what one of the students wrote:

> - Dragging the point A, point B and the rotated point B' remain fixed, differently from the rotated point A'. Moreover, the length of the segment remains the same. Varying A, segment AB varies too, such as segment A'B'.
>
> - Dragging point P, segment A'B' translates so that: the 60° inclination between the line through line segment AB and the line through line segment A'B' is preserved, and PA=PA' and PB=PB'.

The text shows the student's comprehension that dragging the extremes of the segment AB gives as a result the movement of the segment A'B, that endows rotation with the meaning of a punctual correspondence and of an isometric transformation. Moreover, the dragging of point P and the observation of the resulting movement of the segment A'B', has allowed the role of the centre of rotation to be highlighted.

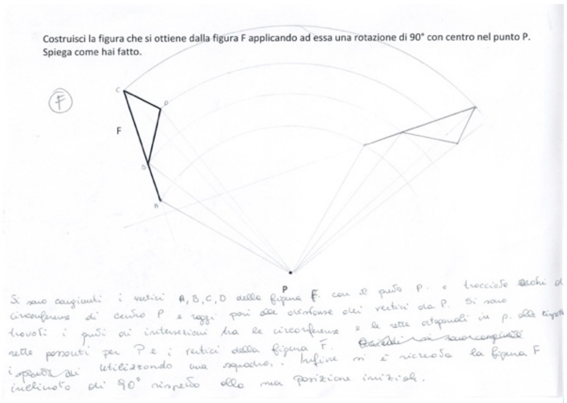

Figure 2a: One of the students' protocols of Task 1

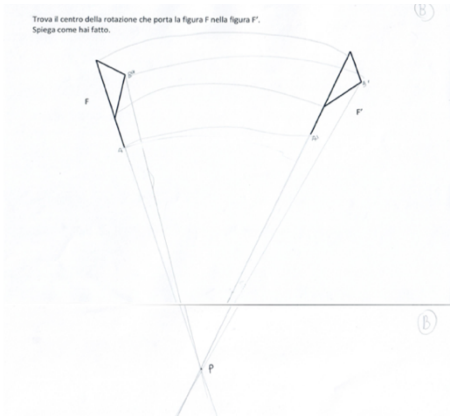

Figure 2b: One of the students' protocols of Task 3

In the final part of the lesson students worked on the third task to identify the centre of rotation that transforms one of the given figures into the other. This resulted in being the most challenging task. In one of the groups, for example, students started reflecting on the idea that a rotation preserves the distances between the centre and each pair of the corresponding points. However, in order to identify the centre, they extended the line through one of the segments of the figure and focused on one of the vertices. Along this line, they looked for the point which has the same distance from the corresponding point of the second figure. In this way, the centre was outside the piece of paper and they decided to take another sheet and extend the lines on that one (see Fig. 2b). To conclude, students didn't succeed in accomplishing the third task, and this became the main focus of the next class discussion.

Class discussion of the three tasks

According to the TSM, during the second lesson, the teacher initiated a class discussion with the aim to focus on the aspects of the rotation on which students were required to reflect on, in order to fully construct the mathematical meaning. In particular, the students' attention was brought to the perpendicular bisector of the segment joining a point of one figure to the corresponding point of the rotated figure. As the centre of the rotation belongs to the perpendicular bisector for any pair of corresponding points, indeed, this allows us to find the centre as the intersection point of two different perpendicular bisectors. For this purpose, students were firstly asked to reflect on how to accomplish the first task without any tools. The required folding of the paper allowed the students to focus on the triangles obtained joining two corresponding points between them and with the centre. In this way, it was easy to pay attention to the fact that these triangles are isosceles.

At this point, the teacher moved on to discuss the second task. Asking students to report on what they had done working in groups during the first lesson, she pointed on what moves and what doesn't move when dragging A or P. Students were able to make the right hypothesis concerning the movement of A' with respect to dragging of A. The discussion on how A'B' moves when dragging P was based on the use of the trace (Fig. 3a). The teacher's aim was to let students give meaning to the "translation" of the segment A'B'. As the following quote shows, students claimed that the segment AB has to have always the same direction as the segment A'B':

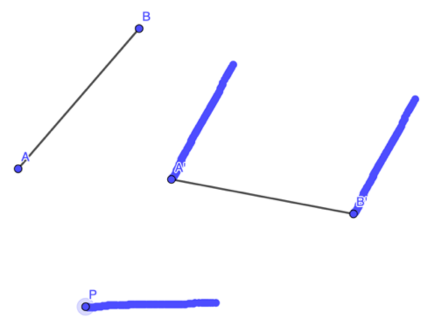

Figure 3a: GeoGebra screenshot taken during the discussion – dragging P on the left

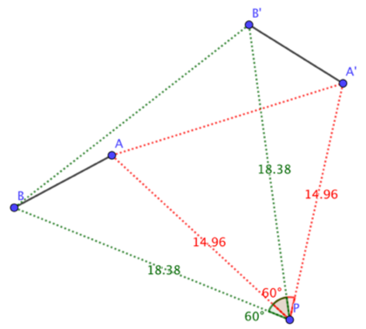

Figure 3b: GeoGebra screenshot taken during the discussion – highlighting the triangles

Giusy: A'B' has to move along the direction of 60° and has to move down as much as I drag on the left in order to keep constant the distances PA' and AP

Felice: it is right that it goes down because if it had gone up the angle would have been contracted

Susanna: indeed, we paid attention to the distances but not to the angles, so [for the segment] to go up the angle contracts, as he said, and the angle is not preserved of course

The discussion continued with the teacher aiming to focus once more on the triangles obtained joining two corresponding points between them and with the centre. Different colours were used to draw the segments joining different pairs of corresponding points with the centre (see. Fig. 3.b). It was an easy consequence, thus, to perceive the centre as the intersection point of two different perpendicular bisectors of any two pairs of corresponding points. This allowed the request of the third task to be accomplished according to the following observations:

Stefano: if we consider the circumferences on which the points move, the centre of the circumference that transform this point into this other will be on the perpendicular bisector of the chord

Susanna: once the points are connected, we should find the midpoint of these segments

Giusy: and so, it comes that what is given by the intersection of the perpendicular bisectors is the centre of the rotation

The excerpt shows how, through the mediation of the digital artefact, the students realised that the centre of rotation can be found as the intersection point of the perpendicular bisectors of two segments joining a point of one figure to the corresponding point of the other figure.

Class discussion on the use of a similar activity to introduce rotations at high-school level

During the last session, students were asked to reflect on the experience they had had with the teaching activity. They recognized that the activity had helped them to give meaning to the notion of rotation, focusing not only on its definition as an isometric transformation but also on its properties. During the discussion, in particular, they paid attention to the relationship between rotations and axial symmetries: it happens, indeed, that, in the attempt to obtain the rotated figure, one of the students folded twice the paper and used a pin. Then the students were required to reflect on the way the activity, conveniently changed, can be developed in a high-school class in order to teach rotations through a synergic use of digital and non-digital resources. They recognised that GeoGebra, and in particular the trace tool, reveals itself to be effective in order to highlight the notion of rotation as an isometric transformation, which preserves the distances and the amplitudes of the angles and transforms segments into congruent segments. However, they believed it was important to start the sequence of tasks with paper and pencil. In particular, they started discussing on the idea to introduce the activity starting from the request to do a double axial symmetry with respect to two lines having in common a point, and they thought to make use of a further artefact, a transparency:

Teacher:	so, I ask you to do the two axial symmetries, and then? What are the questions to pose?
Stefano:	then I put a transparency on
Mario:	what is happened to the figure?
Stefano:	putting the transparency on and tracing the starting figure, try to overlap rotating

As the quote above shows, the idea was to guide students to build the notion of rotation as a combination of two axial symmetries with their axes that meet in the centre of the rotation, using paper, pencil and a transparency.

Then they moved to the use of GeoGebra in order to exploit its potential to focus on the dependence of the final figure from the starting one and from the intersection point of the symmetry axes. It also emerges that using GeoGebra to look at the rotation as a double axial symmetry can allow high-school students to easily recognise that rotations preserve the distances of any pair of corresponding points from the centre. However, they underlined the importance of guiding students to focus on the triangles obtained joining two corresponding points between them and with the centre, as they recognised this aspect to also have been important in their experience. It is by means of this observation that they were able to find the centre of the rotation, given the figures, and the third task, as it was, was considered to be a valuable activity in concluding the sequence.

CONCLUSION

The episodes of the implementation of the teaching activity, discussed in the last section, reveal that the combined, intentional and controlled use of the digital and non-digital artefacts enhanced students' mathematical content knowledge. Indeed, the mediation of the artefacts resulted to be fundamental in order to let personal meanings emerge during the interaction with the artefact in the accomplishment of the tasks. These meanings evolved towards mathematical meanings throughout collective discussion. The most evident aspect is the one concerning the characterization of the centre as intersection of two perpendicular bisectors.

Moreover, results show how the activity was useful for the prospective teachers to develop professional knowledge and skills. The teaching activity was the occasion for them to discuss specific mathematical and pedagogical aspects. For example, the notion of rotation as a combination of two axial symmetries was seen as important to help high-school students to highlight the properties of the rotation. The role of digital resources emerged while the prospective teachers interacted with it and reflected on their experience. They experienced how digital resources can foster the construction of meanings and can be integrated by the teacher in order to serve her didactic objectives.

To conclude, we can say that the activity seemed to have enhanced the students' knowledge (in terms of awareness of the various aspects and the properties of the geometrical concept) and expertise on the use of digital resources in mathematics teaching and learning (in terms of semiotic potential of the resources).

REFERENCES

Bartolini Bussi, M.G., Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In: *Handbook of international research in mathematics education*, New York. 746-783.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (2014). *The mathematics teacher in the digital era,* Dordrecht: Springer.

Faggiano, E., Montone, A. & Mariotti M. A. (2018). Synergy between manipulative and digital artefacts: a teaching experiment on axial symmetry at primary school. *International Journal of Mathematical Education in Science and Technology*, 49(8), 1165-1180.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: the role of the teacher. *ZDM*, 41(4), 427-440.

Trgalová, J., Clark-Wilson, A. & Weigand, H. G. (2018). Technology and resources in mathematics education. In: *Developing Research in Mathematics Education.* Routledge. 142-161.

A commognitive approach for teaching functions: The discursive change of pre-service teachers in a technology-rich environment

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This paper attempts to investigate the contribution of digital tools when pre-service teachers were introduced to functions. The overarching goal was to improve their subject matter knowledge of functions through a commognitive approach using an technology rich environment. The study was conducted using student's narratives from written tasks and oral observation. A claim is that a discursive change can be observed, and that technology played an important part.

Keywords: Functions, mathematical discourse, design research, commognition

INTRODUCTION

With a new curriculum, to be implemented in Norwegian schools in autumn 2020, a further emphasis is put upon functions and modelling. Results from the national Norwegian exams have proved functions to be challenging, and thus call to future teachers do something about that. As a teacher educator, I am also challenged to provide my students with the needed knowledge. That gave grounds for a research project on how to design a first course in functions for pre-service teachers (PST). The research method was an adoption of design research (DR) (e.g. Gravemeijer, 1994; Design Based Research Collective, 2003) and commognitive theory (Sfard, 2008). In this paper I concentrate on how technology contributed to the design.

An important part of the future teacher's knowledge will be to interpret functions as reificated objects with the characteristics of univalence and arbitrariness. My research question for this paper is:

How can a technology rich environment contribute to PST's reification of a function object that includes univalence and arbitrariness?

In the frame of the theory presented later, the reification should end with an abstract d-object.

THEORETICAL BACKGROUND

About functions and MKT

Teaching will require a Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008). One important part of the MKT is the reification of the function as an object (Sfard, 1991, 2008). In mathematics, we operate with the function as an object. Composed functions, derivatives and integration are all examples where functions are treated as objects. Reification is thus required.

Freudenthal (1983) put emphasis on two other properties of the modern function: univalence and arbitrariness. Univalence, or one-valuedness, is a necessary requirement to avoid ambiguousness. A modern definition of a function does not restrict the nature of the dependency between the variables. There is also no restriction of the variables themselves. Both are arbitrary. The two properties are the result of the historical development of the function (Boyer, 1959; Kleiner, 1989). The same properties are also among the demands for the MKT by Even (1993).

A commognitive approach

A commognitive approach was used on several levels. First, the learning goal for functions is formulated within a commognitive framework. Second, the analysis is done with a commognitive approach.

Sfard regards communication and cognition as intertwined: “[t]hinking is an individualised version of (interpersonal) communication” (2008, p. 81). With the blending of communication and cognition, a new word is merged of the two nouns: commognition. An implication is that narratives can be regarded as expressed thinking.

Sfard (2008) describes mathematics as a special discourse with four properties that makes it distinct: word use, visual mediators, narratives and routines. Visual mediators, visualisations used as mediators, are operated upon as an important part of communication. As participants in a discourse the mathematical activity consists of producing narratives that can be endorsed. According to Sfard (2008) and Lavie, Steiner and Sfard (2018) mathematical narratives are governed by metadiscursive rules split into three categories of routines, corresponding to their use: rituals, deeds, and explorations. An exploration will end with an endorsed narrative about change of mathematical objects containing the story of what, and why, the change appeared. Before the learners end up with the full-fledge explorations, they will undergo a development through deeds and rituals. As interlocutors in the discourse, learners will meet mathematical objects. A mathematical object is defined as a signifier with a corresponding realisation tree (Sfard, 2008). A realisation tree is a hierarchical organisation of realisations (representations in other frameworks) of the signifier. An example is the signifier “quadratic function” where realisations can be the expression x^2, the corresponding graph or a table of values. All of them are perceptually accessible and called primary objects (p-objects). Sfard (2008) argues for an objectification as a discursive process where discursive objects (d-objects) are individualised. The participants must create narratives about p-objects. Simplifying is a driving force in the communication and a next step will be to signify p-objects by assigning a noun or pronoun. Thus, a concrete d-object is created. More advanced processes are involved when d-objects are put together and individualised: saming, encapsulation, and reification. Through saming, “the act of calling different things the same name” (Sfard, 2008, p. 170), a search for common attributes will end by assigning one signifier to many realisations; e.g., when “linear function” is used in communication both about linear polynomials on the form $ax + b$ and the graph as a straight line. When a noun, or pronoun, will signify a specific set of objects, the act is called encapsulation. Several objects are turned into a single entity. This act can be observed as a change from narratives in plural being changed to singular when talking about a property of all the set members. Finally, reification happens by introducing a noun or pronoun for processes on objects to create narratives about relations between objects (Sfard, 2008). An abstract d-object will substitute narratives about processes. In the discourse of functions a crucial reification will be that $f(x)$ signifies the function as a d-object and will not be realised as a process of "putting values into an expression and calculate a value".

THE DESIGN EXPERIMENT

According to DR the implementation of the teaching is a design experiment, build upon a conjectured Local Instruction Theory (LIT) as a foundation for a Hypothetical Learning Trajectory (HLT). I will now describe how the design experiment took place, the participants, and the basis for the planning.

The students and data collection

The participants of the design experiment were twenty-six PST in their second year of teacher training. All had chosen mathematics as their primary subject and completed a mathematics course of thirty ECTS in their first year. Combined with the second year course of thirty ECTS, the PST shall be prepared for teaching mathematics to 11-16 year old pupils in Norwegian schools. None of them had any experience from teaching except for the practice during teacher training education.

The collected data consist of written material and recordings of the group- and plenary discussions. Both the participants and me wrote logs. As attendance was voluntary, the number of students who took part of the different lectures may vary.

Some of the narratives from students, who can typically exemplify the discourse, were chosen. These narratives will serve as examples to illustrate the changes of the whole discourse. These students are designated *S1* to *S5*.

The planning and the implementation

A considerable constrain, caused by national, institutional, and collegial premises, was a time limit of four plenary lectures. The duration of each lecture was three hours. Between each lecture the PST had to read and complete assignments. As a consequence of the limited lecture time two instructional videos, were the teacher's monologue played the main part, were made. After watching the videos the students were required to write short summaries and to write down questions for the next lecture. The foundation for the teaching was that participation in an orchestrated discourse would contribute to a transition from processes, and p-objects, to a reification of the function object. Thus, a discursive practice, with all PST engaged as participants, had to be created. Then the role of more experienced participants, as the teacher, becomes important. All narratives, both orally and written, has to be carefully considered by the teachers. The use of "function" has to signify a reified object. Hence, the discourse must be guided in the same direction by allowing the participants to produce narratives that can be collectively corrected.

Digital tools are important mediating artefacts for the learners. In this design they include Dynamic Geometry Software (DGS), graph plotters and Computer Algebra System (CAS). The main digital tools were GeoGebra and Desmos. A conjecture was that these artifacts can contribute to a rich use of realisations of the function and provide visual mediators for the interlocutors.

RESULTS AND ANALYSIS

I will now present an overview of what happened during the teaching periode. First, the discursive change in the narratives of functions, then some episodes that argues for an individualisation of the function object, and last the role technology played.

The students' initial definitions of a function

Before the experiment, the PST had to write down an answer to the question: What is a function? All writings were collected, and a plenary discussion took place. This served as a pre-test for the experiment. The narratives were analysed according to how they responded to the signifier "function". *S1* wrote: "I would say that the concept of a function is how to do calculations from a formula where you can use variables. The result will be a graph in a coordinate system." In this narrative "function" can be analysed as a signifier for a process where the result of the process is a p-object, the graph. When *S2* responded: "It is an expression that can be written with letters and numbers", there is a reference to a p-object with a limited realisation tree. The same applies for the narrative of *S3*: "Something that shows a picture/line of values that changes during certain time periods (hours, days, years)". The p-object is a graph and limited to a time unit, and "function" is not used as a signifier. These narratives are typical for sixteen, of the twenty students who took part in the pre-test. None of these respondents' narratives made explicit requirements of univalence and arbitrariness, and none showed an object-driven use of "function" and a reified function object. The rest of the respondents produced vague, or incorrect, answers.

The students' definitions of a function at the end

At the end of the last lecture a plenary discussion on how a teacher should perceive a function, was conducted. Before the discussion everyone had to write down their own description of a function. These written responses were analysed with the narratives given at the plenary discussion. Twenty-two students participated. The signifier "function" was used as a noun describing a covariation, or coherence, between variables, in a majority of the respondents. Sixteen of the PST showed that clearly. The utterance of *S1*: "A function is a connection between various factor, often x and y. x depends on y. Each value of x gives one y", is an example of that category. A total of seventeen narratives express explicit one-valuedness.

None of the narratives contain a direct statement about "function" as a process, but three can be interpreted as implicit utterances of a process. An example is *S4*: "A function is about relationship of values. Every argument value will affect the value of the function. It is exactly one dependent value". The use of "is about" is imprecise, but interviewed later the respondent corrected the narrative to: "It is the function that expresses the co-variation between the values".

The written, and oral, narratives show a change in the discourse. The signifier "function" had changed from signifying a process to signify a function object. The different narratives of *S2* provides an example. In the first, *S2* expressed a process. The last was: "In order to call something a function there has to be correspondence between the value you use as a variable and the value given by the function. For each input only one output". This is a phrase-, or object, driven use of a d-object with explicit univalence. Thus, an example of the development of the majority of the students.

Arbitrariness was also examined, but is not as evident as the discursive change from processes to objects. It was more clearly drawn into the plenary discussion.

A change of the discourse

During the experiment, there was a change in the discourse. The narratives of functions transformed from being about processes and p-objects into the use of the function as an object, including the characteristics of univalence and arbitrariness. Of course, endorsed narratives cannot give an exact answer on how the object has been individualised – they may be phrase driven. As shown by others (Tall & Vinner, 1981; Vinner & Dreyfus, 1989) there may be a distinction between definitions and by how they are actually treated. On the other hand, incorrect narratives would imply direct shortcomings in the MKT. The change in discourse and the proper narratives are indications of correctness of the hypothesis in the LIT and HLT about the facilitation of the discourse. At the end, most narratives can be categorised as explorations.

An objectification can also be observed from the use of signifiers. Both "function", and various use of symbols like $f(x)$, gives rise to narratives where they are used to signify a function object with a noun. This is typical for an individualisation of the function object by the participants. The narratives also show a rich use of realisation trees.

HOW DID TECHNOLOGY PLAY A ROLE?

The result of the experiment is due to many factors and the commognitive approach regards learning as both situated and distributed. Thus, it is not easy, or even possible, to isolate each factor. Nevertheless, I will try to elaborate on the role and contribution of ICT without insisting on a direct isolated effect.
First, ICT was important as an organiser. A Learning Management System (LMS) was used as tool to structure the learning experiment. It was also used to assign, and collect, responses. The LMS offered the use of a diary and logbooks which proved important for the production of the narratives. These narratives sought to increase both intra- and interpersonal communication. As the narratives could be both peer reviewed, and read by me, they contributed to many productive plenary talks, comments, and served as a reflection tool for interpersonal communication. With the immediate access in the LMS, they also served as an important guide line for adjusting the discourse by the

teacher. Another contribution of technology was the use of instructional videos and the collection of the responses to the videos in the LMS.

Digital tools were also important mediating artefacts and supportive for an inquiry process. I will now outline some of these contributions. The challenges reported on conversions between realisations (cf. Duval, 2006) were considered for the experiment. In the commognitive framework transitions can be considered as including realisations in a discourse and let these play a part in the objectification. The dynamic nature and the easy transitions of realisations offered by the tools constitute an important part of the narratives. Through saming, encapsulation and reification, realisations are crucial for the individualisation of the function object. The digital artefacts can supply realisations in form of objects which can serve as visual mediators. In the function discourse, examples of visual mediators are $f(2)-f(3)$ and $f(g(x))$. Both should be regarded as objects and treated that way. I will provide some examples. At the first lecture, the PST were given a task with a graph of a third degree polynomial function in a coordinate system. The students should solve the equation $f(x)=0$ and find $f(2)$ without access to the expression of the function. Only nine out of twenty-two student interpreted $f(2)$ as the correct –2. Seven did not answer. In the same task, they were also given a function by $g(x)=x^2+1$ and supposed to find $g(3)$. In the plenary discussion that followed they explained that as an easy task, but without an expression given, they were unsure about what to do. The students' explanations can be taken as evidence of a lack of realisation of the signifier $f(x)$, and an incorporated routine for calculating function values. During the experiment the use of digital tools as artefacts for producing visual mediators sought to improve the students' realisation tree. Examples are conversions of physical situations into realisations, tracing graphs with coordinates visualised, and an extended use of symbols. In reflections about the initial task of this example, most of the students expressed astonishment about the fact that they were unable to solve the task.

In a modelling task with the draining of a water tank, the students had to find the rate of change in intervals. They could do that by calculating $\frac{f(t_2)-f(t_1)}{t_2-t_1}$. First, the students calculated every value, but by the tools they saw the use of simplifying and treating $f(t)$ as an object. There was also a change in the discourse as the students started to form narratives like "what is being drained between the third and the second minute" as a replacement of $f(3)-f(2)$. At the end, all participated in a discourse where symbols were replaced the function values. That supports the claim of objectification as well as the contribution of ICT to provide function values without the process of calculation as interference. In another context, the signifier $h(t)$ was used as "all the heights" in a situation concerning the height of a tree over time. Another contribution for the objectification is the object nature of CAS commands. CAS syntax often demands a need to operate on the function object given by an expression. The operation is carried out on the function as an object, and the result can either be properties of the function or a new function. Tasks that emphasised both the result, and reflections about operations on the function object, showed important for the reification. An example is the command *Derivative(f)* where the input is the declared function $\boxed{f}$ and the result is a new function object – the derivative. Some of the reflections about the nature of these commands were crucial for the individualisation of the function object. The same example also provided discussions about univalence: What if univalence was not required? With a CAS command as a visual mediator narratives like "it is obvious that the function cannot give multiple values" were typical. The arguments were found by the impossibility of ambiguous input and output of commands.

A modelling approach was used during parts of the experiment. With digital tools, the students had to collect data from real life situations to make a model. As they had to produce narratives about the process, several important aspects for the properties of the function came up. Arbitrariness could be discussed when an expression was found by regression: Did the expression really delineate the covariance correct? In some cases, the conclusions were that regression gave the best fit, but the covariance could not be expressed through an expression. The gap between the model found, and reality was used for discussions about the arbitrary property of a function.

Throughout the experiment, a main goal was to support how different discourses could be subsumed into one about functions. As ICT provided easy, and rapid, transitions between physical situations, graphs, expressions, and tables, narratives of the each realisation could be transformed into narratives of the function object. This helped the different discourses to be subsumed into one.

CONCLUSION AND DISCUSSION

An explicit change in the discourse could be observed and a conclusion is that technology provided an important contribution to the change. All PST had been taught functions before teacher education and mastered common process related to function, but important properties and a reified function object was missing. The discursive change supports a claim of an individualisation of the function object. Without the support of digital tools this change is hard to imagine within the limited time available.

Naturally, there are several adjustments I would have made. One is to make more videos for the students to see between lectures. That could have provided more time for discussions and work in groups. A more thorough examination of each individual would also be useful. That could have revealed a phrase driven use of narratives as an approach to provide what as intended by the teacher.

REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of teacher education*, *59*(5), 389–407.

Boyer, C. B. (1959). *The history of the calculus and its conceptual development*. New York: Dover Publications.

Design Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5–8.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1), 103–131.

Even, R. (1993). Subject-Matter Knowledge and Pedagogical Content Knowledge: Prospective Secondary Teachers and the Function Concept. *Journal for Research in Mathematics Education*, *24*(2), 94–116.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. ProQuest Ebook Central.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, *25*(5), 443–471.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal*, *20*(4), 282–300.

Lavie, I., Steiner, A., & Sfard, A. (2018). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Studying mathematics teachers' documentational and identity trajectories over time

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We study the evolution of long-term interactions between one mathematics teacher and resources, especially digital ones, with a dual focus on his documentational and identity trajectories. We combine a theoretical framework on teachers' work with resources with a framework based on social practice theory. The analysis allowed us to describe what digital resources the teacher interacted with and when, as well as to explain how and why he interacted with these resources the way he did over time.

Keywords: Digital resources, DAD, documentational trajectory, Patterns-of-Participation, identity trajectory.

INTRODUCTION

The aim of this paper is to study the evolution of long-term interactions between mathematics teachers and resources – especially digital resources (DRs) – with a dual focus on professional and identity development. A multiplicity of studies focuses on mathematics teachers' professional development as an outcome of their participation in specific, often short-term initiatives, while only fewer studies have examined teacher development over a long period of time as a result of their participation in daily activities (e.g., Little, 2002). Recently, calls have been made for extended studies, for instance by Trouche (2019), who argues that teachers' deep instructional changes often need long time. Taking into account the transformative potential of DRs in terms of stimulating learning experiences to students, motivating teacher collaboration and design capacity (Pepin, Gueudet, & Trouche, 2017), we consider the study of teachers' long-term interaction with DRs as a promising research area.

Our literature review revealed only few studies focusing on teachers' long-term development with DRs, predominantly within the framework of *Documentational Approach to Didactics (DAD)* (Trouche, Gueudet & Pepin, 2018) that concentrates on teachers' work with resources. For instance, Rocha (in Loisy, 2019) analyses when, where, why, and how teachers interact with which resources at critical points during their professional life. This analysis provides mainly overall descriptions of teachers' interactions at the macro level by focusing on special events outside classrooms or schools and the corresponding resources. Though contributing valuable insights, this analysis focuses on when and where teachers interact with resources and what promote their actions, and to a lesser extent on why and how teachers interact in the described ways by taking broader social perspectives into account.

In this paper, we adopt a social practice framework, *Patterns-of-Participation (PoP)* (J. Skott, 2019), that favours understandings of how broader social constellations and modes of reasoning inform teacher action and meaning making. PoP allows a social perspective on a teacher's professional identity that is defined as his/her experiences of being, becoming and belonging as a teacher. We aim to combine (Prediger, Bikner-ahsbahs, & Arzarello, 2008) DAD with PoP to get a multi-faceted insight into a Greek mathematics teacher's (Victor's) long-term development and especially into the hows and whys of Victor's interactions with DRs over time. For this, we analyse his documentational trajectory (i.e. his interactions with resources over time) in conjunction with his identity trajectory (i.e. his identity formation over time).

THEORETICAL APPROACHES

DAD (Trouche et al., 2018) acknowledges the crucial role of resources for teachers' work and professional growth retaining a broad meaning for *resources* as material and non-material elements (e.g. textbooks, discussions with colleagues) and considers that teachers' work with and on resources constitutes a dialectic process where design and enactment are intertwined. Teachers use different kinds of resources that shape not only the mathematical content and the ways it is (re)presented, but also students' mathematical learning. Teachers adapt their appropriation and use of resources to their needs and customs. This dynamic process of (re)-design and interpretation continues during enactment of the resources and orients teachers' *documentation work* leading to their creation of *documents*. The set formed by all the resources used by a teacher defines his/her *resource system* (RS). Rocha (in Loisy, 2019) introduces the concepts of *documentational experience* (i.e. a teacher's appropriation of professional events acquired during his/her interactions with resources, that were remarkable to his/her documentation work from his/her perspective) and *documentational trajectory* (i.e. the set of professional events of both individuals and collectives that ground teachers' documentational experiences) to study the long-term evolution of interactions between teachers and resources. She also proposed two new methodological tools: a *Reflective Mapping of a teacher's Documentational Trajectory* (RMDT) made by the teacher and an *Inferred Mapping of a teacher's Documentational Trajectory* (IMDT) made by the researcher. These tools map a teacher's experience throughout his/her professional life by indicating on a timeline crucial events/collaborations to his/her documentation work (Fig. 1).

Trouche (2019) indicates a need to consider the broader contexts in which teachers operate as they make decisions about what resources to use and how to use them. To take into account how social constellations and modes of reasoning are reflexively related to and co-determiners of teachers' acts and meaning-making, we use PoP (J. Skott, 2019). In line with other studies of teacher identity, PoP draws on the notions of practice and figured worlds in social practice theory, where practice "connotes doing [but] doing in a social and historical context that gives structure and meaning to what we do" (Wenger, 1998, p. 47). Figured worlds are "socially and culturally constructed realm[s] of interpretation, in which particular characters and actors are recognised, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland, Lachicotte Jr., Skinner, & Cain, 1998, p.52).

PoP differs from other identity studies using social practice theory by focusing on shifts in a teacher's ways of participating in social interactions over time (e.g., classroom interactions) and not focusing on how a teacher move towards a more comprehensive participation in a specific, pre-established community or figured world. In PoP, a teacher's participation in social interactions is understood as influenced by his/her interpretation of the immediate situation and his/her simultaneous meaning-making where he/she continuously interprets "others' actions symbolically, including their actual or possible reactions to one's own behaviour" (J. Skott, 2019, p. 472). In terms of symbolic interactionism, the teacher takes the attitude to him/herself of others. The others can be a colleague or a parent, but it can also be a social group or community in which case the teacher takes the attitude to him/herself of generalized others. In PoP, practices and figured worlds are interpreted as possible generalized others. Using PoP, we define Victor's professional identity as his experiences of being, becoming and belonging as a teacher at his school and beyond (e.g. research projects), and Victor's identity trajectory as his identity formation over time. Our research questions are:

(1) What are the major professional events and corresponding resources in Victor's long-term documentational trajectory? How do these events/resources contribute to his professional development at the macro level?

(2) How do Victor's experiences of being, becoming and belonging influence his interactions with DRs? How do these experiences contribute to forming his identity trajectory at the macro level?

METHODOLOGICAL APPROACH

We study Victor's professional life focusing on his interactions with DRs over 18 years from when he starts working as a new mathematics teacher in his own auxiliary school to his current position as a teacher in a multicultural public school in northern Greece. Victor's case is illustrative as he explores the different opportunities for interacting with DRs through his participation in multiple research projects and development initiatives. We aim to understand how Victor's participation in such major events at the macro level influence his interactions with resources, especially digital ones, but also, why Victor engages in these events and what motivates him in the long run. In line with the principles of reflective investigation (Trouche et al., 2018, p. 8) we (a) analyze Victor's long-term documentation work ("long-term follow-up" principle) at different time moments in- and out-of-class ("in- and out-of-class follow-up"), (b) address his reflections on his choices and experiences ("reflective follow-up"), (c) take into account a "broad collection of the material resources" produced and used throughout the follow-up, and (d) "permanent confronting Victor's views on his documentation work and the materiality of this work". For this paper our data is: 8 individual interviews, e-mails on his documentation work in different time periods, resources for his teaching of a specific unit in grade 8 (e.g., worksheets, lesson plans), and an activity template where he described aspects of his design and experiences from the implementations. Based on the interviews and teaching resources, we made first drafts of IMDT including major events and resources. We then asked Victor to provide written comments (wr) on the IMDT to (a) examine this version and let him modify it by adding missing events, (b) highlight the most critical events for him and explain why, and (c) provide examples of resources related to these events. Based also on a follow-up discussion (fd), we constructed a combined IMDT and RMDT (we name it I/RMDT).

For the analysis of Victor's documentational trajectory, we cross-referenced I/RMDT with other data and identified key points operating as 'entrances' into his trajectory and examined if and how they influenced his professional activity. For the analysis of his identity trajectory we used mainly his written reflections on I/RMDT and the follow-up discussion. This analysis was carried out in two steps. First, under a broadly grounded approach we analyzed line-by-line Victor's utterances and practices in relation to his orientations towards important actors, significant acts and values, and gathered these into distinct practices, characters and figured worlds. Second, we analyzed how the roles of these practices, characters and worlds shifted in dominance to Victor's experiences of being, becoming and belonging when interacting with DRs, drawing corresponding landscapes of their constellations. By identifying shifts among these landscapes, we constructed Victor's identity trajectory.

VICTOR'S DOCUMENTATIONAL TRAJECTORY

Our analysis of Victor's I/RMDT shows four categories of events and corresponding representative resources (Fig. 1) that constitute his professional life: (a) taking part in professional exams (E2, E4); (b) teaching secondary students and pre-service teachers (E1, E3, E6, E9); (c) engaging in research and developmental work in mathematics education especially with the use of DRs (E5, E8, E10, E11, E12, E13, E14, E15, E16, E17); (d) participating in scientific and professional associations (E7). By combining the I/RMDT with our data, we identify three overall/key events and their corresponding time periods: participation in the national exams (E2) (2000-2001), collaboration with teachers and researchers (2004-2010), and participation in teams designing curriculum/curriculum resources such as in the projects *New School* and *Digital School* (2010-2015). These events provide the basis for three major documentational experiences underlying his trajectory: collaborating with colleagues and researchers; collaborating within specific frames/contexts and norms regarding resource design especially with DRs; approaching mathematics education research.

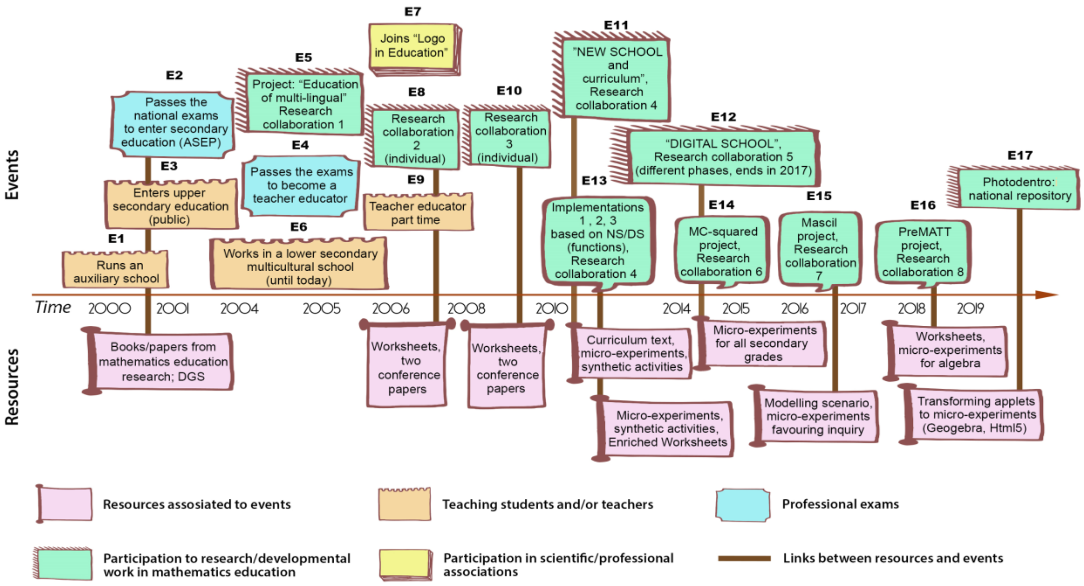

Figure 1: Victor's I/RMDT

The reform aspects of New School and Digital School concerned multiplicity of curriculum resources and conceptualization of teachers as active partners in designing teaching with DRs. Furthermore, the new curriculum was structured in learning trajectories and used two novel DRs: 1) *micro-experiments*: interactive applets developed through different categories of proliferating digital media (CAS, programmable software), and 2) *synthetic activities*: generic scenarios promoting project-work based on modelling, inquiry and integrated representations. A set of principles guided the work of those two projects like combining formal math representations (e.g., graphs), text and/or simulations (e.g., math behaviour), promoting students' dynamic manipulation of math objects in order to explore and discuss their relationships/properties, and promoting inquiry. Victor's collaboration with researchers in the projects affected deeply his interactions with DRs and his RS. We highlight three points related to a new constitution of this system. (1) The new curriculum design and the enrichment of textbooks with DRs mark Victor's entrance to design DRs and to integrate them systematically in his RS. As a result, this set of new resources gives birth to a new RS that includes novel resources such as *Enriched Worksheets* consisting of theory (e.g., definitions), different tasks (e.g., problems, exercises), multiple representations and links to DRs (e.g., micro-experiments). (2) Although worksheet was an innovative resource in Victor's initial system, micro-experiments occupy dominant positions in all subsequent forms of his RS. (3) The two projects mark Victor's passage from looking for new resources (e.g., by reading research literature for ASEP) to aligning his RS with resources designed in the projects and later on to being able to design such resources himself. In terms of structure, we consider micro-experiments as pivotal resources (Trouche et al., 2018) used by Victor in various activities (e.g., lesson planning, tasks). Also, they seem to play a central role in helping Victor to restructure and reorganize his RS. Finally, Victor's participation in New School and Digital School seems to influence his professional development as a consequence of the evolution of his RS. Resources designed in the projects are different from the resources (tasks, worksheets) normally used in classrooms. The new resources aim to introduce teachers to new design approaches as regards functionalities and use of available DRs as well as to open discussions and to share ideas for future use. Further, developing such resources is a demanding process requiring associated documentation work that in Victor's case led to deep reorganization/restructuring of his RS at the level of both technical (e.g., design capacity in Geogebra) and pedagogical demands (e.g., synthetic activity).

VICTOR'S IDENTITY TRAJECTORY

Preparing for the national exams to enter secondary education (1997-1998).

As a new teacher Victor works in his own auxiliary school (1997-2001) and belongs to a world of *Para-education* (e.g. a parallel private educational system behind the formal school system in Greece that supports students in passing the higher education exams). This world is characterised by "stereotyped perceptions of math" (wr, p.6), and it does not encourage use of DRs as this is not required for obtaining its purpose: helping students to pass exams. However, when preparing for his own exam to teach at secondary level, Victor realises that "something was wrong [in his teaching] as it did not help all students" (wr, p.1). His realisation marks a critical shift in his identity formation. He begins to orientate himself towards a world of *Reform mathematics teaching* and a new character of an *Embracing mathematics teacher* emerges that combines the inquiry part of the Reform-world with his political ideas of education for all. Thus, to teach student-centred, Victor starts designing worksheets with mathematical graphics which is new in Para-education. These worksheets also represent Victor's first steps in using DRs in his teaching, which is rather rare in Greece by this time. Figure 2 shows his first landscape of practices, characters and figured worlds (a dark blue arrow shows strong prominence).

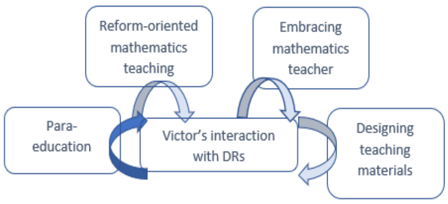

Fig. 2: Victor's first landscape

Collaborating with teachers and educational researchers (2004-2010)

The equity issue gains importance for Victor, and hence his devotion to become an embracing character, when he begins to teach at a multicultural school with heavy language problems. Two types of experience prove important in this period: (1) Victor gains experience with designing resources that include DRs that students can use (e.g. dynamic geometry systems); (2) he starts collaborating with colleagues and researchers: "Our [collaboration] was based on my classroom teaching. She [researcher] supported me theoretically providing me a framework for functions" (wr, p.2). He experiences being recognised by colleagues and researchers as competent collaborator and teacher due to his affiliation with the Reform-world. Fig. 3 shows Victor's second landscape of practices, characters and worlds.

Participating in New School and Digital School (2010-2015)

This period is the most important for Victor's identity formation in relation to DRs, "It was a very big step … one of the most important things I have done so far" (fd, p.5). While Victor transforms his ways of collaborating and designing, his landscape of practices, characters and worlds is similar with the previous period. We therefore focus on the two practices. Regarding collaborating, Victor experiences being part of a complex and diverse collaboration both in terms of more participants, multiple actors (i.e. researchers, teachers, ministry advisors), larger scope (e.g. grade 1-9) and comprehensive purposes. By engaging in new ways of collaborating especially with researchers, Victor gains experience that helps him become an even more competent teacher "I knew a lot, but it was first when I consider the curriculum [in terms of learning trajectories] that I understood it deeper" (fd, p.5). Regarding the second practice, designing, Victor gains experiences with creating novel DRs in new ways, especially micro-experiment, that is "an autonomous application" with text, questions and digital constructs integrated and "starting with a problem. You need to design the problem to create space for the next step, the mathematics" (fd, p.8). By being required to engage deeply with the affordances of DRs relative to the targeted reform-intensions Victor experiences how to use DRs to facilitate these, "I understood deeper what using technology means … the role of representations, the idea of exploration, the dynamism of dragging a slider and observing what happens … Before, all these ideas circulated around and teachers always asked "Who can do all this?"" (fd, p.6). At the same time, Victor tries out the micro-experiments in his classroom. By engaging in these new ways of collaborating and designing DRs and experimenting in classrooms, Victor gains experiences with implementing the Reform-world in classrooms. He thus becomes more like the embracing character.

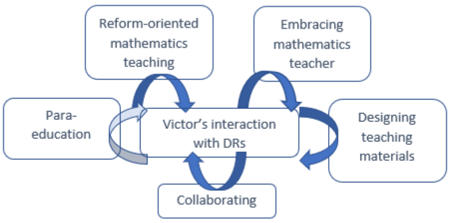

Fig. 3: Victor's second landscape

In summary, Victor's identity formation undergoes a big transformation during these 18 years. From belonging to the traditional Para-education world and not using DRs, to becoming an embracing teacher who uses DRs as a tool to implement the Reform-world in his classrooms. The motivation for Victor's identity is his insistence on becoming the embracing character. By engaging in the two practices, collaborating and designing of DRs in New School and Digital School, Victor gains experiences with and recognition for his ways of implementing the Reform-world in his classrooms. These experiences contribute further to explain how and why Victor becomes like the embracing character.

CONCLUSION

We used two theoretical constructs, documentational and identity trajectories, to study a Greek teacher, Victor's, interactions with DRs over time. The DAD analysis shed light on major events and corresponding resources that promoted Victor’s interaction with DRs. It stressed the evolution of his RS and how it was influenced by his collaboration with specific agents or collectives. The PoP analysis highlighted Victor’s trajectory focusing on big shifts in his identity formation related to his long-term experience of being, becoming and belonging as a teacher interacting with DRs. The analysis shows that his dominant motivation was to become an embracing math teacher and how this influenced his ways of participating in social interactions, and furthermore that the roles shifted between the practices, characters and figured worlds that were crucial to his professional experience. Our combined theoretical focus thus allowed to describe which DRs Victor interacted with and when, and to explain how and why he developed his interactions which these DRs.

REFERENCES

Holland, D., Lachicotte Jr., W., Skinner, D., & Cain, C. (1998). *Identity and Agency in Cultural Worlds*. Cambridge, MA: Harvard University Press.

Little, J. W. (2002). Locating learning in teachers ’ communities of practice: opening up problems of analysis in records of everyday work. *Teaching and Teacher Education*, *18*, 917–946.

Loisy, C., et al. (2019). Analyzing Teachers’ Work with Resources: Methodological Issues. In L. Trouche, G. Gueudet, & B. Pepin (Eds.), *The 'Resource' Approach to Mathematics Education* (pp. 257-321). Cham: Springer.

Pepin, B., Gueudet, G., & Trouche, L. (2017). Refining teacher design capacity: Mathematics teachers’ interactions with digital curriculum resources. *ZDM - Mathematics Education*, *49*(5), 799–812.

Prediger, S., Bikner-ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM Mathematics Education*, *40*, 165–178.

Skott, J. (2019). Changing experiences of being, becoming, and belonging : Teachers’ professional identity revisited. *ZDM Mathematics Education*, *51*(3), 469-480.

Trouche, L., Gueudet, G., & Pepin, B. (2018). *Documentational approach to didactics.* https://arxiv.org/abs/2003.01392.

Trouche, L. (2019). Evidencing Missing Resources of the Documentational Approach to Didactics. In L. Trouche, G. Gueudet, & B. Pepin (Eds.), *The “Resource” Approach to Mathematics Education* (pp. 447–489). Cham: Springer.

Wenger, E. (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge, UK: Cambridge University Press.

Teachers as task designers in the digital age: teaching using technology

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The aim of the paper is to present and analyse the case of one teacher attempting to introduce his students to fractals using digital technology. His task design process has been made explicit through the writing of a storyboard. It has been analysed in order to focus on the stages of the process, identifying prominent elements in it by using the knowledge quartet framework. Results can be useful to inform teacher educators about his needs with respect to the development of his ability in task design. The importance of this aspect, particularly worth of note in the digital age in which teachers have many opportunities to access teaching resources online, has been amplified by the constraints to which educational systems have been subjected during the Covid-19 pandemic emergency.

Keywords: Task design, digital technology, teacher knowledge, teacher professional development.

INTRODUCTION

Digital technology is usually pointed as having the potential to promote the students' development of mathematical thinking. However, according to Thomas and Lin (2013), this development is more a consequence of the tasks proposed to the students and of the way these tasks explore the potentialities of the technology, than of the technology alone. Nevertheless, designing tasks that enhance the potential of technology is a complex and difficult achievement (Joubert, 2017) and even a major pedagogical activity (Leung, 2017). There is the need to design tasks promoting mathematical learning and understanding, and which take advantage of technology. This can challenge the curriculum and the teaching trajectories, changing the more traditional approaches. Often, mathematical tasks aim at achieving results or answers, emphasizing procedural skills instead of promoting conceptual understanding and the development of problem solving competencies. The use of technology changes the way of having access to results and facilitates a focus on conceptual understanding (Rocha, 2020). However, this might result in a need to new prerequisites for designing tasks (Olsson, 2019).

This paper presents and analyses a case of one teacher attempting to integrate digital technology in his teaching. We intend to reflect on how he was engaged in the development of digital technology rich learning trajectory concerning fractals. Fractals has been chosen for their potential to address several mathematical contents, and also because it is possible to find on-line many different kinds of resources concerning them. We address the research question: what are prominent elements characterising stages of the teacher's task design process that can be identified? Although the paper concerns a single case study and a specific mathematical topic, the findings can be useful to inform teacher educators about teachers' needs related to the development of their task design ability. The importance of this aspect, particularly in the digital age in which teachers have many opportunities to access teaching resources online, has been amplified by the constraints to which educational systems have been subjected during the Covid-19 pandemic emergency.

THEORETICAL FRAMEWORK

Teachers always design tasks in order to promote their students learning. According to Leung (2017, p. 4), "mathematics task design can be thought of as designing activities situated in pedagogical environments that provide boundaries within which students engage in doing mathematics leading to the construction of mathematical knowledge". The decisions teachers take during this process of creation are guided by the learning goals they define for their students (e.g., planning an exploration activity or a moment to practice). This process becomes more complex when digital technologies are part of it and, in these circumstances, for students to take advantage of all the potential provided by technology, the tasks should require them to explore, reconstruct and explain mathematical concepts and relations (e.g., Olsson, 2019).

When referring to different types of task sequences, the global idea given by Watson et al. (2013) is that the initial tasks of a sequence somehow offer a basis for the development of mathematics knowledge needed to address the later tasks. However, most of the research focuses on isolated tasks, trying to characterize them. For example, Burkhardt and Swan (2013) give attention to the difficulty of the task, analysing the task according to different factors: complexity; unfamiliarity; mathematical procedural demand; student autonomy and level or kind of guidance. Other authors, e.g. Ponte (2005), classify tasks according to their level of difficulty but also according to their level of structure (from closed to open-ended tasks).

Also, when considering the use of digital technologies in a task, there are different roles they can assume (Rocha, 2020). Actually, technology can be integrated into a task as a way of doing part of the mathematics, as a way of allowing for exploration of a situation and the development of conjectures, or in several other roles. Laborde (2001) classifies the tasks according to the role assumed on it by technology, but in a different way: tasks that are facilitated by technology but not modified by it; tasks where technology facilitates exploration and analysis; tasks that can be done with paper and pencil, but where technology allows new approaches; tasks that cannot be accomplished without technology. In this case the focus is not so much on what technology does, but more on how the use of technology impacts the task.

According to Leung (2017), the teacher knowledge is central in the options assumed by the teacher in the process of task design. The author assumes this knowledge as a complex construct resulting from the interactions among different knowledge domains. And from these interactions Rocha (2013) emphasizes the impact of technology on the pedagogical options of the teacher and also on the mathematical content addressed.

Rowland et al. (2005) also value the teacher knowledge and highlight four dimensions of what they call quartet knowledge: foundation, transformation, connection, and contingency. While the last dimension can be described as the ability to “think on one’s feet” during the contingent classroom events, the first three dimensions have been further characterized by Tanışlı et al. (2019, p. 136) in the following way: “foundation refers to a repertoire of the teacher's academic knowledge for teaching and learning mathematics including his/her beliefs regarding why mathematics is important and why it should be taught”; “transformation refers to the transformation of theoretical knowledge into practice by designing and planning pedagogical tasks in terms of choosing appropriate examples and activities for the construction of mathematical meanings”; “connection refers to the coherence of designed parts of a lesson or series of lessons through deliberately chosen activities and domain specific tasks. Such pedagogical task sequences enable students to make a connection between different concepts as well as to interplay between different representations”. These dimensions give to the conceptualization a close connection to the teachers' practice, particularly suitable for this study.

METHODOLOGY AND CONTEXT

As we were interested in identifying prominent elements in the stages of the teacher task design process, we used storyboards as research tools. The storyboard we analysed in this paper was written by an Italian mathematics teacher, with a Master degree in Mathematics, working at the high-school level for 7 years. Although he was not enrolled at that moment in any research or training programme, in order to become a teacher, he was involved in a two-year teacher education program ending with an examination. Thanks to this program he acquired basic notions of mathematics education and of the use of technology in the teaching practices. He designed a teaching sequence of tasks which involves digital technology and concerns fractals, attempting to exploit the opportunity to approach them in different ways and at different levels. His task design process was made explicit through the writing of a storyboard. Excerpts of the storyboard are presented and analysed using the dimensions of the quartet knowledge framework. Although the framework was thought mainly to focus on the analysis of mathematics teaching, we believe it could be useful to develop some understanding also about the way teachers are engaged in their task design process in a context of technology integration.

RESULTS

In the storyboard analysed in this paper, the sequence of tasks was conceived to be hypothetically developed in four hours in the laboratory, so that pairs of students could share a computer. Herein, we focus on the teacher's task design process starting with the presentation of the teacher’s plans related to his hypothetical task sequence.

According to what the teacher wrote in the storyboard, he starts his teaching sequence by posing a problem. The chosen problem seems not to be immediately connected with fractals, but it is pivotal in the development of the sequence. Indeed, his aim is to let emerge from the discussion that the curve "broken infinite times" presents a difference with that "broken 1000 or 100000 times". At this point he plans to show a short part of a video, explaining the property of the Roman cabbage of "remaining equal to itself" on any scale, and hence to give a definition of fractals using this property, so to introduce the concept of self-similarity.

In the next phase, he plans to introduce the students to the many examples in which nature uses fractal structures. In a brainstorming session he intends to invite students to propose possible explanations as to why nature often uses these structures. The final answer is entrusted again to a short part of a video that shows the increase of the surface in a limited volume. This makes students reflect on the "relationship between dimensions (surface – volume)". Then, he plans to show fractal constructions that can be obtained by recurrence through geometric transformations, such as the Koch curve and the Sierpinski triangle. In doing these constructions he is particularly interested in highlighting the role of the affine transformations and determining their equations:

> It will be shown how to use the self-similarity of fractals to determine the minimum number of portions "equal to themselves" that allows us to obtain the whole figure and how to apply the transformations to these parts. This will be done for the Koch curve and the Sierpinski triangle and, after having collectively identified the parts and transformations, students will be invited to write their equations.

Successively, he plans to show some tutorials –founded online– presenting the creation of ad hoc tools to reproduce the minimal part of the similarity –in order to directly involve the students, in pairs or in small groups, in the construction of fractals using GeoGebra. He intends to underline how the identification of the transformations is the basis on which the ad hoc tools are created.

> The possibility that GeoGebra gives to zoom in on portions of areas will allow students to better understand how fractals are related to the concept of infinity.

Finally, his idea is to come back to the starting problem to explain how fractals can be useful to tackle these kinds of issues.

In what follows, we show how the foundation, transformation and connection dimensions of the quartet knowledge framework can be identified in the teacher's storyboard on the task design process. The last dimension, contingency is exclusively related to the implementation in the classroom and would not be addressed here.

Foundation

The teacher identifies mathematical content that can be addressed using fractals. This can be seen as characterizing his knowledge with respect to the foundation dimension. For example, he writes:

> Fractals involve different mathematical concepts (and not only): geometry (in a broad sense), proportions, geometric transformations, concept of dimension, arithmetic, trigonometry, successions, functions, limits and convergences, set theory, logic, ...

> The intention in this case is not to be limited to purely exposition: the presentation of these mathematical objects with the playful, artistic and anecdotal aspects, in my view, could be combined with the use of fractals as application and study of geometric transformations in the Cartesian plane. On this aspect, therefore, my teaching intervention intends to be more mathematically punctual and “operational”. Moreover, from a more cultural point of view, I would like to take this opportunity for a (possibly different) reflection on the concept of infinity and its implications with mathematics.

He also underlines that the Internet offers a surprisingly large variety of software to create fractals, but he considers the idea of generating fractals to come out of the purpose of a limited educational intervention aimed at a mathematical learning goal. However, in some software he sees aspects that can be useful from the didactical point of view. For example, he found a software that has a rich gallery of interactive fractals: “you can zoom in (in fact the program recompiles always guaranteeing excellent definition and richness of details), select portions of the image, rotate them, vary the colours of the convergence sets”. He recognises in this interactive functionality a didactical potentiality.

The foundation dimension of the teacher's knowledge can also be seen in the way he looks for online resources and in the comments he writes concerning the choice of the resources with respect to his beliefs regarding mathematics teaching. Two examples are given below:

> - This is one of the first websites I found (of some interest). The most interesting part is a section in which it proposes programs written in BASIC that should build fractals showing their iterations. This thing is of potential interest, but in addition to the fact that I already had doubts about the usefulness of running a code that for students means nothing (not knowing the language), honestly by doing the tests I was not even able to run them.
>
> - This website is absolutely pertinent to the aims of the educational intervention that I have in mind, given that it emphasizes exactly the related transformations. The website is divided into three parts: related transformations in the plan (this part should be a prerequisite for the students, but it is convenient that it is also present on the website); fractal geometry (it is the main and most interesting part for my purposes); insights (especially centred on the relationship between fractals - golden section - spirals).

Transformation

When the teacher describes the way he analysed the resources he found online, he makes explicit some assumptions which reveal his way of perceiving the knowledge in action. As the quote below shows, his analysis of the resources reveals not only his pedagogical point of view but also the mathematical one. Concerning some of the resources he took into consideration he makes the following kind of comments:

> - This video presents fractals giving excellent ideas and using a captivating video editing.
>
> - The intent of the video seems to me to bring out the mathematics that “underlies” the fractals in a way that can be used by non-professionals, but the speed of the exposure

makes, in my opinion, the whole video inadequate for students who, in addition to being captivated, should also learn something.

- Some issues that can be tackled by high school students are completely left out or excessively trivialised by this video.

- The attempt to keep content within a few minutes is appreciable but, dealing with so many aspects in such a short time, it does not make this video suitable even for a simple “first presentation” of the topic.

This foundation knowledge led him to do some transformation and he decided to choose small parts of videos to address specific aspects that can benefit from the visualization allowed by the video format.

He also declared the willingness to let students experiment by themselves with the construction of fractals. Taking into account that his aim was mainly to show the relationship between the related transformations and the generation of fractals, he thinks about using GeoGebra and behaving “by hand”, so that students can “touch” the transformations involved and the necessary iterations.

Connection

The way the teacher structures the teaching sequence reflects choices based not only on the knowledge of structural connections within mathematics itself, but also on his pedagogical content knowledge. For instance, his foundation knowledge makes him value an approach based on problem solving as a way to involve students, and his connection knowledge makes him decide to start by what he called a “stimulus” problem. The particularity of the problem he chooses was that it could be faced by the students considering two approaches which come out as two different and conflicting answers. This was, in his hypothetical teaching sequence, the occasion to promote a brainstorming that could bring students to the need of discovering fractals. He values the relevance for the students of what is addressed (foundation knowledge), as so, he tries to lead students to recognize fractals around them. In this sense he moves from the initial (mathematical) approach towards some issues that allow students to connect fractals around them again with mathematics point of view.

The role of technology in the teacher task design

A focus on the role of technology in the task design process of the teacher can be added to the analysis in terms of the quartet knowledge framework. This would add some understanding to the case of a task design process involving technology. It is possible to identify some stages on the teacher's task design that are directly related to technology and grounded in the teacher's foundation knowledge: looking for digital materials concerning fractals; choosing some of the most “interesting”; and reflecting on the potentialities of them. Concerning transformation knowledge, some other stages can be identified: focusing on a mathematical content that can be mediated by the use of them; and recognising aspects of the content that can or cannot be taken into consideration when using them. Finally, the building of a sequence of tasks using the chosen digital material, can be seen as the final stage grounded in the teacher's connection knowledge.

CONCLUSION

Nowadays, there are many resources available online. This circumstance increases the relevance of the teachers' knowledge to allow them to be able to explore and use resources to promote learning. With the aim to identify prominent elements characterising the stages in the task design process, our conclusions, based on the quartet knowledge framework, point to a significant impact of the teachers' foundation knowledge. Although we have only analysed the task design of one teacher, as far the presented case study is concerned, it is this knowledge that guides the initial options of the teacher, defining what he considers relevant to use and what he does not. This is the starting point for modifications to the resources, guided by the teacher's transformation knowledge. And this is the source for the alignment of the sequence of tasks, where the connection knowledge defines the learning trajectory proposed to the students. The prominent elements characterising the stages related with the use of technology, as they emerged through the analysis, can be grounded in all the three dimensions. However, our findings highlight that foundation knowledge has a strong influence over the initial choice of digital technologies and over the reflection about their potentialities. This could be useful to inform teacher educators about teachers' needs with respect to the development of their ability in task design. In particular, in order to promote a rich selection of digital technologies, it seems important, within the teachers training programs (initial and continuous), to give attention to the development of foundation knowledge. In this way it will be possible to develop a deep integration of digital technologies on teachers' practice. Nevertheless, this requires attention also to the transformation and connection knowledge of the teachers, in order to transform technology into an irreplaceable part of the tasks (according to the view presented by Laborde, 2001), where tasks are changed by the technology and cannot be implemented without it.

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REFERENCES

Burkhardt, H. & Swan, M. (2013). Task design for systemic improvement: principles and frameworks. In C. Margolinas (Ed.), *Task design in Mathematics Education – Proceedings of ICMI Study 22* (pp. 431-440). ICMI.

Joubert, M. (2017). Revisiting theory for the design of tasks: Special considerations for digital environments. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital* *technologies in designing mathematics education tasks - Potential and pitfalls* (pp. 17-40). Springer.

Laborde, C. (2001). Integration of technology in design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning*, 6, 283-317.

Leung, A. (2017). Exploring techno-pedagogic task design in the mathematics classroom. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital technologies in designing mathematics education tasks - Potential and pitfalls* (pp. 3-16). Springer.

Olsson, J. (2019). Relations between task design and students' utilization of GeoGebra. *Digital Experiences in Mathematics Education*, 5, 223-251.

Ponte, J. (2005). Gestão curricular em Matemática. In GTI (Eds.), *O professor e o desenvolvimento curricular* (pp. 11-34). APM.

Rocha, H. (2013). Knowledge for Teaching Mathematics with Technology – a new framework of teacher knowledge. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th PME* (vol. 4, pp. 105–112). PME.

Rocha, H. (2020). Graphical representation of functions using technology: a window to teacher knowledge. *Teaching Mathematics and its Applications*, 39(2), 105-126.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8, 255-281.

Tanışlı, D., Köse, N., & Turgut, M. (2019). Professional development through a webbased portal: The progress of mathematics teachers teaching algebra based on hypothetical learning trajectories. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of CERME 11* (pp. 2741-2748). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Thomas, M. & Lin, C. (2013). Designing tasks for use with digital technology. In C. Margolinas (Ed.), Task design in Mathematics Education – *Proceedings of ICMI Study 22* (pp. 109-118). ICMI.

Watson, A., et al. (2013). Introduction. In C. Margolinas (Ed.), Task design in Mathematics Education – *Proceedings of ICMI Study 22* (pp. 7-14). ICMI.

Bi-national survey on mathematics teachers' digital competences

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We aim at investigating digital competences of mathematics teachers in France and in Israel. A questionnaire based on proficiency levels of teacher's competences was designed and implemented. Results suggest similar patterns of teachers' use of technology in both countries, yet different with respect to various competences. Insufficient instrumental genesis resulting in didactic instruments may explain these results and suggests a path for professional development efforts.

Keywords: Teacher digital practices, Digital competence, Instrumental genesis.

INTRODUCTION

Research findings converge on that specific knowledge and skills are required for teaching mathematics with digital technology. Yet, these knowledge and skills are not defined clearly enough to help devise efficient teacher development programs toward technology integration. Indeed, most of the reports about such initiatives conclude with unsatisfactory outcomes in terms of gaps between teachers' expectations and needs, and the program contents (Hegedus et al., 2016).

Several theoretical frameworks have been developed to provide a conceptual frame defining teachers' knowledge specific to the use of digital technology. Yet, Neubrand (2018) claims that knowledge-driven approaches are limited because of both "the gap between knowing and acting" and the lack of the "affective component" (p. 609) deemed as important as the cognitive one. As Kunter et al. (2013) put forward, "aspects beyond knowledge may be important in determining teacher success. These aspects include teachers' beliefs, work-related motivation, and ability for professional self-regulation" (p. 807), and delineate the concept of teacher professional competence. This study aims at investigating digital competences of mathematics teachers in France and in Israel.

THEORETICAL FRAMEWORK

We outline our meaning of competence, the instrumental approach (Rabardel, 2002) used to investigate ways mathematics teachers use digital technology, and the three frames of teachers' digital technology use (Abboud-Blanchard & Lagrange, 2007).

Competence

Klieme et al. (2008) consider competences to be "context-specific dispositions for achievement that can be acquired through learning. Furthermore, they functionally relate to situations and demands in specific domains" (p. 8). Ala-Mutka (2011) suggests that competence is "an ability to use knowledge and skills with responsibility, autonomy and other appropriate attitudes to the context of work, leisure or learning"

(p. 18). These definitions converge on that competence involves an ability to act in a particular context or situation. We thus consider competence as a means for acting in different professional situations, and put aside affective and other aspects because of space limitation.

We draw more particularly on the European framework for the digital competence for educators (DigCompEdu, Redecker, 2017). The framework describes educator-specific digital competences that are grouped in six areas: (1) professional engagement, (2) digital resources, (3) teaching and learning, (4) assessment, (5) empowering learners' competences, and (6) facilitating learners' digital competence. Areas 2-5 represent "the core of the DigCompEdu Framework" (p. 16).

Instrumental genesis

The instrumental approach (Rabardel, 2002) is used to understand processes by which a user transforms a digital tool - an artifact, into an instrument enabling her to achieve her goals. While the artifact (material or symbolic) is available to the user, the instrument is a personal construct created by the user during her activity with the artifact. This process is called instrumental genesis and comprises two interrelated processes: instrumentation leading to the development by the user of schemes of use of the artefact, and instrumentalisation during which the user adapts the artefact according to her knowledge and beliefs. The development of schemes of use manifests itself in a user's invariant behaviour in a given class of situations.

Three frames of teachers' professional use of digital technology

An important part of teacher's activity occurs outside the classroom, e.g., preparing lessons, searching for resources or communicating with colleagues. Teachers also use different kind of technology: mathematical software for teaching, but also general technology such as Internet or text editor for preparing students' worksheets. Abboud-Blanchard and Lagrange (2007) distinguish three frames of digital technology uses:

- Frame 1 is "the personal sphere of activity wherein the teacher uses ICT with no direct connection with his/her classroom activities" (e.g., communication with colleagues, or use of specific software not directly linked to students' learning).
- Frame 2 "refers to prep work, the teacher having in mind what knowledge and know-hows he/she wishes the students acquire. For instance, a teacher might be using general tools (Internet, spreadsheets) and more specific ones strongly connected with subject teaching".
- Frame 3 is "in classroom, ICT use being intimately bound to subject teaching and being subservient to the students' learning".

One can expect that activities within different frames imply different teachers' competencies. This leads us to specify teachers' activities when defining their digital competence. We fine-tune these frames to the purposes of our research. In particular, we believe that instrumental genesis encompasses a reflectivity on professional actions taken, which the three frames do not capture. We adapt the delineation of the three frames as follows: first, we distinguish between teacher's professional activity done outside the classroom (frames 1 and 2) and in the classroom (real or virtual) implying teacher's interactions with learners (frame 3). We consider frame 1 including teacher's professional activity not directly aimed at planning or implementing a lesson, whereas frame 2 referring to preparation activity, as well as reflection on the lesson implementation leading to the improvement of the lesson and possibly impacting the subsequent lesson preparation. Whereas frame 3 relates to the didactical use of technology, this may not be the case for frame 2: a teacher can prepare students' worksheets using technology and implement them in a printed paper format in the classroom.

We investigate mathematics teachers' digital competencies as they manifest themselves in their practices in the three frames. Our research question is: Are there differences in practicing teachers' use of technology according to the three frames?

METHOD

Classes of situations where digital technology can be used

In line with the instrumental approach, to define classes of situations where teachers can use digital technology, we refer to the three frames of teachers' professional use of digital technology and to the areas of the DigCompEdu Framework that define teachers' digital competence, in particular the areas 2 and 3.

We suggest that *searching for and selecting resources* is a class of teachers' professional situations belonging to frame 1, as the purpose of this activity may not be directly related to lesson preparation. Frame 2 encompasses activities related to lesson planning and reflecting after lesson implementation: *creating and modifying digital resources* and *designing learning tasks, sessions or sequences of sessions*. Frame 3 comprises activities involving teachers' interactions with learners: *implementing and managing digital learning activities*, *assessing students learning and performance*, and *monitoring the class and following the students*. Note that the last two activities belong to the frame 2 as well.

Design of the questionnaire and data collection

We conducted a survey in France and in Israel involving mathematics teachers. For each class of situations, we listed items taken from the DigCompEdu Framework (Redecker, 2017), particularly from statements related to educator's proficiency levels: we consider stage A - educators assimilate new information and develop basic digital practices; stage B - they apply, further expand and structure their digital practices; and stage C - they pass on their knowledge, critique existing practice and develop new practices. After having tested the English version with a few practicing mathematics teachers in Israel and in France, we refined the items and translated them to Hebrew and French. The online questionnaire was distributed nationwide in both countries. Participation to the survey was anonymous and voluntary.

FINDINGS

Responses from 79 teachers in Israel and 434 teachers in France were collected. The participants are the teachers who are in contact with the teacher centres; hence we see them as the more proficient. We cannot claim that the data is representative of practicing mathematics teachers in either country; yet, this sample allows us getting some insight into teachers' digital competences. Next, we present findings related to the first four classes of situations, as they are associated to one of the frames. Due to the space limitation, we only refer to part of the findings.

Searching for and selecting digital resources (frame 1)

Question: Which of the following items characterize your practice when you search for and select digital resources? Please tick the corresponding cells.	Israel	France
(A) I only rarely, if at all, use internet to find resources for teaching and learning.	5 (6.9%)	34 (7.8%)
(A) I use simple internet search strategies (e.g., keywords) and common educational platforms to identify digital content relevant for teaching.	33 (45.8%)	271 (62.4%)
(B) I evaluate the quality of digital resources based on basic criteria, such as e.g. place of publication, authorship, other users' feedback.	29 (40.3%)	123 (28.3%)
(B) I adapt my search strategies to identify resources, e.g. searching and filtering by license, filename extension, date, user feedback etc.	10 (13.9%)	55 (12.7%)
(C) I evaluate the reliability and suitability of content for my learner group and specific learning objective based on a combination of criteria, verifying also its accuracy and neutrality.	39 (54.2%)	161 (37.1%)
(C) In addition to search engines, I use a variety of other sources, e.g. collaborative platforms, official repositories, etc.	38 (52.9%)	229 (52.8%)

Table 1. Results related to searching for and selecting digital resources

We see this class of situations in frame 1 because searching for resources is not always linked with lesson planning. Interestingly, the responses in both countries are coherent: the items appear in the same order according to percentages. A small percentage of the teachers (about 7% in Israel, 8% in France) declare rarely, if at all, use Internet to search for resources. Hence, Internet is a source of digital resources for the majority of teachers. Teachers rather use simple search strategies or search for resources on common educational platforms (around 46% in Israel, 62% in France). Only about 14% Israeli and 13% French teachers adapt their search strategies to identify resources. Two hypotheses can explain this finding: either simple strategies provide satisfactory results or teachers complement these strategies with other ways of searching for resources. The fact that about 53% of teachers in both countries declare using other sources, such as collaborative platforms or official repositories, corroborates the latter hypothesis. Such platforms are created by the mathematics teacher education centre in Israel; in France several well-known repositories are frequently used, like those of the Institutes for Research on Teaching Mathematics. Some teachers use specialized search engines, e.g., Google Scholar or Publimath (French search engine dedicated to mathematics resources). Regarding resource selection based on quality evaluation, about 40% of Israeli and 28% of French teachers say using basic criteria, while 54% and 37%

respectively evaluate resource reliability and suitability with respect to their educational goal and their students.

Creating and modifying digital resources (frame 2)

Question: Which of the following items characterize your practice when you create and modify digital resources? Please tick the corresponding cells.	Israel	France
(A) I may make use of digital resources, but I do not usually modify them or create my own resources.	10 (13.9%)	38 (8.8%)
(A) I use office software to design and modify resources (e.g. worksheets and quizzes) and presentations.	40 (55.6%)	374 (86.2%)
(B) When I create digital resources (e.g. presentations), I integrate some animations, links, multimedia or interactive elements.	37 (51.4%)	154 (35.5%)
(B) I modify and combine existing resources, including interactive elements, to create learning activities that are tailored to a concrete learning context and objective, and to the characteristics of the learner group.	36 (50.0%)	205 (47.2%)
(C) I employ design principles for increasing accessibility for the resources and digital environments used in teaching, e.g. as concerns font, size, layout, structure.	24 (33.3%)	129 (29.7%)
(C) I create my own apps or games to support my educational objectives.	29 (40.3%)	53 (12.2%)

Table 2. Results related to creating and modifying digital resources

The findings are again quite coherent between the countries. The first item received the least number of responses: these teachers usually do not modify digital resources, which suggests that they use them as they are. The highest percentage of teachers chose the second item (respectively about 56% and 86%), suggesting that the use of office software to modify resources or create worksheets or presentations is common. Half of the teachers declare modifying and combining existing resources to adapt them to their context and objective. When creating resources, respectively around 51% and 36% of teachers integrate multimedia. A third of teachers in both countries consider themselves sensitive to accessibility issues when creating resources. Surprisingly, quite a big number of Israeli teachers (40%) design their own apps or games, comparing to only 12% of French teachers. This can be explained by the use, in Israel, of platforms to which teachers can apply their own content (e.g., Kahoot).

Designing a learning activity, a session or a sequence of sessions (frame 2)

Question: Which of the following items characterize your practice when you design a learning activity, a session or a sequence of sessions? Please tick the corresponding cells.	Israel	France
(A) I do not or only very rarely use digital devices or digital content in my teaching.	6 (8.3%)	35 (8.1%)
(A) I choose mathematical digital technologies according to the learning objective and context.	53 (73.6%)	308 (71%)
(A) When designing learning activities, I consider the importance of ensuring equal access, both physical (i.e., access to hardware and software) and intellectual (i.e., necessary technical knowledge) to the digital technologies used for all students	16 (22.2%)	102 (23.5%)
(B) I choose the most appropriate tool for fostering learner active engagement in a given learning context or for a specific learning objective.	41(56.9%)	198 (45.6%)

(B) I select and use some learning activities, e.g. quizzes or games, that allow learners to proceed at different speeds, select different levels of difficulty and/or repeat activities previously not solved adequately.	28 (38.9%)	123 (28.3%)
(B) I design learning sessions or other interactions with a mathematical digital technology.	19 (26.4%)	134 (30.9%)
(C) When I set up learning activities in digital environments, I let my students choose digital technology.	3 (4.2%)	15 (3.5%)
(C) I continuously evaluate the effectiveness of digitally enhanced teaching strategies and revise my strategies accordingly	19 (26.4%)	83 (19.1%)
(C) I reflect on, discuss, re-design and innovate pedagogic strategies for personalizing education through the use of digital technologies.	16 (22.2%)	121 (27.9%)

Table 3. Results related to lesson planning

Only about 8% of respondents confess not to use, or only very rarely, digital devices or content in their teaching. As regards the choice of digital technology while preparing their teaching, almost 3 teachers out of 4 claim choosing appropriate technology with respect to the learning goal and context. The percentage decreases when active learners' engagement is at stake, which might imply that the remaining percentage of teachers adopt rather teacher-centred pedagogy. Even lesser part of the teachers pays attention to differentiation (around 40% and 28% respectively) when planning their teaching. Not surprisingly, only about 4% of the teachers in both countries plan to let their students choose digital technology to be used. Less than one third of teachers (around 26% and 31% respectively) design learning activities with mathematical digital technology, which contrasts with much higher percentage of teachers who claim choosing mathematical digital technologies according to the learning objective and context (around 74% and 71% respectively). This finding suggests again the prevalence of teacher-centred pedagogy. A quarter of teachers declares being sensitive to ensuring equal access to technologies. This amount of answers should however be nuanced as some teachers confess not to have understood the meaning of the item. Finally, regarding the reflectivity about own digital strategies, about 26% of Israeli and 19% of French teachers say continuously evaluating its effectiveness and 22% of Israeli and 28% of French teachers declare innovate their pedagogical strategies through the use of technology.

Implementing and managing digital learning activities (frame 3)

Question: Which of the following items characterize your practice when you implement and manage digital learning activities?	Israel	France
(A) I use available classroom technologies, e.g. digital whiteboards, projectors.	47 (65.3%)	393 (90.6%)
(A) I use digital technologies to visualize and explain new concepts in a motivating and engaging way, e.g. by employing animations or videos.	41 (56.9%)	287 (66.1%)
(A) When implementing collaborative activities/projects, I encourage learners to use digital technologies to support their work, e.g. internet search or present results.	26 (36.1%)	108 (24.9%)
(B) Putting learners' active use of digital technologies is important in my instructional process.	27 (37.5%)	88 (20.3%)

(B) I implement collaborative activities, in which digital technologies are used by learners for their collaborative knowledge generation, e.g. for sourcing and exchanging information.	13 (18.1%)	58 (13.4%)
(B) I require learners to document their collaborative efforts using digital technologies, e.g. digital presentations, videos, blog posts.	6 (8.3%)	22 (5.1%)
(B) I use a range of digital technologies to create a relevant, rich and effective digital learning environment, e.g. by addressing different sensory channels, learning styles, by varying activity types and group compositions.	15 (20.8%)	45 (10.4%)
(B) When sequencing and implementing digital learning activities, I allow for different learning pathways, levels and speeds and flexibly adapt my strategies to changing circumstances or needs.	16 (22.2%)	85 (19.6%)
(C) I implement learning sessions so that different (teacher-led and learner-led) digital activities jointly re-inforce the learning objective.	13 (18.1%)	105 (24.2%)
(C) I select, design, employ and orchestrate the use of digital technologies within the learning process according to their potential for fostering learners' active, creative and critical engagement with the subject matter.	23 (31.9%)	75 (17.3%)

Table 4. Results related to implementing and managing digital learning activities

This class of situations gives insight into the declared use of technology in class. Indeed, around 65% of the Israeli and 91% of the French teachers declare using available technology (projector, IWB, PC). Teacher-centred use of technology seems to be dominant in both countries, as was marked by about two thirds of teachers. In contrast, the percentage is much lower for items clearly indicating the use of technology in the hands of students; only around 38% and 20% of teachers declare paying attention to learners' active engagement with technology. This percentage is even lower (32% and 17%) when students' active, creative and critical engagement is at stake. When collaborative activities are implemented, around 36% and 25% of teachers declare encouraging students to use technology to support their work, whereas only 18% and 13% take profit of such activities for learning purposes. However, only about 8% and 5% of teachers require learners to document digitally their collaborative efforts. About a fifth of teachers declare exploiting technology for differentiation, which is less than regarding planning (40% and 28%), which might mean that when planning, more teachers might use technology for preparing differentiated paper-based worksheets. Only around 21% and 10% of teachers combine various digital technologies to create a rich learning environment. Finally, about 18% and 24% of teachers declare implementing sessions in which different digital activities reinforce learning goal.

DISCUSSION AND CONCLUSION

The four classes of situations show similar pattern in responses between the two countries, however very different according to the three frames. The class of situations in frame 1 - *searching for and selecting digital resources* (Table 1) - shows increased percentage of teachers' responses along proficiency levels. A plausible hypothesis explaining this finding is that nowadays searching on the internet is part of the digital literacy of any citizen; hence teachers are proficient in this competence. Classes of situations in frame 2 - *creating and modifying digital resources* and *lesson planning*

(Tables 2 & 3) - show a pattern that resembles normal distribution, suggesting that most of the teachers are at intermediate proficiency level B regarding these competences. Finally, the class of situations in frame 3 -*implementing and managing digital activities* (Table 4) - shows decreasing pattern along the proficiency levels. These findings suggest a gap in uses of technology in frames 1 & 2 and in frame 3, i.e., although teachers use it to search for resources and to prepare their teaching, they use it much less in the classroom. This finding is similar to what Abboud-Blanchard and Lagrange (2007) observed in student-teachers' use of technology. We suggest that the difficulty with integrating technology in classrooms continues, as teachers fail to transform available technology into didactic instruments for their teaching. This may suggest avenues for teacher professional development.

REFERENCES

Abboud-Blanchard, M., & Lagrange, J.-B. (2007). Uses of ICT by pre-service mathematics teachers: towards a professional instrumentation? *International Journal for Technology in Mathematics Education*, 13(4), 183-19.

Ala-Mutka, K. (2011). *Mapping digital competence: Towards a conceptual understanding*. JRC Technical Note. Luxembourg: European Union.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H. S., Tabach, M., Trgalova, J., & Moreno-Armella, L. (2016). *Uses of technology in upper secondary mathematics education.* Switzerland: Springer.

Klieme, E., Hartig, J., & Rauch, D. (2008). The concept of competence in educational contexts. In J. Hartig, E. Klieme, & D. Leutner (Eds.), *Assessment of competencies in educational contexts* (pp. 3–22). Göttingen: Hogrefe.

Kunter, M., Klusman, U., Baumert, J., Richter, D., Vos, T., & Hachfeld, A. (2013). Professional Competence of Teachers: Effects on Instructional Quality and Student Development. *Journal of Educational Psychology*, 105(3), 805–820.

Neubrand, M. (2018). Conceptualizations of professional knowledge for teachers of mathematics. *ZDM Mathematics Education, 50*(4), 601–612.

Rabardel, P. (2002). *People and technology — a cognitive approach to contemporary instruments*. Université Paris 8.

Redecker, C. (2017). *European Framework for the Digital Competence of Educators (DigCompEdu)*. Joint Research Centre Science for Policy Report. EU: Luxembourg Publications Office.

Theme 1

Mathematics Teacher Education and Professional Development in the Digital Age

Posters

Blended-learning in mathematics teacher professional development

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The project is located at the University of Education Schwäbisch Gmünd, Germany as cooperation between the University and the aim (Akademie für Innovative Bildung und Management) in Heilbronn, Germany. As part of the project, a teacher professional development program on the topic of language in arithmetic lessons is designed, carried out and evaluated. The program started in October 2019 with 14 arbitrarily selected teachers of mathematics (voluntary registration for the course). It ends in June 2020. The program is designed as blended-learning to provide a maximum of flexibility in time and space for the participants.

Keywords: professional development, blended-learning, language in mathematics learning.

RESEARCH ON PROFESSIONAL DEVELOPMENT FOR TEACHERS

For more than ten years, research on teacher professional development has concluded that successful professional development programs for teachers include some specific aspects. These are e.g. long-term duration instead of so called “one-shot” programs and referring to school topics and curricula. Professional development programs for teachers have to address teachers and generate subjective importance (Bogler and Nir, 2008; Lipowsky and Rzejak, 2012). Moreover, the meta-analysis of Darling-Hammond, Hyler, and Gardner (2017) reports further features of successful programs: they have to focus on active learning on the side of the participants what implies that the programs are illustrated with authentic examples and tasks. Furthermore, the authors advise to support collaboration, give best practice models, provide coaching, feedback and reflection. However, professional development programs in Germany often do not take up these features.

RESEARCH FINDINGS ON LANGUAGE IN MATHEMATICS LEARNING

Since the first survey, PISA has reported weaker mathematics’ skills for migrant students than non-migrant students (Gebhardt et al., 2013, p. 275). There is no question that differences in the level of competence achieved in mathematics can be attributed to linguistic aspects. This is not a phenomenon limited to German, but applies to all processes of second language acquisition as, for mathematics, three different language registers need to be learned (technical language, educational language and everyday language) (Cummins, 1979). In addition, research has shown that characteristics of the mathematical terminology and specialties of the German language also cause problems on different levels (Prediger, 2015).

IMPLICATIONS FOR THE DESIGN OF THE RESEARCH PROJECT

The project aims to develop a professional development program for teachers that fits current research in several ways: to overcome the gap between research findings on successful professional development for teachers, the designed program refers to the following aspects: it is planned long-term to provide the possibility for the participants to connect, collaborate, try out and reflect. To provide a maximum of flexibility in time and space, it is held as a blended-learning-course what means that it is started and ended by a classroom event. In between, a longer online stage will present authentic application examples. The topic language in arithmetic lessons is chosen to take into account the research that found out, that professional development programs have to provide authentic content that is close to the curricula. With this specific topic, we paid attention to this aspect. Since this topic is relevant for teachers in primary and secondary level, the participants will be from both types is the style. Based on the theoretical considerations and empirical findings described above, the following main research question arises for the presented research project: How does a professional development program, designed as a blended-learning course, influence the satisfaction of the teachers and their willingness to attend further PD of this kind? This research question raises further sub-questions that will be presented on the poster.

REFERENCES

Cummins, J. (1979). Cognitive/Academic Language Proficiency, Linguistic Interdependence, the Optimum Age Question and Some Other Matters. *Working papers on bilingualism*, 19, p. 197–205.

Darling-Hammond, L., Hyler, M. E., Gardner, M. (2017). *Effective Teacher Professional Development*. Palo Alto, CA: Learning Policy Institute.

Gebhardt, M., Rauch, D., Mang, J., Sälzer, C. & Stanat, P. (2013). Mathematische Kompetenz von Schülerinnen und Schülern mit Zuwanderungshintergrund. In M. Prenzel, C. Sälzer, E. Klieme, & O. Köller (Eds.), *PISA 2012: Fortschritte und Herausforderungen in Deutschland* (p. 275– 308). Münster: Waxmann.

Prediger, S., Wilhelm, N., Büchter, A., Benholz, C. & Gürsoy, E. (2015). Sprachkompetenz und Mathematikleistung – Empirische Untersuchung sprachlich bedingter Hürden in den Zentralen Prüfungen 10., In *Journal für Mathematik-Didaktik, 36*(1), p. 77-104.

Lipowsky, F., & Rzejak, D. (2012). Lehrerinnen und Lehrer als Lerner – Wann gelingt der Rollentausch? Merkmale und Wirkungen effektiver Lehrerfortbildungen. *Schulpädagogik heute, 5*(3), p. 1–17.

Nir, A. E., & Bogler, R. (2008). The antecedents of teacher satisfaction with professional development programs. *Teaching and Teacher Education*, *24*(2), 377–386.

Potential explanations to the opposition of curricular digitalisation: a case study of Egyptian mathematics teachers

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This poster is grounded on a wider empirical study that was conducted with Egyptian mathematics teachers in view of their perceived relation to students and to mathematical content knowledge. Based on a contextual and dynamic understanding of the instructional triangle, it presents a potential explanation to the opposition that was recently nationally recorded, with regards to the digitalisation of the teaching and learning platform of school mathematics education. Findings suggest the opposition may be rooted in a distorted self-image of mathematics teachers as authority figures in the classroom.

Keywords: digital platform, teacher role, student self-image, mathematical content

INTRODUCTION

In 2018, the Ministry of Education in Egypt launched a national initiative to digitalise the teaching, learning and examination platform for mathematics education. Teachers and students were to be provided with a common digital interface, from which they had common access to mathematical knowledge, explanation and practice tools as well as access to central government examination, when needed. This initiative was faced with a lot of opposition. In this poster, I use the instructional triangle dynamic as a theoretical lens. I present findings of an empirical investigation that uncovers how teachers relate to students and to mathematical tasks in the classroom and how their perceived self-image as authority figures in the classroom might be disrupted by the introduction of a unified digital platform for knowledge exchange.

THEORETICAL FRAMEWORK

In their study of mathematical resources and their utilisation for instructional purposes, Cohen, Raudenbush and Ball (2003) plot the complexity of the contextual interaction between students as peers, students and the teacher as well as students, teachers and the content of instruction. According to the authors (Cohen et. al., 2003) instruction happens as an interaction between multiple agents. Firstly, there is the teacher knowledge, both of the diversity of their student body and of the mathematical content being taught. Secondly, there is the student-peer interaction, the student- teacher interaction and the student's own thinking and learning process in relation to mathematical content learning. The interactions happen within a context that is bound by a classroom environment, a school environment and a wider societal environment that either endorses or weakens the learning operation.

LITERATURE REVIEW

In his investigation of the learning environment within the framework of the centrally controlled and unified national agenda for teaching and learning in Egypt, Naguib (2006) argues for the existence and reproduction of a culture of despotism that views teaching and learning content as static and solely owned by the figure of authority. A passive depiction of knowledge transfer is passed on hierarchically to school via the centrally governed national policy. This culture is in turn reproduced at school and then again in classroom. As a result, the teacher relates to the content taught in the lesson as being static in nature and as teacher ownership. Similarly, the teacher relates to students as a block of passive, non-diverse classroom audience.

EMPIRICAL EXPLORATION AND FINDINGS

As part of a wider study that aimed to explore Egyptian teachers' perceptions of math problem solving tasks and their classroom integration, math teacher groups were presented with a set of different math problem solving task integration scenarios, each depicting a different experience of the math task in the classroom. Teachers were prompted to discuss which of the scenarios best related to their perception of their role in the classroom, both in relation to students and to math content. Patterns in the findings were mapped against the same analytical framework that was devised to put together the scenarios. Repeatedly, findings revealed a teacher self-image of her role as the main authority figure in the classroom, solely responsible for the transmission of content knowledge to a passive audience of students: "The task belongs to me. I am the teacher. I oversee the classroom and know exactly how to teach it. The student, he doesn't know."

DISCUSSION

In line with Naguib's (2006) depiction of culture of despotism that reproduces itself at school and classroom level, it becomes understandable why equal access to a digital interface of infinite math knowledge is considered threatening to the teacher and hence opposed. The understanding of how the student passively relates to the mathematical content also explains why a student would oppose an open digital platform for learning. The teacher self- perceived role in the classroom in relation to both content and the student seems to have, over time, led to a limited student self-image in relation to themselves, their peers and the mathematical content studied.

REFERENCES

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resource, Instruction, and Research. *Educational Evaluation and Policy Analysis*, 25 (2), 119-142.

Naguib, K. (2006). The production and reproduction of culture in Egyptian schools. In L.Herrera, & C. Torres (Eds), *Cultures of Arab Schooling* (pp. 53-81). New York: State University of New York Press.

Surveying prospective teachers' conceptions of GeoGebra when constructing mathematical activities for pupils

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In this poster, we present an ongoing study about prospective mathematics teachers' conceptions about the relationship between mathematics, problem solving and GeoGebra. The context of our study is a curriculum reform in Sweden that emphasizes the use of digital tools mathematics education. In that respect, we will investigate prospective upper-secondary teachers′ conceptions when participating in a geometry course at university level. During the course, participants will construct mathematical activities for pupils by using GeoGebra.

Keywords: mathematics teaching and learning, prospective teachers, dynamic geometry software, problem solving.

BACKGROUND

Despite a substantial emphasis on the potential of digital technologies for mathematics education in the last two decades (e.g. Hoyles & Lagrange, 2010), a meaningful implementation of digital tools in the teaching of mathematics is not an unproblematic issue (e.g. Drijvers, 2013). Following the international trend, a relatively recent curriculum reform in Sweden highlights the use of digital technology in mathematics education in upper-secondary school. Importantly, according to the curriculum, digital tools should be used by pupils to develop their competences in problem solving and mathematical modelling.

Mathematical problems and problem solving are considered central to mathematics education. When discerning problems from routine tasks, the relationship between the solver and the proposed task, in combination with the challenge that the solver faces when solving the task, are essential (Carlson & Bloom, 2005). Consequently, to prepare prospective teachers to use digital tools in their teaching, these perspectives should be included in the teacher education programme. Additionally, prospective teachers′ relatively limited experience in working with digital technology in mathematics, should be addressed in the context (Misfeldt, Szabo & Helenius, 2019).

THE STUDY

By focusing the educational programme of prospective teachers, the main goal of the present study is to investigate their conceptions about the relationship between school geometry, problem solving and GeoGebra. Consequently, we designed a survey related to various aspects of and intersections between these subjects. The survey is mainly based on previous studies about mathematics teachers facing challenges related to new digital technology or to the use of Dynamic Geometry Software (DGS) (e.g. Fahlgren & Brunström, 2014; Misfeldt, Szabo, & Helenius, 2019). Some questions focus prospective teachers' use of DGS in the perspectives of instrumental distance and professional genesis (Haspekian, 2014). To achieve an appropriate level of reliability and validity of the responses we use a four-alternative Likert scale.

The 22 participants are prospective upper-secondary teachers enrolled in a geometry course at university level. Prior to the course, they had no formal education in and very little experience of DGS. During the course, participants applied GeoGebra to construct activities for pupils that include problem solving. In order to develop participants' critical reflection, mentioned activities underwent peer assessment and were tested by peers, acting as pupils.

Due to the limitations of this poster, we present only one result from our study. The analysis shows that 95% of participants feel enthusiastic – by stating that DGS is important and meaningful for mathematics education – and 70% feel well-prepared to use DGS in their teaching. On the other hand, 85% of participants think that they will meet pupils who are more skilled in DGS than they are. This indicates that prospective teachers' conceptions related to DGS should be discussed further in the light of instrumental distance and professional genesis (Haspekian, 2014). That is, it is not unreasonable to assume that, despite feeling optimistic and relatively well-prepared, participants will face challenges when implementing DGS in their teaching.

REFERENCES

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75.

Drijvers, P. (2013). Digital Technology in Mathematics Education: Why It Works (Or Doesn't). *PNA, 8*(1), 1–20.

Fahlgren, M. & Brunström, M. (2014). A model for task design with focus on exploration, explanation, and generalization in a dynamic geometry environment. *Technology, Knowledge and Learning, 19*(3), 287–315.

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era* (pp. 241–275). Dordrecht: Springer Media.

Hoyles, C., & Lagrange, J.-B. (Eds.). (2010). *Mathematics education and technology – Rethinking the terrain*. New York, NY/Berlin, Germany: Springer.

Misfeldt, M., Szabo, A., & Helenius, O. (2019). Surveying teachers' conception of programming as a mathematics topic following the implementation of a new mathematics curriculum. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 2713–2720). Utrecht: Utrecht University.

Theme 2
Mathematics Curriculum Development and Task Design in the Digital Age

Papers

Preservice teachers' perceptions on outdoors education using a digital resource

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This paper refers to a study that aims to understand the potential of digital technology in outdoors mathematics education from the perspective of future teachers. We followed a qualitative approach and collected data through observation, questionnaires and photographic records. The participants were forty-eight preservice teachers that used Math City Map to do a math trail in Viana do Castelo. Results show that they valued the experience, having the possibility to solve realistic problems, developing cooperative work, critical thinking and establishing mathematical connections. They found the app to be user friendly and motivating, mentioning its contribution for students' engagement through active learning, spatial orientation, autonomy and being more interactive than the paper version.

Keywords: Math trail; Problem solving; Mathematical connections; STEM education; Teacher training.

INTRODUCTION

This paper is based on previous work developed by the authors in the scope of outdoor mathematics education. We have been carrying out several studies conducted with preservice teachers (e.g. Barbosa & Vale, 2016; Barbosa & Vale, 2018; Vale, Barbosa & Cabrita, 2019) which show that the outdoors can be seen as a privileged educational context, promoting positive attitudes and additional engagement/motivation for the study of mathematics. In particular, the use of math trails has great potential in unveiling the connections between mathematics and everyday life, specifically with the environment that is close to us. These studies focused mainly on a particular detail of the math trails, which was task design, approaching different aspects of problem posing and obviously problem solving, using a mathematical eye to formulate tasks that highlight connections with daily life. Along with this research interest, being part of the Consortium of the Project Math Trails in School, Curriculum and Educational Environments in Europe (MaSCE[3]), gave us the opportunity to contact with a different approach to math trails, other than task design, adding the possibility to resort to digital technology, specifically mobile devices. It is important to state that the use of Math City Map (MCM), a project of the working group MATIS I (IDMI, Goethe- Universität Frankfurt) in cooperation with Stiftung Rechnen, has been reported as having a positive impact in supporting teachers and students in the process of teaching and learning mathematics outside the classroom, acting as a resourceful tool to explore the outdoors in a mathematical perspective (e.g. Cahyono & Ludwig, 2019; Ludwig & Jablonski, 2019). We are convinced that these approaches are extremely relevant in mathematical education and also to the development of a set of skills expected from students in 21st century, so it is our purpose in this study to understand the potential of digital technology in outdoor mathematics from the perspective of future teachers. Based on this problem, the following research questions were formulated: 1) Which potentialities and limitations are recognized in MCM by the participants?; 2) How can we characterize the reactions of the participants to a math trail?

OUTDOOR EDUCATION: THE CASE OF MATH TRAILS

One of the main ideas of this paper is that of Math Trail. Hence, it is pertinent to begin by defining this concept. We consider a math trail to be a sequence of tasks along a pre-planned route (with beginning and end), composed of a set of stops in which students solve mathematical tasks in the environment that surrounds us (Vale et al., 2019, adapted from Cross, 1997). This is a privileged context to offer rich learning experiences to the participants, that also enables the exploration of mathematical concepts stated in the curricular guidelines, which can be considered as an advantage in the teachers' perspective (e.g. Vale et al., 2019). While experiencing a math trail, students can use and apply mathematical knowledge learned in school but, at the same time, mobilize informal daily life knowledge. Beyond this possibility there is a wide range of skills that are naturally in line with outdoor education like problem solving, critical thinking, collaboration, communication, reasoning or the establishment of connections. For all the stated arguments, we believe that it is important to complement the work developed inside the classroom with experiences in the scope of outdoor mathematics, allowing students to discover and interpret the world beyond the classroom walls and accepting that education can take place in different places and contexts (Bonotto, 2001).

During a math trail the participants contact with realist problems that illustrate the usefulness of mathematics, but more than that amplify the possibility of establishing connections between mathematics and reality. This feature can be crucial to induce positive attitudes towards this discipline (e.g. Bonotto, 2001; Borromeo-Ferri, 2010), relying specially on curiosity, motivation and interest. Beyond solving realist problems, in this non-formal context we must not forget the influence produced by movement in students' attitudes. The body plays a decisive role in the entire intellectual process. Alongside cognitive engagement, math trails imply two other dimensions: physical and social engagement (Hannaford, 2005). The interaction between these dimensions, facilitated by a math trail, is in line with an active learning approach, known by committing students to the learning process, hence promoting positive attitudes towards mathematics (e.g. Vale et al., 2019).

Richardson (2004) proposes a series of steps for the preparation of a math trail: (1) first comes the selection of the site. It can be anywhere, as long as it is rich in mathematics. The teacher must observe the elements of the chosen context and look for aspects like patterns, shapes, things to measure, count or represent; (2) then, we take photos at each chosen location to later use them in the design of the tasks; (3) select the photos, create a map and identify the chosen places to carry out the tasks in order to verify the distribution of the stops and the distance of the route; (4) formulate the different tasks and the instructions to reach the different stops. These tasks must have different cognitive levels of demand (Smith & Stein, 2011) and admit different mathematical approaches. The tasks must be solved with knowledge previously learned in the classroom; (5) whenever possible, it is interesting to establish connections between mathematics and other curricular areas through the tasks. Regarding the task design, Richardson (2004) recommends that questions should arouse the curiosity, forcing the students to observe the environment to achieve a successful solution. There are other aspects to consider on a math trail. According to Shoaf, Pollak, and Schneider (2004): they should be for everyone, regardless of age and experience, since it is intended that they discuss and compare their reasoning and strategies; they require collaboration and not competition; the participants must be able to manage time; participation must be voluntary, given that participants must feel involved and interested; they should be presented in any safe public place, since mathematics is everywhere; and they are temporary, since the places are subject to changes over time. After completing the trail, the participants must carry out their assessment, in order to expose the difficulties felt, as well as the aspects to maintain and improve.

DIGITAL TECHNOLOGY AND OUTDOOR MATHEMATICS

Nowadays, mobile devices are fully integrated in our daily lives and, consequently, in the lives of students starting from very young ages. Teachers should be more aware of this fact and try to follow this trend using resources of this nature in their teaching practices. In addition to keeping up with the development and needs of contemporary society, it is also important to state that mobile devices are becoming a resource with great potential both in classrooms but also in outdoor learning (Sung, Chang & Liu, 2016). This is due to the rapid developments in mobile devices and also in the creation of a diversity of educational apps, which increases the window of opportunities for teachers to use these tools with their students.

The diversity of learning opportunities offered by this type of technology can make STEM education more interesting, significant and enjoyable for students, enhancing the possibilities for their engagement in STEM subjects inside but also outside the classroom (e.g. Sung et al., 2016). The extension of the classroom to the outdoors is facilitated by the portability and wireless functionality of the mobile devices, which presents students with a more authentic and appropriate context, making it easier to explore the surrounding environment (Cahyono & Ludwig, 2019). Digital technology can help develop a deeper understanding of mathematics, acting as a mind tool that facilitates inquiry, decision making, reflection, reasoning, problem solving and collaboration (Fessakis, Karta & Kozas, 2018).

METHODS

This study follows an interpretative qualitative methodology (Erickson, 1986). The participants are forty-eight future teachers that attend an undergraduate teacher training course in primary education (6-12 years old), which includes a unit course on Didactics of Mathematics that acts as the context for the development of the study. Knowing that, at the beginning of the semester, the participants did not have any significant experiences working mathematics outside the classroom, we chose to start with an activity of this nature, a math trail. Initially they completed a questionnaire (Questionnaire I) that aimed to access their perceptions about the teaching and learning of mathematics outside the classroom and also about the use of technology in that type of context. Then they had the opportunity to do a math trail using the Math City Map (MCM) app, which was organized and designed by the researchers to be solved in the historical centre of the city of Viana do Castelo. The trail was planned and the tasks were designed based on the ideas of Richardson (2004) and Smith and Stein (2011), seeking, in general, to propose diversified tasks with regard to the mathematical contents involved and with different cognitive levels of demand. To implement the trail the preservice teachers worked in groups of 3 or 4. They attributed the responsibility of the use of the app/smartphone to one of the group elements, while the others were in charge of the measurements, calculations and registers. After doing the trail they completed a second questionnaire (Questionnaire II), applied with the purposed to analyse eventual changes on the perceptions of the participants about outdoor mathematics and the use of technology, specifically the MCM app.

Data was collected in a holistic, descriptive and interpretive manner and included observations (of the preservice teachers doing the math trail), questionnaires, photographs and written productions (solutions of the tasks). The later were not analysed for this specific paper. The researchers accompanied the participants during the trail, a choice that facilitated the accomplishment of the direct observation. Since we had forty-eight participants, to maximize the observation, we chose to divide the group in half and do the math trail with each group separately. The questionnaires contained mainly open-ended questions, so the content analysis focused on finding categories of responses regarding the perceptions evidenced by the participants, which were crossed with the evidences collected with the observation. In this process we reached categories mainly influenced by the research questions: reactions to the math trail; potentialities of MCM; limitations of MCM.

RESULTS AND DISCUSSION

We started by analysing the results of Questionnaire I, to be aware of the initial perceptions of these future teachers about outdoor education and the use of technology in such a context. In this process we used percentages but only as a mere indicator of trends in the answers. We concluded that the majority of the participants (91%) considered that it is possible to teach and learn mathematics outside the classroom. The examples cited varied between: tasks related to real life situations; counting activities;

money related tasks; shopping activities; games; competitions; clubs; field trips; observing architecture/artwork/shapes in the outdoors; finding mathematics in nature, like patterns/shapes; doing a trail/peddy paper. 87% of the participants revealed that they never experienced a mathematics class outdoors, which in a certain way may explain the general and vague ideas they had about how to do it. Considering these results, we believe that preservice teachers need to experience certain methodologies before they are able to incorporate them in their future practices. As for technology knowledge, 60% of the participants stated that they did not know any digital resources to explore mathematics outdoors. The 40% that admitted knowing resources of this nature mentioned digital games, apps and robots, but none of the examples given allowed the exploration of the surrounding environment, they only had a playful strand.

Before going to the city centre to do the math trail with MCM, the participants had a brief session about the use and the main features of Math City Map. Then the researchers accompanied them to the location of the trail and supervised the activity, which, as mentioned, facilitated the observation of certain aspects. Regarding the use of the app, we can say that they didn't show noteworthy difficulties. They found it to be very intuitive and were extremely autonomous throughout the trail. The gamification feature of the app was definitely an extra motivating factor: on one hand it caused excitement when the solution was correct; and implied greater care before the introduction of the answers, which was reflected on several situations were the participants tried to make sure of the validity of the answer discussing it within the groups. The dynamics of the math trail using MCM naturally promoted collaborative work, within each group, leading them to share responsibilities (e.g. carry and use the smartphone; measurement; recording data; calculations), or even among different groups cooperating with the same goal in mind (e.g. joining several articulated meters to find the measure of a certain length). In Figure 1 we can observe different moments of the trail implementation that illustrate the preservice teachers' work, where they had, for example, to: determine the volume of a flower pot; estimate the length of an avenue based on a pattern of lamps; discover the probability of hitting the white area of a no entry sign with a dart; or characterize the rotation symmetries in a stained glass window. These are only four of the tasks of the math trail but they are representative of the other tasks used in the trail.

Figure 1: Preservice teachers doing the math trail with Math City Map

Throughout the trail it was possible to witness reactions and comments made by the preservice teachers that we think are relevant and must be emphasized since they reveal engagement: the trail gave them the opportunity to get to know better certain aspects of the city, related to historic and architectural features that they did not know of; many expressed interest in using the app with their future students; we identified a generalized satisfaction throughout the activity; they valued the need to move around, opposed to the sedentary work traditionally developed inside the classroom.

After experiencing the math trail with MCM, the preservice teachers completed Questionnaire II. From the analysis of the results we were able to conclude that all the participants recognized the importance of teaching and learning outside the classroom, especially as a way to complement the formal educational context. Contrary to the results obtained through Questionnaire I, they were all convinced, with no exception, that teaching and learning mathematics outside the classroom is possible, showing that some of these preservice teachers changed their opinion about this issue. Those who already thought that this strategy was a possibility, stated it with even more emphasis, admitting that it exceeded their expectations. We found several arguments supporting these ideas: it follows the principles of active learning, promoting intellectual, social and physical engagement; learning is more meaningful for students because they are directly involved; it increases motivation and enthusiasm; it helps understand the usefulness of mathematics, realizing its application in real life problems; it allows to increase the knowledge of the cultural and natural heritage; it facilitates collaborative work and helps develop communication skills, as well as critical thinking; it can lead to the use of technology.

The majority of these participants expressed that they enjoyed solving all of the tasks presented along the trail, which is consistent with the observed enthusiasm. The tasks pointed as favourites corresponded to those considered as the most challenging or the ones that presented information/curiosities/historic aspects about certain elements of the city that they did no know about. On the other hand, the least favourites were the ones that required too many steps during the solution process.

In this questionnaire the participants also commented on the use of MCM and its features. From the users/students perspective they highlighted as potentialities: the possibility to use curricular contents in real life situations; being user friendly, easy to understand, promoting autonomy; facilitating cooperation; it helps to get to know the local environment; it develops spatial orientation; being more practical and interactive than the paper version; the possibility of getting immediate feedback; and the gamification feature. As for the teachers' perspective, the participants mentioned as potentialities: the possibility to design tasks adapted to the local environment and publishing them; addressing different mathematical contents and promoting interdisciplinary tasks; a way to diversify educational contexts; it allows the teacher to supervise and accompany the work developed by the groups, due to the autonomy it provides the user. When asked about the limitations of the app, these preservice teachers only referred to the possible lack of access to Wi-Fi, the fact that students of younger ages normally do not have smartphones and, in terms of the tasks, the limitation of the answer formats to either a value or multiple choice.

CONCLUDING REMARKS

Based on previous studies developed with preservice teachers (e.g. Barbosa & Vale, 2016; Barbosa & Vale, 2018; Vale et al., 2019) we had already concluded that designing and implementing math trails can promote positive attitudes towards mathematics and help gain a broader view of the connections we may establish with the surrounding environment. This type of experience develops the "mathematical eye" of the trail designers as well as of the trail users (e.g. Vale et al., 2019), bringing out the usefulness and applications of mathematics.

Unlike the above mentioned studies, this one focused only on the perspective of the trail user and not the designer. We intended to understand the potential of the MCM app in outdoor education from the point view of preservice teachers. Globally they valued the math trail experience as a meaningful pathway to engage students in realistic problem solving (Richardson, 2004), that presents a diversity of opportunities for the establishment of connections between mathematics and other content areas, as well as with real life (e.g. Bonotto, 2001; Borromeo-Ferri, 2010). Active learning was also pointed out by the participants as a fundamental attribute in a math trail, allowing intellectual, physical and social engagement, whose interaction normally generates positive attitudes (e.g. Hannaford, 2005; Vale et al., 2019). Math City Map was used as the means to present and execute the trail. This was the additional dimension of this study, trying to perceive its impact. These preservice teachers valued the use of the app, finding it user friendly and motivating, especially due to the gamification feature. They also mentioned as positive its contribution for developing spatial orientation, cooperation, students' autonomy and being more practical and interactive than the paper version. The only limitations recognized by the participants were related to constraints like the absence of Wi-Fi or smartphones and also the limited possibilities for answer formats.

To conclude, when implementing the math trail we recognized an additional motivation associated to the digital and interactive features of the MCM app, which facilitated and made more interesting the exploration of the outdoors (e.g. Cahyono & Ludwig, 2019). Being preservice teachers, the participants other then going through this experience as users, they also had the opportunity to assess the potential of the strategy (math trail) and the resource (MCM app) and analyse how could they, as teachers, implement it in the future. Recognizing the importance of keeping up with the technological development and society requirements they considered the possibility of integrating this resource, and the math trail strategy, in their practices.

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REFERENCES

Barbosa, A., & Vale, I. (2016). Math trails: meaningful mathematics outside the classroom with preservice teachers. *JETEN, 12*, 49-63.

Barbosa, A., & Vale, I. (2018). Math Trails: a resource for teaching and learning. In Gitirana, V., Miyakawa, T., Rafalska, M., Soury-Lavergne, S., & Trouche, L. (Eds.), *Proceedings of the Re(s)sources 2018 International Conference* (pp. 183-186). Lyon: ENS de Lyon.

Bonotto, C. (2001). How to connect school mathematics with students' out-of-school knowledge. *Zentralblatt für Didaktik der Mathematik, 33*(3), 75-84.

Borromeo-Ferri, R. (2010). Estabelecendo conexões com a vida real na prática da aula de Matemática. *Educação e Matemática, 110*, 19-25.

Cahyono, A. N., & Ludwig, M. (2019). Teaching and Learning Mathematics around the City Supported by the Use of Digital Technology. *Eurasia Journal of Mathematics, Science and Technology Education, 15*(1), 1-8.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York, NY: Macmillan.

Fessakis, G., Karta, P., & Kozas, K. (2018). Designing Math Trails for Enhanced by Mobile Learning Realistic Mathematics Education in Primary Education. *iJEP, 8*(2), 49-63.

Ludwig, M., & Jablonski, S. (2019). Doing Math Modelling Outdoors - A Special Math Class Activity designed with MathCityMap. *In 5th International Conference on Higher Education Advances (HEAd'19)* (pp. 901-909). Editorial Universitat Politecnica de Valencia: Valencia.

Richardson, K. (2004). Designing math trails for the elementary school. *Teaching Children Mathematics, 11*, 8-14.

Shoaf, M.,Pollak, H., & Schneider, J. (2004). *Math Trails.* Lexington, MA: COMAP-Stake.

Smith, M. & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions.* Thousand Oaks, CA: Corwin Press.

Sung, Y., Chang, K, & Liu, T. (2016). The effects of integrating mobile devices with teaching and learning on students' learning performance: A meta-analysis and research synthesis. *Computers & Education, 94*, 252-275.

Vale, I., Barbosa, A., & Cabrita, I. (2019). Mathematics outside the classroom: examples with preservice teachers. *Quaderni di Ricerca in Didacttica (Mathematics), 2*(3), 138-142.

Curricular learning with MathCityMap: creating theme-based math trails

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We introduce the math trail method as an opportunity to teach math outdoors with digital tools. Regardless of proven positive effects on motivation and learning, math trails are so far rarely used to teach a specific math topic. In order to promote the use of math trails as part of regular math lessons and to meet the requirements of outdoor learning, we have developed the concept of so-called theme-based math trails. They are a collection of conceptually related tasks focusing on a specific topic in the mathematics curriculum. This way we try to provide a curriculum-based outdoor learning experience for mathematics teaching. To exemplify the creation of theme-based trails, we have identified characteristic tasks for a trail around the concept of linear functions. By developing design principles, we intend to support teachers in creating a curriculum-based math trails using the MathCityMap system.

Keywords: Generic Tasks, MathCityMap App, Mathematics Curriculum, Mobile Learning, Theme-based Math Trails.

INTRODUCTION

One promising approach for mathematical education outdoors is the math trail method. While working on realistic and authentic tasks about real existing objects, students explore their own environment from a mathematical perspective. The MathCityMap project revives this idea and combines it with the potentials of digital media, e.g. GPS navigation and systemic feedback (Ludwig & Jesberg, 2015).

So far, the math trail method is mainly used for the revision of already learned topics. Based on local conditions, teachers create mathematical tasks about interesting objects in their own environment. As consequence, the tasks refer to various mathematical contents, especially from geometry. However, as Zender (2019) pointed out, the math trail method can also be used for the targeted work on a specific curriculum topic. To enable teachers to easily integrate math trails into their regular teaching units, we point out possibilities to create so-called theme-based math trails within the MathCityMap system. Therefore, the following terms will be used.

(1) Generic task: A generic task represents on the one hand a characteristic problem for a specific mathematical topic. On the other hand, a generic task can be understood as blueprint, i.e. the task can be performed on a variety of outdoor objects. For example, the concept of slope can be experienced outdoors through ramps, handrails or spiral staircases (Ludwig & Jablonski, 2020).

(2) Theme-based trail: A theme-based trail can be considered as a collection of generic tasks on one particular mathematical topic.

Within the concept of theme-based trails, math trails are directly linked to the mathematics curriculum. Therefore, theme-based trails can be seen as the adoption of the regular math lessons in outdoor settings. In this paper we introduce the idea of math trails and the MathCityMap app. Subsequently, we develop design principles for the creation of theme-based trails and illustrate them by focusing on the concept of linear functions. In this way we want to give a substantial contribution to implement MathCityMap math trails for the teaching and learning a specific curriculum topic.

CONCEPTUAL FRAMEWORK: MATH TRAILS & MATHCITYMAP

Mathematical outdoor education can be implemented by using the math trail method. A math trail is a walking route consisting of several place-bound math tasks. These tasks treat mathematical questions about real existing objects in one's environment, which allows people from all ages to perceive their own environment from a mathematical perspective (Shoaf, Pollak, & Schneider, 2004). The method generates "an appreciation and enjoyment of mathematics in everyday situations, usually to complement work in the classroom" (Blane & Clarke, 1984, p. 1).

According to Cross (1997), the math trail method offers several advantages for mathematics education: By working on realistic and authentic tasks, students firstly experience relevance of mathematics in everyday life. Consequently, they learn to apply their theoretical knowledge in a wide variety of practical situations and can develop strategic problem-solving skills. Secondly, Cross stresses the value of group cooperating and communicating, which enables students to clarify and structure their mathematical knowledge. Thirdly, solving a math trail task usually requires to collect and record data – a worthwhile skill that is rarely fostered in regular math class. Finally, the method also promotes learning about the immediate environment, so math trails offer interdisciplinary and multi-faceted learning opportunities (ibid.).

MathCityMap: Digitalization of the Math Trail Method

The MathCityMap project revives the 'old' idea of math trails and supplements it with mobile learning (Ludwig & Jesberg, 2015), which can be defined as usage of mobile devices like tablets or smartphones in an educational context (Park, 2011). The computing power and portability of these devices as well as the possibilities of wireless communication and digital tools offer great potential for both traditional teaching and outdoor learning (Sung, Chang, & Liu, 2016).

MathCityMap is a two-component system consisting of a web portal and a freely available app. It provides a simplified, digital way to create, share and to run math trails (Ludwig & Jablonski, 2020). The first component, the MathCityMap web portal (www.mathcitymap.eu), is a database for finding and creating mathematical tasks and to combine them to math trails. Both, tasks and trails can be shared with members of the worldwide MathCityMap community. This aspect of sharing can be named as one of the core features of the web portal. For every math trail, a math trail guide is available. It can be downloaded as PDF file or accessed via the MathCityMap app. This guide includes a map of the trail, all tasks and one picture of each task situation.

The MathCityMap app enables users to access math trails, which were created on the web portal. After downloading the trail guide to the mobile device, it is possible to run the math trail via app even without an internet connection. Only one smartphone or tablet with the installed app is necessary per group.

Features of the MathCityMap App

According to Ludwig and Jesberg (2015) mobile-supported math trails offer several advantages in comparison to 'classic' math trails: Firstly, the MathCityMap app eases the task localization by navigating students to the tasks via GPS. Secondly, the app provides hints to support students' problem-solving progress. Thirdly, students receive a systemic feedback on their calculated solution (ibid.). Adding the value of pedagogical gamification, we will describe the listed benefits.

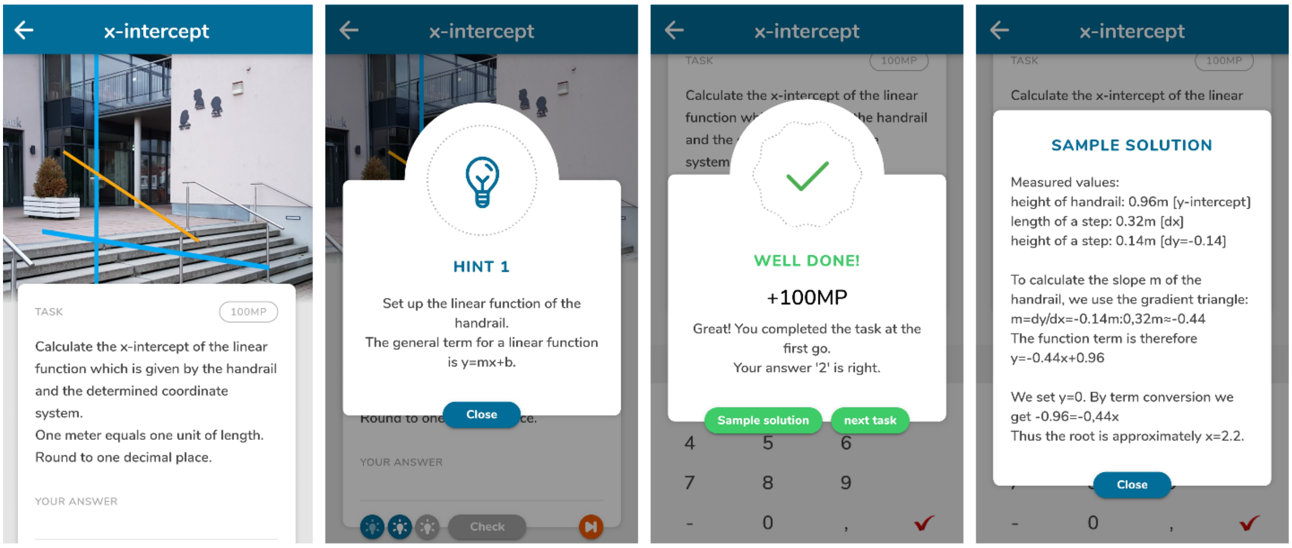

Figure 1: The Task "x-intercept" in the MathCityMap App: Task Formulation, a Hint, Validation of the Solution and Sample Solution (from left to right).

Stepped hints: For each task, the app displays up to three stepped hints to support students' problem-solving progress (Fig 1, center-left). While students can call up those hints independently, they can determine the difficulty of the task. This enables learners to adapt task to their individual performance and motivation level. Consequently, the hints allow both higher and lower performing students to work on the given tasks (Franke-Braun, Schmidt-Weigand, Stäudel, & Wodzinski, 2008).

Feedback: According to Reinhold (2018), a major potential of digital tools is the ability to give students immediate feedback on their work progress. In addition, through their feedback, digital tools should enable the recognition of possible errors and misconceptions in the solution process (ibid.) On one hand, the MathCityMap app gives students an immediate feedback regarding the numerical correctness of the solution entered (Fig 1, center-right). On the other hand, a look at the sample solution, which can be viewed after solving a task or after at least six wrong answers, allows the independent detection of errors in the solution process (Fig 1, right). All in all, the MathCityMap App can validate the calculated solutions of the students and enables the learners to identify potential errors or misconceptions by analyzing the sample solution.

Gamification: The study of Lieberoth (2014) indicates a positive motivational impact of the gamification of activities. By solving MathCityMap tasks each group receive up to 100 points per task, depending on the quality of their solution. By using the optional function Leaderboard, the groups are listed in a local ranking. Those gamification elements lead to both an increase in the number of completed tasks per hour and a reduction of blind guessing (Gurjanow, Olivera, Zender, Santos, & Ludwig, 2019).

RESEARCH INTEREST: CREATION OF THEME-BASED MATH TRAILS

Although recent studies about MathCityMap math trails have shown positive motivational effects (Cahyono, 2018; Gurjanow et al., 2019) as well as positive learning effects (Zender, 2019), math trails have so far mainly been used for a methodical variation of math teaching: They often aim at a wider revision of already learned topics. Consequently, the math trail method is – until now – rarely used for practice lessons with an explicit connection to a topic of the math curriculum.

This lack could be caused by the low amount of so-called theme-based math trails. Even though it is possible to develop curriculum-related math trails for many mathematical topics (Cross, 1997; Zender, 2019), the creation of such trails has an inherit challenge: As the tasks of the 'classic' math trail arise from local conditions (and with respect to the curriculum), the possible task types of a theme-based trail are predetermined by the curriculum. Therefore, a suitable object for a predefined task has to be found at the chosen location. At the same time, those topic-related tasks should not only allow the learning of a specific topic, but still remain an authentic problem concerning a real existing object a given place. In conclusion, the difficulty of creating theme-based trails is to identify objects in one's environment that raise realistic questions related to a particular curriculum topic. Within this paper, we aim to show possibilities for the curriculum-based use of the math trail method.

DESIGNING A THEME-BASED TRAIL

By taking up the concept of generic tasks, we identify design principles for the creation of theme-based trails linking the math trail method and curriculum-based topics in arithmetic, algebra, analysis and stochastic. By following these principles, teachers should be enabled to create their own theme-based trails on current classroom topic.

Generic Tasks and Theme-based Trails

A generic task is a mathematical task which can be applied to frequently occurring objects, e.g. the slope of a ramp or a handrail. These objects offer the possibility to easily transfer existing tasks to other locations (Ludwig & Jablonski, 2020). Within the Erasmus+ project MoMaTrE (Mobile Math Trails in Europe), a catalogue of generic tasks was created and translated in six languages, including English, French and German. Those tasks can be considered as best-practice examples for MathCityMap tasks with a specific reference to the mathematics curriculum.

A theme-based trail is a collection of generics tasks of one common topic, e.g. fractions in arithmetic, percentage calculation in algebra and linear functions in analysis as well as combinatorics in stochastic theory. Within a theme-based trail, a specific curriculum topic is addressed and can therefore be directly connected to regular math class. Zender (2019) already followed this approach during his study about stereometry for ninth graders by examining math schoolbooks for recurring topic-related tasks.

Design Principles for Theme-based Trails

The development of theme-based math trails requires design principles which are presented in the following. First, a detailed look into the school curriculum is necessary to identify the sub-areas of the current mathematics topic, e.g. the concept of slope as part of linear functions. Based on these concepts, a suitable method is to examine math textbooks for common task types. This ensures that all major concept-related task types used in the school are covered by the theme-based trail. After studying the curriculum and identifying the characteristic tasks, the required data and results have to be defined for all these task types: Which data is given in the task formulation? How is the result calculated? What is the expected solution process?

Finally, the author can search for suitable objects outdoors. An object must fulfil several conditions in order to be usable. First, it must be suitable to collect the required data, e.g. by measuring and counting. It is important to recognizes which data are directly measurable for the students and which data must be obtained by calculations. Furthermore, the task objects must be publicly available and clearly identifiable. Otherwise it is possible that the students will neither find nor reach the object during the lesson. In some cases, reference sizes are helpful or even necessary, e.g. auxiliary lines in the task picture, or reference objects as a lantern if the intercept theorem is used to calculate the height of a building.

Regarding to Zender (2019), we recommend the creation of filling tasks (non-topic-related tasks), in order to offer students a variety during the theme-based math trail. If students realize that they work on several similar tasks in a row, e.g. measuring the diameter of a circle for calculating the circumference, they first lose their motivation and then remain unfocused. Consequently, Zender (2019) suggests the creation of filling tasks to interrupt the sequence of constantly recurring tasks so that students do not blindly work through algorithms. To ensure that the topic-based focus is still maintained, every third or fourth task should be designed as a filling task.

In summary, Table 1 presents the developed design principles for theme-based trails:

Step	Object
1	Analyze the curriculum to identify the sub-area of the chosen topic.
2	Examine textbooks searching for common task types.
3	Define required data and results.
4	Search for suitable objects outdoors.
5	Create theme-based tasks and filling tasks in a 3:1 or 4:1 ratio.

Table 1: Design Principles for Theme-based Math Trails.

An Example: Theme-based Trail on Linear Functions

In order to illustrate the described design principles, we introduce the concept of linear function as a possible topic for a theme-based math trail in eighth grade. By examining math textbooks, we have identified five characteristic tasks and defined their required data (Tab. 2). Since all these task types can be applied on frequently occurring objects, they can be considered as generic tasks for linear functions. Therefore, a theme-based trail on linear functions can be easily created almost at any place by using these five generic tasks. In the following, we present exemplary tasks for all five identified sub-areas (Fig. 2) for the object "handrail".

Sub-Area	Object	Task Type	Required Data
Proportional Relationship.	e.g. Price list.	Calculate the costs of z pieces, e.g. balls of ice-cream.	Change in x, y.
Slope.	e.g. Ramp.	Calculate the slope of the ramp. Give the result in percentage.	Change in x, y.
Slope-intercept form.	e.g. Slide.	Define the linear function given by the slide.	Change in x, y & y-intercept b.
x-intercept.	e.g. Handrail.	Calculate the root of the linear function given by the handrail.	Change in x, y & y-intercept b.
Point of intersection.	e.g. Gable roof.	Find the point of intersection of two lines given by the gable roof.	Equations of two lines.

Table 2: Generic Tasks for the Concept of Linear Functions.

Focusing on the task types "slope" and "x-intercept", we describe students' solution process and possible hints given by the MathCityMap app.

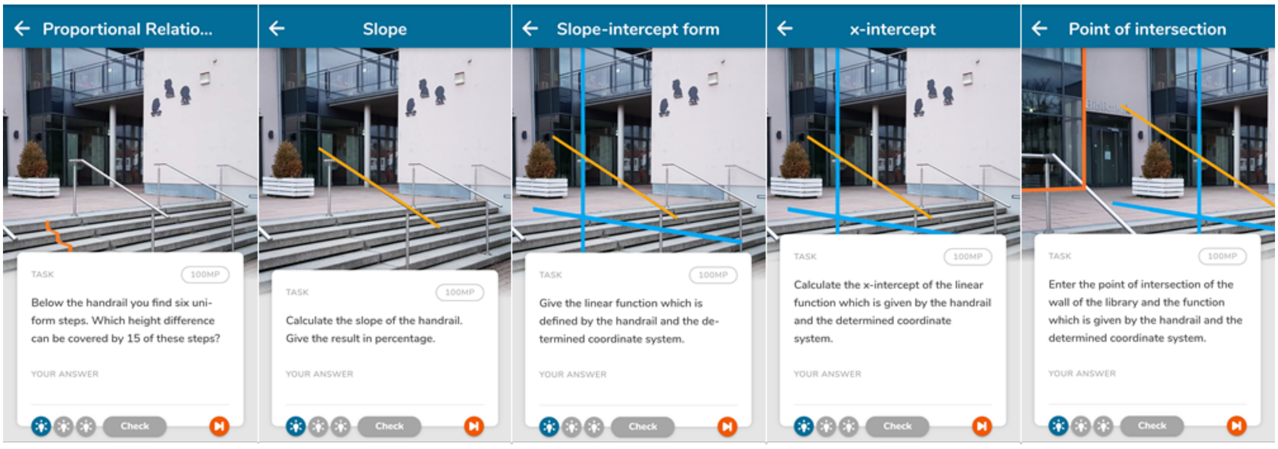

Figure 2: Selection of Tasks for Linear Functions within the MathCityMap App.

Slope (Fig. 2, center-left): To identify the slope of the handrail in percent, students have to measure the its length x and its maximum height difference y. Subsequently, they have to divide y by x and convert the fraction into percentage. This process could be guided by stepped hints: (i) Use a gradient triangle. Hint (ii) could include a picture of a gradient triangle, whereas hint (iii) aims at the percentage notion: The result should be given in percentage. For example, m = 0,21 equals 21 percent.

x-intercept (Fig. 2, center-right): In order to identify the x-intercept, the handrail function has first to be defined by calculating the slope m and measuring the y-intercept b. To structure students' solution process, the general term of the linear function and the required data could be given in the first two hints. Hint (iii) could clarify the further proceeding: Students have to equate the handrail function with zero and thereby identify its x-intercept.

CONCLUSION

The math trail method enables learners and teachers to explore their environment in a mathematical way. The usage of the MathCityMap system for creating and working on theme-based math trails adds the benefits of mobile learning to the 'classic' math trail idea: The app guides learners through a math trail, providing GPS, the task formulation, hints, feedback and a sample solution. Teachers can create their own math trails within the MCM web portal or use public available math trails.

Considering the positive effects on motivation (Cahyono, 2018; Gurjanow et al., 2019) as well as learning growth (Zender, 2019), the math trail method is underrepresented in regular school lessons. So far, trails are mainly used for a broader repetition of already learned topics, but not for the targeted work on current math content. We explain this by the lack of curriculum-related math trails, which – so far – prevents a more frequent embedding of the math trail method in the regular teaching units. To address this issue, we have introduced the idea of a theme-based math trail that covers the content of a regular math lesson through curriculum-related tasks. For this purpose, we have developed design principles to support teachers in creating theme-based trails. These include the analysis of specific mathematical sub-areas in textbooks and the identification of generic tasks that can be applied on common objects in one's own environment. In this way, we offer a possibility to make the math trail method usable for many, if not all, contents of the math curriculum. Enabling teachers to create their own theme-based trails fulfils an important requirement for the long-term integration of the math trail method into regular math lessons.

REFERENCES

Blane, D. C., & Doug, C. (1984). A mathematics trail around the city of Melbourne. Monash: Monash Mathematics Education Centre, Monash University.

Cahyono, A. N. (2018). Learning Mathematics in a Mobile App-Supported Math Trail Environment. Cham: Springer.

Cross, R. (1997). Developing mathematics trails. Mathematics Teaching, pp. 38-39.

Franke-Braun, G., Schmidt-Weigand, F., Stäudel, L., & Wodzinski, R. (2008). Aufgaben mit gestuften Lernhilfen. In R. Messer, & W. Blum, Lernumgebungen auf dem Prüfstand. Kassel: Kassel University Press, pp. 27-42

Gurjanow, I., Olivera, M., Zender, J., Santos, P., & Ludwig, M. (2019). Mathematics Trails: Shallow and Deep Gamification. International Journal of Serious Games, 6(3), pp. 65-79.

Lieberoth, A. (2015). Shallow Gamification - Testing Psychological Effects of Framing an Activity as a Game. Games and Culture, (10), pp. 229-248.

Ludwig, M., & Jablonski, S. (2020). MathCityMap - Mit mobilen Mathtrails Mathe draußen entdecken. MNU Journal, (1/2020), pp. 29-36.

Ludwig, M., & Jesberg, J. (2015). Using Mobile Technology To Provide Outdoor Modelling Tasks - The MathCityMap-Project. Procedia - Social and Behavioral Sciences, 191, pp. 2776-2781.

Park, Y. (2011). A Pedagogical Framework for Mobile Learning: Categorizing Educational Applications of Mobile Technologies into Four Types. International Review of Research in Open and Distance Learning, 12(2), pp. 78-102.

Shoaf, M. M., Pollak, H., & Schneider, J. (2004). Math Trails. The Consortium for Mathematics and Its Applications (COMAP).

Sung, Y.-T., Chang, K.-E., & Liu, T.-C. (2016). The effects of integrating mobile devices with teaching and learning on students' learning performance: A meta-analysis and research synthesis. Computers & Education, (94), pp. 252-275.

Zender, J. (2019). Mathtrails in der Sekundarstufe I. Der Einsatz von MathCityMap bei Zylinderproblemen in der neunten Klasse. Münster: WTM-Verlag.

PrimarWebQuest for content and language integrated learning classes

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Using PrimarWebQuests can motivate pupils to discuss mathematical terms during different learning sequences, which can support pupils' mathematical language abilities, particularly in bilingual classes. In a research project concerning the use of Information and Communication Technology (ICT) in bilingual settings, pupils were observed while working with bilingual PrimarWebQuests. In the following paper, the adaptation of WebQuests for primary education will first be described. Following this, the meaning of Content and Language Integrated Learning (CLIL) will be explained, as well as how the combination of CLIL and ICT fits in this special context. This combination is achieved in the approach of bilingual PrimarWebQuests. The framework of an ongoing study and first results will also be described.

Keywords: WebQuest, bilingual education, online resources, primary education, symmetry

FROM WEBQUESTS TO PRIMARWEBQUESTS

The method WebQuest, invented by Dodge and March in 1995, is an inquiry-oriented and web-based learning approach (see webquest.org 02.02.2020). WebQuests are offered on the Internet. However, pupils can use both, online and offline sources. The sources are chosen by the teacher in advance, so the learners do not get 'lost in cyberspace'. March describes a WebQuest with the following important aspects:

> A real WebQuest is a scaffolded learning structure that uses links to essential resources on the World Wide Web and an authentic task to motivate students' investigation of an open-ended question, development of individual expertise and participation in a group process that transforms newly acquired information into a more sophisticated understanding. The best WebQuests inspire students to see richer thematic relationships, to contribute to the real world of learning, and to reflect on their own metacognitive processes. (March, 2004, p. 42)

The method WebQuest was invented for dealing with internet resources in adult education as a challenging pattern of teaching. WebQuests are based on a constructivist theory of cognition in which knowledge can only be acquired by action. WebQuests should give a structure for using the internet sources in an efficient and target-oriented way. This method focuses on the use of information instead of searching for it (Moser, 2008; Schreiber & Kromm, 2020). In regard to mathematics education, Bescherer (2007) emphasizes that implementing WebQuests can foster mathematical communication and argumentation. Furthermore, she maintains a possible realization of inquiry-learning by using WebQuests.

Schreiber (2007) trialled WebQuests in primary schools and observed some difficulties for pupils. He adapted the method for primary school children and called it PrimarWebQuest. A mathematical PrimarWebQuest contains digital and analogue sources in order to deal with mathematical concepts and issues. It is important while working with internet resources that one considers the host of information and its reliability. The structure of a PrimarWebQuest can set a focus on important aspects and reduce the complexity of the available information. Therefore, the sources linked to a mathematical issue still have to be chosen by the teacher in advance. With those sources, the pupils research their topic in small groups and finally present their results. As the pupils should be enabled to self-evaluate their learning process and the learning product, the requirements of a PrimarWebQuest must be made transparent for the pupils from the beginning of the working process (Schreiber & Kromm, 2020; see also Baschek in preparation).

PRIMARWEBQUESTS IN BILINGUAL CLASSES

The structure of bilingual PrimarWebQuests is similar to those which are monolingual. However, in bilingual PrimarWebQuests (in this case German/ French), pupils can read each instruction in both languages provided in two different language columns (see Figure 1).

Figure 1: Part of an Introduction of a Bilingual PrimarWebQuest

Therefore, they can choose the working language independently. The sources are also offered in both languages, so every group, regardless of which working language they choose, has the possibility to use both languages for research. As the sources are mostly realistic websites made by native speakers of the particular language areas, you can find different information about the topic by using both languages. Additionally, there are methodical advantages of group work. The pupils can interact in both languages, which makes a language as well as a content exchange possible, especially by using information from both countries.

There are six sequences involved in using a PrimarWebQuest in class (see Schreiber & Kromm, 2020, pp. 70-73). In the first sequence, (1) *introduction,* every page is explained by the teacher and the pupils are allowed time for looking at the websites. The following sequence is titled (2) *working with sources*. The pupils work with the information texts and different types of tasks, which motivates them to exchange their different ideas and understanding of the topic in their small groups. For discovering common conflicts or difficulties, the teacher encourages the pupils to reflect upon their group work in a sequence titled (3) *balance drawing*. This sequence allows for the emphasis of the use of academic terms and everyday language, for example, by creating a common lexical storage on a poster. In this case, the pupils can negotiate the meaning of different terms and compare their conceptions of those terms. In the next sequence, (4) *presentation planning*, the pupils prepare their posters for a successful presentation. After the (5) *presenting* sequence, another (6) *reflecting* sequence follows during which the teacher reflects the whole working process with each individual group.

For beneficial use of a bilingual WebQuest, one must take three aspects into consideration. First of all, the students could need linguistic support, because a bilingual WebQuest can challenge them. This could be a common mapping or classifying of new terms. Second of all, particularly WebQuests with a bilingual design can motivate the students to switch between the languages as they practice using both languages while switching. Third of all, the students can get into a collaborative dialogue because of the open-ended nature of the task, which offers them the possibility to check their understanding of mathematical terms by discussing the new terms with their classmates. It is necessary to adapt a bilingual WebQuest to the linguistic knowledge of the students to ensure a successful learning experience for all (Baschek, in preparation).

CLIL: CONTENT AND LANGUAGE INTEGRATED LEARNING

In the European context, the term CLIL is used as a generic term for different bilingual learning and teaching models and can be implemented in different ways. It summarizes educational situations in which a subject or even just a selected topic is taught in an additional language than the school language for a fixed or an enduring time. This approach aims to connect content and language learning in an integrated way. It is seen as an effective method for learning a foreign language with the aid of authentic topics. "[…] [A]chieving this twofold aim calls for the development of a special approach to teaching in that non-language subject is not taught *in* a foreign language but *with* and *through* a foreign language" (Eurydice 2006, p. 7, emphasis in original).

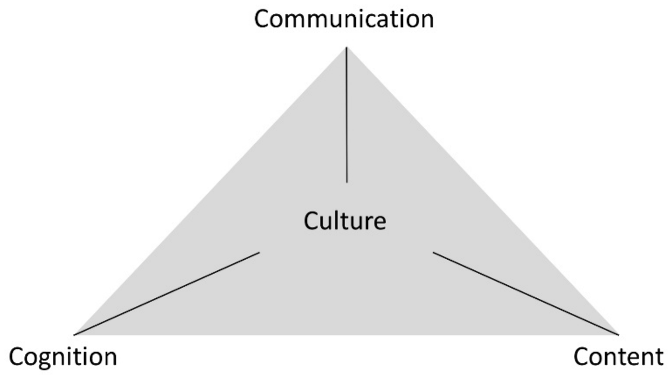

Figure 2: Curricular Framework for CLIL Classes According to Coyle (2006)

Coyle (2006) describes four key components (see Figure 2) for planning CLIL lessons in her "4Cs Framework". The aspect *Content* contains learning new knowledge and an understanding with regard to content for the pupils. The content of bilingual classes is geared toward curricular guidelines of the subject and its aim is a double and profound knowledge acquisition. *Communication* means extending the pupils' language knowledge and abilities, which qualify them for an academic interaction in class. The aspect *Cognition* means all cognitive abilities which can be established, such as metalinguistic knowledge or strategy learning. As for *Culture,* the pupils have to develop a reflexive attitude vis-à-vis their own and other cultures. It is important to think about multiple perspectives and to switch between them (Baschek, 2019b; Baschek in preparation). In this study, the framework was used for creating suitable PrimarWebQuests for CLIL classes. The implementation succeeded the best for the topic of point symmetry because one aspect of this topic is dealt with differently in French and German schools. That is why the pupils are offered the chance of an intercultural learning which was evident for them and which deepens Coyles component *Culture* even more than in the two other PrimarWebQuests.

PILOT STUDY

The aim of this pilot study is to describe the interaction between pupils, teacher and material to explore which opportunities of language and content learning are offered by the approach of PrimarWebQuest.

Different bilingual PrimarWebQuests with the topic of symmetry sections were tested in the fourth grade of a German primary school offering bilingual classes (German/ French) in multiple subjects. The school follows the German curriculum and uses German textbooks. However, mathematics is generally taught exclusively in German. Most of the pupils' mother tongue is German. As can be seen in Table 1, the class has been split into three different sections of symmetry (line symmetry, rotational symmetry and point symmetry) and each topic has been split once again into two groups.

Line symmetry	Rotational symmetry	Point symmetry
Group G	Group G	Group G
Group F	Group F	Group F

Table 1: Group Division in class

One group had the requirement to present the topic in German (Group G), the other one in French (Group F). They were offered to use bilingual PrimarWebQuests where the working language was chosen freely by the pupils. All groups were video and audio recorded. As the topic *point symmetry* was the only completely new topic for this learning group and appeared to include the most fruitful discussions because of the cultural differences, these two groups were chosen for analysis. Crucial sequences, such as discussions of one or two groups of which the negotiation of mathematical terms seemed to be in focus, were transcribed later [1]. In table 1, a short overview of the groups is provided. For interpreting the utterances, we used the interaction analysis which is based on the ethno-methodological conversation analysis and developed by Bauersfeld, Krummheuer and Voigt. It deals with processes of interaction that take place in school (Bauersfeld, Krummheuer & Voigt 1988).

FIRST RESULTS

Within the aim of this investigation to explore the possible learning opportunities regarding language and content, pupil interactions are described and analysed. The different sequences of PrimarWebQuest usage offer several possibilities for supporting the mathematical language skills of bilingually taught pupils. During the first working sequence, the pupils start the procedure of answering their task. While reading and interpreting the given sources, they must gather and check the suitable information details from the authentic sources. This is shown in the following originally German dialogue:

Pupil	**Translated Utterance**
Dana:	*(reads)* Point symmetric shapes. The red marked spots are the symmetry point of each shape. This remains the same all the time.
Finn:	Look, it is explained here.
Dana:	If you turn the N, yes turn it half. So, when the N is like this is and here that spot, then you do half a turn, so like that *(shows a turning gesture with her hands)*. Then it is a N again.
Evelyn:	Right.
Dana:	It always has to be the same if you turn it one time. It is just the same for that, for that and for that *(points at the shape examples)* and even for those here. If you do a half turn, then it looks the same. Then it looks like you’ve got it from the front.

Evelyn:	Right.

In this sequence, the pupils try to understand in a collective way which general and mathematical information the given text contains. The group work helps them to communicate successfully about the new information. As the pupils must describe their own thinking processes as well as their approaches during the group work, they learn to verbalize their understanding of the informative texts on their own. The open-ended task supports the subject-based discourse while preparing the poster presentation, because the pupils need to come to a mutual result at the end of this sequence. This result must be discussed by the pupils while comparing their understanding of the different information. The pupils could express assumptions or recognize connections (Baschek, 2019a).

For example, group F discussed about a text for their poster. They wanted to use the German internet information that the term *symmetry* is Greek and can also be described with *regularity*. The dialogue is originally in German and French terms are typed in SMALL CAPITALS:

Pupil	**Translated Utterance**
Anne:	SYMÉTRIE…
Fabienne:	You forgot CENTRALE.
Inés:	SYMÉTRIE CENTRALE, got it.
Anne:	Yes, write it down. No no, it has to be without CENTRALE.
Fabienne:	Why?
Anne:	Because eeh the, CENTRALE means point. But eeh regularity means symmetry. That could be the line symmetry, too. Symmetry, but not line. Do you understand? Neither axial, nor point, nor rotational.

While discussing, Anne argued why they must use the term *symmetry* instead of *point symmetry*. She did not want to write down the French term CENTRALE because she understood that *symmetry* is a generic term to *point symmetry* (Baschek, 2019a). In addition to her arguing abilities, she shows a different or deeper understanding of this term compared to her classmates. Due to the continuous change of both languages, it is possible that the pupils learn terms in both languages and beyond that, they could fix them in a cognitive bilingual way. During the poster presentation sequence, a proper use of mathematical terms and symbols is necessary, especially for reflecting their approaches and strategies.

Regarding the comparison of both groups, group G shows a more static use of academic terminology. As they worked quite activity-oriented, they looked for many examples in the class room. That group utilized everyday terminology and periphrases for describing their insights in a flexible way. Group F showed a more proper use of academic terminology and compared both languages in an intensive way. In fact, they spoke more German than French, but they thought of the French terms repeatedly when starting their work. From the foreign language learning perspective, it must be mentioned that group G didn't show a large increase in new productive French speaking skills, but they did deepen their decoding skills in French because of the French sources and French presentations of the other groups. It was possible to observe a subject-based discourse in both groups (Baschek, 2019b).

There were discussions between the two groups which contributed the negotiation of some terms. For group F, the French language was an obstacle as well as a motivator. In the beginning, they had difficulties getting into the topic and struggled with expressing their thoughts during the sequence. However, those difficulties were a stimulus for the pupils to work with both languages. They compared terms and discussed their meanings for understanding the topic. Those pupils expanded some strategies in foreign language learning and showed both a productive and receptive expansion of vocabulary. Group F was able to explain the idea of the point reflection in addition to mutual knowledge, such as checking the congruence of two figures or naming examples (Baschek, 2019b).

CONCLUSION

To conclude, the method PrimarWebQuest can support the use of mathematical language in bilingual classes (see also Baschek in preparation). The pupils felt safe because of the open choice of working language. Specifically, the preparation of the presentations encourages the pupils to think about their informative texts. The open-ended task allows a motivating individual focus during the working sequences. If the content of the sources is too complex for the pupils, a PrimarWebQuest can guide them purposefully in working on the task successfully. In retrospect, PrimarWebQuests result in the pupils dealing with different terms in both languages in a language-aware way. The unknown content of the sources and the authentic materials motivate the pupils to negotiate new terms with their groups. The pupils work with mathematical terminology and are able to compare their understanding of terms in multiple ways thanks to the group work. This cooperation and communication can support language learning as well as proper language use. During the presenting sequence, the pupils are able to communicate adequately with mathematics classes.

Seen from a technical perspective, the two groups were able to learn the mathematical content by using academic terms in both languages. This integrated method of learning follows the idea of CLIL classes. In addition to the language and mathematical learning of the pupils, they also worked on their media competences while using PrimarWebQuests with multiple internet sources. They were able to select the relevant information from internet sources and prepare them for a presentation. In this case, the presentation was an analogue poster presentation. In other cases, it can be a digital presentation using suitable software. The mode of presentation depends on the task in the PrimarWebQuest and previous knowledge of the pupils.

NOTES

1. You can find the bilingual PrimarWebQuest here: https://pwq-punktsymmetrie-symetriecentrale.weebly.com

REFERENCES

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H.-G. Steiner and A. Vermandel (Eds.), Foundations and methodology of the discipline mathematics education (pp.174-188). Antwerp: University of Antwerp.

Baschek, E. (2019a). Mit PrimarWebQuests Sprache fördern [Supporting language with PrimarWebQuests]. *Mathematik differenziert, 3,* 10–13.

Baschek, E. (2019b). Using ICT in Bilingual Mathematics Classes – an Example. In J. Novotná & H. Moraová (Eds.), *International Symposium Elementary Mathematics Teaching. Proceedings. Opportunities in Learning and Teaching Elementary Mathematics.* (pp. 455–457). Prague, Czech Republic: Charles University, Faculty of Education. Retrieved from: https://www.semt.cz/proceedings-19.pdf [02.02.2020]

Baschek, E. (in preparation). Content and language Integrated learning with primarwebquests – Using Information and Communication technology in Bilingual Primary Mathematics Classes. *ICME-proceedings*. China: Shanghai 2021

Bescherer, C. (2007). WebQuests und Mathematikdidaktik. WebQuests als Methode für projektorientierten Mathematikunterricht. [WebQuest and Mathematics Education. WebQuest as a Method for Inquiry-Oriented Mathematics Classes] Computer + Unterricht, 67, 18-19.

Coyle, D. (2006). Content and language integrated learning. Motivating learners and teachers. *Scottish Languages Review, 13,* 1–18.

Eurydice (Eurydice European Unit) (2006). *Content and language integrated learning (CLIL) at school in Europe*. Retrieved from http://www.indire.it/lucabas/lkmw_file/eurydice/CLIL_EN.pdf [02.02.2020]

March, T. (2004). The learning power of WebQuests. *Educational Leadership*. *Vol. 61(4)*. 42 – 47.

Moser, H. (2008). *Abenteuer Internet. Lernen mit WebQuests* [Adventuring internet. Learning with WebQuests]. Donauwörth, Germany: Auer.

Schreiber, Chr. (2007). WebQuests für die Grundschule: Prima(r)WebQuest [WebQuests for primary school: Prima(r)WebQuest]. lehrer-online. Retrieved from https://www.lehrer-online.de/artikel/fa/webquests-fuer-die-grundschule-primarwebquest/ [02.02.2020]

Schreiber, Chr. & Kromm, H. (2020). *Projektorientiertes Lernen mit dem Internet – PrimarWebQuest* [Inquiry-oriented learning with the internet – PrimarWebQuest]. Baltmannsweiler, Germany: Schneider.

Re-design of digital tasks: the role of automatic and expert scaffolding at university level

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In this study we present the re-design of a digital task for university students attending to a probability course. The re-design, directed toward the overcoming of specific critical issues highlighted in previous studies, is mainly aimed at providing students (in particular low achievers) with hints and feedback as tools of scaffolding and meta-scaffolding. Thanks to the analysis of a low achiever's interaction with the re-designed task, we investigated the limits of the automatic scaffolding and the key- role of expert's interventions in fostering students' overcoming of possible impasses.

Keywords: digital scaffolding, task design, university level, role of the expert

INTRODUCTION

The research presented in this paper is part of a wider study focused on the individualization of teaching-learning paths at university level (Alessio, Demeio & Telloni 2019, Cusi & Telloni 2019a, Cusi & Telloni 2019b). In particular, in Cusi & Telloni (2019a, 2019b) we presented two teaching experiments, focused on the design of online teaching-learning paths, developed with formative assessment purposes, involving groups of engineering students of the Polytechnic University of Ancona (Marche, Italy), attending to Calculus and Probability courses. Both studies highlighted students' inadequate awareness about their difficulties with mathematical topics and about their needs and a widespread lack of metacognitive control. As a result of these lacks, students are not able to activate appropriate strategies to overcome their difficulties while they work within digital environments. Starting from these results, here we propose a re-design of one of the digital tasks of the teaching-learning path presented in Cusi and Telloni (2019a). This re-design aims at creating a digital environment that could enable students to activate themselves at the metacognitive level, offering them feedback that support their use of hints to scaffold their work. Moreover, we propose the analysis of the interaction of a low achiever with the re-designed task to reflect on further difficulties that could arise and on the key-role that the expert plays in supporting students' overcoming of these difficulties.

THEORETICAL FRAMEWORK

In our design of individualised teaching-learning paths, we refer to Baldacci's (2006) definition of *individualization* as the act of differentiating the didactical paths in order to enable all the students to reach common objectives. This is particularly relevant at university level, where students need to overcome gaps of knowledge due to the heterogeneity of their background. A possible way of realizing individualization at this level is focusing on the use of digital environments, where a fundamental formative assessment process can be flexibly activated: *providing feedback* (Hattie & Timperley, 2007). Giving feedback could be also conceived as a possible means to realize *scaffolding*, that is the "act of teaching that (i) supports the immediate construction of knowledge by the learner; and (ii) provides the basis for the future independent learning of the individual" (Holton and Clarke, 2007, p.131). When scaffolding is realized within digital environments, the focus is on *computer-based scaffolding* (Belland, 2017), that is the "computer-based support that helps students engage in and gain skill at tasks that are beyond their unassisted abilities" (p.26).

Research in mathematics education has distinguished different scaffolding domains. Holton and Clarke (2007), for example, introduce two domains: (a) *conceptual scaffolding*, which relates to specific contents; and (b) *heuristic scaffolding*, which relates to the development of heuristics for learning or problem solving. Types of scaffolding could be also identified in relation to the agents that provide it (Holton and Clarke, 2007): *expert scaffolding* (provided by an expert), *reciprocal scaffolding* (provided by peers), and *self-scaffolding* (provided by an individual to himself). This last type of scaffolding plays a key-role in fostering the following *fading*, that is the appropriation of the scaffolding by the learner (Shvarts & Bakker, 2019).

Students' effective use of the provided scaffolding and subsequent development of awareness about the role of scaffolding requires that they activate themselves at the *metacognitive level* (Holton & Clarke, 2007). The fundamental role of metacognitive aspects is stressed also within digital environments, where a good balance between procedural and metacognitive-scaffolding is needed (Sharma & Hannafin, 2007).

Research has highlighted that, within digital environments, the support provided by facilitators (teachers, tutors or, more in general, human experts) in activating meta-scaffolding is particularly crucial (Pea, 2004). To analyze this role, we refer to Wood, Bruner and Ross' (1976) main scaffolding functions: *recruitment* (enlisting learner's interest and the adherence to the requirements of the task), *reduction in degrees of freedom* (simplifying the task by reducing the number of constituent acts required to reach the solution), *direction maintenance* (keeping learners in pursuit of a particular objective), *marking critical features* (accentuating relevant features or parts of the activity), *frustration control* (reducing learners' stress, without creating too much dependency on the tutor) and *demonstration* (modelling solutions to a task).

RESEARCH QUESTIONS AND RESEARCH METHOD

The analysis developed in Cusi & Telloni (2019a, 2019b) highlighted some critical issues that prevent students from fruitfully exploiting the hints provided by digital environments to scaffold their work. This study is aimed at facing two main research questions: (1) What criteria can guide a re-design of digital tasks to overcome these critical issues and foster an effective scaffolding of students' work? (2) When a digital task is re-designed according to these criteria, what factors inhibit the overcoming of the critical issues that have been highlighted? In case of students' impasses due to these inhibiting factors, what kind of support could be provided by the expert scaffolding to effectively integrate the digital automatic scaffolding?

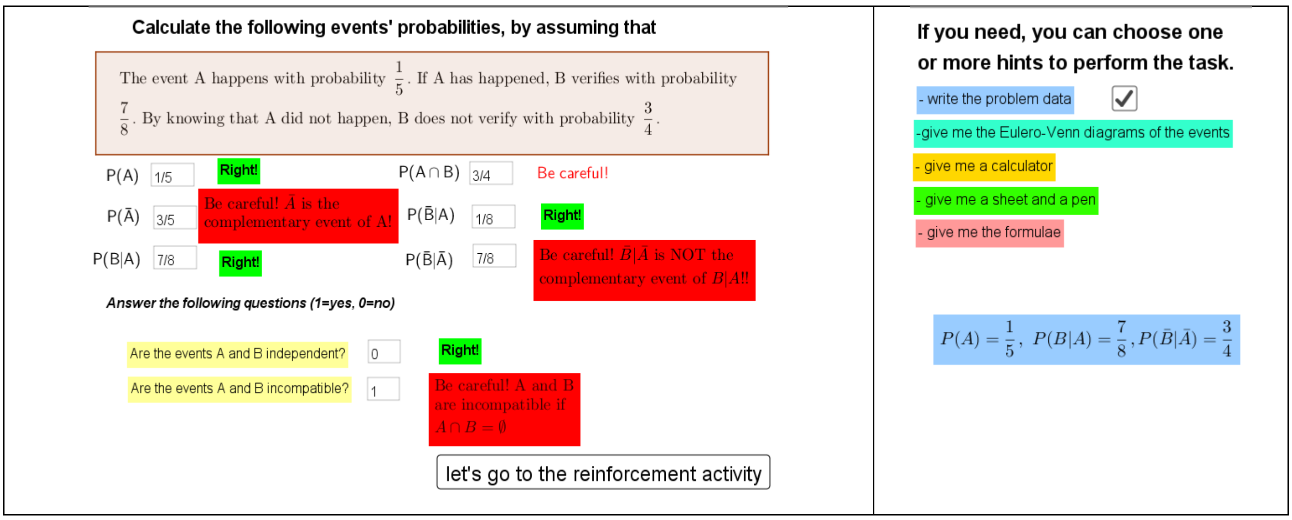
The reflections developed to answer to question 1 enabled us to identify three main criteria, that guided our re-design of one of the digital tasks belonging to the teaching-learning path presented in Cusi and Telloni (2019a). These criteria and their use in re-designing the task will be presented in the next section.

To investigate the aspects connected to question 2, we developed a teaching experiment, with a group of ten first year master-degree engineering students, enrolled on voluntary basis. The students, who were attending to a mini course focused on probability (in the period September-December 2019), in the middle of the course (November 2019) were asked to work on the re-designed version of the digital task, within a laboratorial activity. We collected the video-recordings of students' screens while facing the task. To develop an in-depth analysis of students' use of feedback and hints provided within the digital environment to scaffold their work, we asked them to think at loud while facing the tasks, and audio-recorded their speeches. A tutor (the teacher of the course, one of the authors) was in the computer lab to provide support to students in case of problems.

In this paper, we will focus on the analysis of the interaction of a low achieving student, Maria, with the re-designed task, through the tutor's support. Maria was selected because she displayed, from the very beginning of her work on the task, her awareness about her lack of knowledge. This analysis was developed in two subsequent phases. In the first phase, we identified key-moments, during which the student faced difficulties in her work on the task. In the second phase, we developed an analysis of the interactions between Maria, the digital environment and the tutor to identify: (a) impasses in the student's use of hints to scaffold the development of the resolution process; (b) factors that create impasses; (c) specific roles of the tutor in fostering the overcoming of these impasses.

ANALYSIS OF THE TASK RE-DESIGN

In this section we focus on the re-design of the digital task presented in Table 1.



A text in verbal language introducing some events and their probabilities (their values are random) appears on the screen. The students are required to fill six input fields by inserting the probabilities of some events: three of them are those given in the text and the remaining can be obtained applying probability rules (complementary events, conditional probability). When the six input fields are correctly filled, other two questions (yellow boxes) appear on the screen, concerning the independency and the incompatibility of events. For each answer given by the student, the program reacts by a feedback (red and green boxes).	During the interaction with the task, students can ask for different kinds of hints. We mention, in particular: a summary of the data given in the text (*data hint*), Eulero-Venn diagrams of elementary and compound events (*E-V hint*), a list of useful formulae (*formula hint*).

Table 1: The first version of the task [1]

Thanks to our previous studies (Cusi & Telloni, 2019a, 2019b), we identified three main critical issues and their effects in negatively influencing students' work on digital tasks. The identification of these critical issues suggested us three main criteria that could guide task re-design. These aspects are summarised in table 2.

Critical issues	Negative effects	Criteria for the re-design
Students' lack of awareness or partial awareness about their difficulties/weaknesses and their learning needs.	Students are not able to identify useful hints to scaffold their work on the tasks.	(1) Add explicit written feedback in which possible hints to be used are highlighted.
Students' lack of metacognitive control in monitoring their problem-solving processes.	Students are not able to exploit the provided hints to effectively scaffold their work on the tasks.	(2) Re-structure the tasks in order to guide students in identifying the fundamental steps to solve the tasks. Provide students with explicit feedback by a tutor to enable them to become aware about possible ways of using hints.
Insufficient flexibility in the use and interpretation of different representations (graphical, symbolic, verbal…)	Students are blocked in the interpretation of hints and in the use of hints to face the tasks.	(3) Provide students with multiple representations and explicit feedback to stimulate a flexible use of these representations (written feedback or feedback given by a tutor)

Table 2: Critical issues, their negative effects and criteria for the task re-design

The identified criteria led our re-design of the task, aimed at scaffolding students' work by guiding them into 5 sub-steps that characterize an effective resolution process. The structure of the re-designed task is summarized in Figure 1.

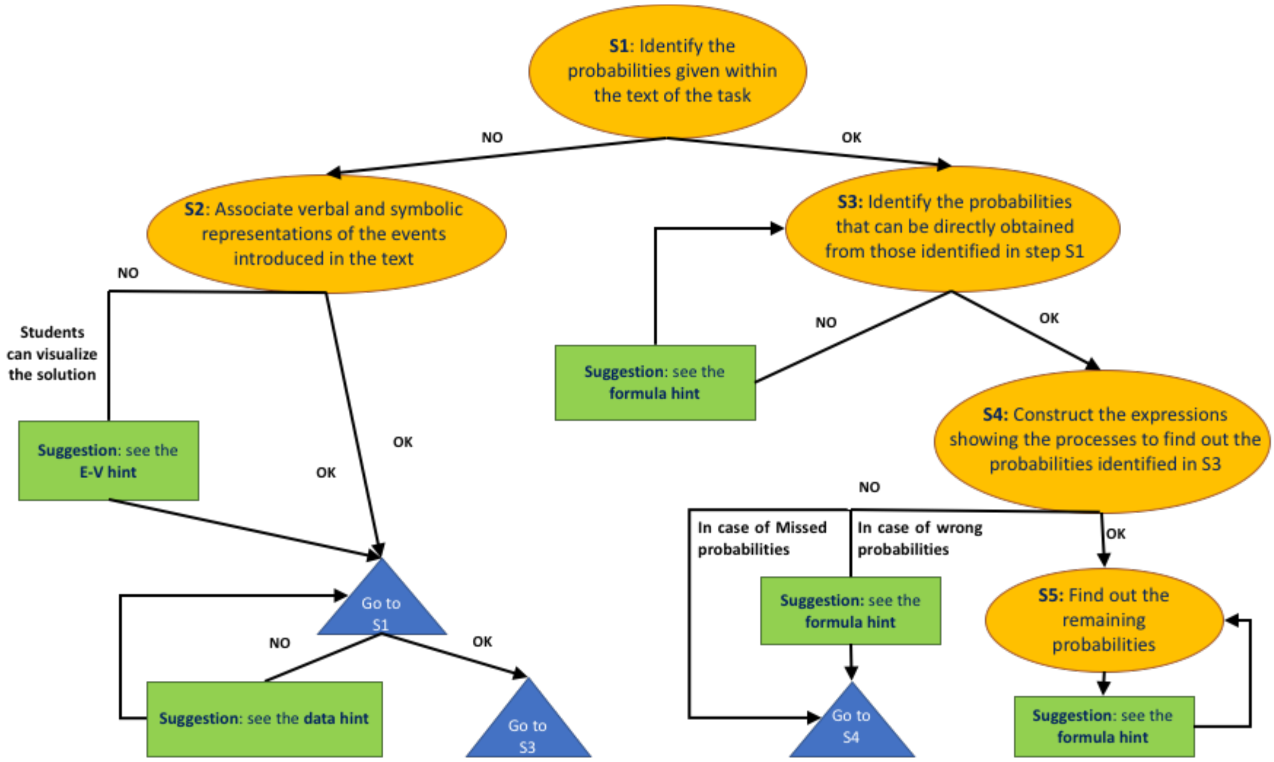


Figure 1: The structure of the re-designed task

The orange rounds in Fig. 1 represent the 5 steps that characterize a possible effective approach to the resolution of the problem. We referred to these steps to organize the scaffolding of students' work on the task (criterion 2), since each step corresponds to a sub-task, whose completion enables the students to progress in their work. In this way, the scaffolding functions *reduction of the degree of freedom* and *marking critical features* (Wood et al., 1976) can be activated.

The individualization of students' paths when facing the task is highlighted by the arrows and the blue triangles, which show the steps to which students are addressed according to their answers, and by the green boxes, which contain suggestions automatically given to students if they fail in specific steps. These suggestions are new elements introduced in the re-design process to scaffold students' work at a meta-level, guiding their choice of the hints, conceived as possible effective tools for conceptual scaffolding (criterion 1). Another element of individualization is the fact that students are free to accept or not the suggestions of asking for specific hints.

The design of the questions in S1 (involving interpretation of verbal texts) and S2 (requiring conversions from verbal to symbolic representations) and our choice of the hints on which scaffolding is focused are aimed at fostering students' flexible use of different representations (criterion 3). This flexible use is also required in S4, where students are asked to represent, through suitable numerical expressions, the processes that lead to determining the required probabilities starting from the known ones.

ANALYSIS OF A LOW ACHIEVER'S INTERACTION WITH THE RE-DESIGNED TASK

In this section we analyze Maria's interaction with the re-designed task. We identified three key-moments that highlight Maria's difficulties in effectively referring to the provided hints to scaffold her work. Because of space limitations, we summarize the main results of our analysis in table 3, in which each key-moment is analyzed in terms of: impasse that is shown, factors creating the impasse, roles played by the tutor to foster the overcoming of the impasse.

Key-moment 1: impasse due to a lack of focus in developing a strategy	
Position of the key-moment within Maria's path	After having failed S1, faced S2 and visualized S2's solution, Maria is facing again S1.
Kind of impasse	Maria is not able to use the data hint to correctly complete S1.
Factors creating the impasse	Maria is confused because her main focus is on her mistakes in S2 in converting from verbal to symbolic representations.
Roles played by the tutor to foster the overcoming of the impasse	The tutor reformulates the request in S1, making Maria observe that the data hint directly shows how to complete this step. In this way, the tutor activates: a *direction maintenance* scaffolding function, making Maria reflect on the role of the data hint; and a *recruitment* scaffolding function, explicitly re-focusing Maria's attention on the request in S1.
Key-moment 2: impasse due to an inadequate strategic use of provided hints	
Position of the key-moment within Maria's path	While Maria is facing S3, she follows the suggestion of using formula hint.
Kind of impasse	Maria is blocked: she is not able to use the formula hint to identify the probabilities that can be directly obtained.
Factors creating the impasse	Maria is not able to activate a strategic approach in identifying formulas that can be used to determine other probabilities.
Roles played by the tutor to foster the overcoming of the impasse	The tutor suggests Maria to focus on one of the probabilities that have to be found, then on one of the data and asks Maria to identify a formula that relates them. In this way, Maria is guided in setting a sub-goal with respect to the request in S3, and the *reduction in degrees of freedom* scaffolding function is activated. The tutor's intervention aims also to activate the *marking critical features* scaffolding function, enabling Maria to focus on a specific formula towards a specific goal.
Key-moment 3: impasse due to difficulties in handling multiple representations	

Position of the key-moment within Maria's path	Maria is working on S4, when she explicitly asks for the tutor's support.
Kind of impasse	Maria does not know what strategic approach she can activate to find out one of the required probabilities.
Factors creating the impasse	Maria refers to the right formula, but she is not able to find out the inverse formula to determine the required probability, because of her lacks in manipulation and interpretation of algebraic formulae.
Roles played by the tutor to foster the overcoming of the impasse	The tutor models an effective strategic approach, activating the *demonstration* scaffolding function to guide Maria in obtaining the right inverse formula, and in understanding how the known probabilities should be substituted in the formula itself. This scaffolding also involves metacognitive aspects, since it regards the use of the hints to perform this step of the task.

Table 3: Key-moments in Maria's interaction with the re-designed task

FINAL REMARKS

The re-design process presented in this paper has been developed considering three criteria, which proved to be effective, especially in the case of average and high achievers, in stimulating students at a metacognitive level, fostering a scaffolding focused on solution processes that require flexibility in making reference to conceptual knowledge and in effectively using it.

The analysis of Maria's interaction with the re-designed task enabled us to highlight that the activated scaffolding is sometimes not effective, especially in the case of low achievers, who, because of their lack in metacognitive control, are often not able in correctly interpreting the given feedback and in autonomously using the hints provided within digital environments. Through our analysis, we showed that the role of the expert (the tutor) becomes crucial in supporting low-achievers in overcoming moments of impasse connected to specific factors, such as lack of focus in developing a strategy, inadequate strategic use of provided tools, difficulties in interpreting mathematical representations and in using multiple representations.

Through our analysis, we identified specific roles that could characterize, also in other contexts, the expert's approach in case there is the need of integrating the automatic scaffolding provided by the digital environment: (a) making the students reflect on the role and use of specific hints; (b) re-focusing students' attention on the task's requirements; (c) focusing students' attention on conceptual aspects; (d) setting sub-goals that could guide students' resolution process; (e) highlighting connections between specific goals and tools to achieve them; (f) modeling effective strategic approaches, focusing on syntactic and semantic control of these processes. In table 3 we also show how these roles can be connected to an effective activation of specific scaffolding functions (Wood et al, 1976).

As an ongoing development of this research, we are focusing on a further tool to scaffold students' learning and to activate them at metacognitive level: the use of the diagram in Fig.1 to support students' a-posteriori reconstruction of their own learning path and consequent reflections on their use of digital tools and hints.

NOTES

1. The task, presented in Cusi&Telloni (2019a), has been designed using the software GeoGebra.

REFERENCES

Alessio, F., Demeio, L. & Telloni, A.I. (2019). A formative path in tertiary education through GeoGebra supporting the students' learning assessment and awareness, *International Journal for Technology in Mathematics Education*, *26*(4), 191-203.

Baldacci, M. (2006). *Personalizzazione o individualizzazione?* Roma: Erickson.

Belland, B. (2017). *Instructional Scaffolding in STEM Education*, Cham: Springer.

Cusi, A. & Telloni, A.I., (2019a). The role of formative assessment in fostering individualized teaching at university level. In U.T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis (Eds.). *Proceedings of CERME 11* (pp. 4129-4126) Utrecht: Freudenthal Group & Institute, Utrecht University and ERME.

Cusi, A. & Telloni, A.I., (2019b). Students' use of digital scaffolding at university level: emergence of utilization schemes. In B. Barzel et al. (Eds.), *Proceedings of ICTMT 14* (pp. 271-278). Duisburg-Essen Publications online.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*(1), 81-112.

Holton, D., & Clarke, D. (2007). Scaffolding and metacognition. *International Journal of Mathematics Education in Science and Technology*, 37(2), 127-143.

Pea, R. (2004). The social and technological dimensions of scaffolding and related theoretical concepts for learning, education, and human activity. *Journal of the Learning Sciences*, *13*, 423-451.

Sharma, P. & Hannafin, M.J. (2007). Scaffolding in technology-enhanced learning environments, *Interactive Learning Environments*, *15*(1), 27-46.

Shvarts, A. & Bakker, B. (2019). The early history of the scaffolding metaphor: Bernstein, Luria, Vygotsky, and before, *Mind, Culture and Activity*, *26*(1), 4-23.

Wood, D., Bruner, J. & Ross, G. (1976). The role of tutoring in problem solving. *The Journal of Child Psychology and Psychiatry*, *17*, 89-100.

A qualitative-experimental approach to functional thinking with a focus on covariation

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Students encounter functional relationships in almost every grade. Nonetheless, they often experience difficulties when dealing with functions and show misconceptions. The prevalent numerical approaches to the topic in school practice lead to a pointwise view of functions which contributes to these problems. Experimental approaches have proven to be beneficial for functional thinking, with simulations inducing greater learning gains than experiments with real material. A closer look at these two methods reveals that each of them promotes a different aspect of functional thinking. The study presented here deals with the question of how both methods can be combined beneficially and proposes two different approaches.

Keywords: functional thinking, experiments, simulations, covariation.

FOSTERING FUNCTIONAL THINKING

According to Vollrath (1989), functional thinking is based on three main aspects: the correspondence of an element of the definition set to exactly one element of the set of values; the covariation of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an object. This differentiation is in line with the developmental perspective on students' conceptualization of functions derived by Breidenbach et al. (1992) using the Action-Process-Object-Scheme (APOS) theory. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky and Wilson 2013).

Learning environments with experimentation activities have proven to be beneficial for functional thinking (Lichti and Roth 2018, Ganter 2013) and motivation (Ganter 2013). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view.

Furthermore, experiment activities enable constructivist settings, that lead to higher learning gains when using digital technologies (Drijvers et al. 2016) and provide embodied experiences, contributing as cognitive resources (Drijvers 2020).

Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and post-process (reflect results) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (ibid.), but a closer look reveals disparities: while hands-on material promotes the correspondence aspect and the association to the real situation, simulations foster covariational thinking, the interpretative usage of graphs and lead to higher overall gains in functional thinking (Lichti 2019).

The instrumental approach (Rabardel 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis (Artigue 2002) - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers 2020). In addition, embodied activities in a task seem to contribute to the instrumental genesis (ibid.). From the viewpoint of instrumental genesis, the results of Lichti (2019) can be interpreted as follows: when using simulations, schemes that develop are concerned with variation and transition, while measurement procedures of the hands-on material induce schemes that concentrate on values and conditions (ibid.). The students working with hands-on material associate their argumentation more often with the material, while the rationale of students using simulations frequently relates to the graph. Again, the instrumental genesis can explain these disparities: the hands-on material stimulates basic modelling schemes, relating the situation to mathematical description. Simulations already contain models of a situation and when used as multi-representational systems (Balacheff and Kaput 1997) illustrate connections between model and mathematical representation (e.g. graph and table) that evoke schemes for these representations and their transfer.

The study presented here attempts to make use of all these beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

SETTING 1: EXPERIMENTS WITH HANDS-ON MATERIAL AND SIMULATIONS

The learning environment is set in a story of two friends preparing to build a treehouse. The student activities are structured in five contexts (see below for details), each one laid out like a scientific experimentation process with preparation, experimentation and post-processing phase. Starting off with hands-on material to activate modelling schemes and enable embodied experience, students are asked to make assumptions about a pattern or relationship and on that basis, estimate values. During experimentation phase they take a series of measurements and data is recorded in a table within a simulation (GeoGebra). The simulation is designed in accordance to the hands-on material and provides the opportunity to create a graph concurrent with the context animation and to display the measurements of the hands-on material (and a corresponding trendline). This gives students the opportunity for systematic variation and parallel observation of the altering quantities, to induce schemes with a dynamic view and covariational thinking. Above, it facilitates the time consuming but little challenging representational switch from table to graph (Bossé et al. 2011). In the post-processing phase the students verify their measurements and analyse the graph (interpreting and interpolating). Subsequently they get back to the real material to check their estimations from preparation phase. Finally, they elaborate on the answer to the overarching task (calculate the amount of material needed to build the treehouse) based on the insights from experimentation activities, bringing together the modelling and representational schemes developed.

SETTING 2: ALTERNATIVE COMBINATION OF ARTEFACTS WITH A FOCUS ON COVARIATION

In setting 1 proposed above the measurement plays a dominant role, which sets a focus on the individual values of quantities and on single states of the relationship. This leads to a pointwise view of functions (Monk 1992), promotes the action concept and concentrates on the correspondence aspect (see above). In accordance with Breidenbach et al. (1992) and Dubinsky and Wilson (2013) it would be desirable to shift this focus to a process concept and to covariation, especially since possible sources of student' difficulties with functional relationships are seen in the dominance of numerical settings in school (Goldenberg et al. 1992). Together with the close relation of covariation to the difficult concept of variables (Leinhardt et al. 1990), this led to the call for a qualitative approach to functions (Thompson 1994; Falcade et al. 2007; Thompson et al. 2013) to facilitate the idea of covariation. Thus, in a second setting explicitly choses a non-numerical approach for experimenting with immediate examination of covariation.

The learning environment of setting 2 is structured accordingly to setting 1, with modifications in the experimental structure of the contexts: in the preparation phases of the first three contexts the students are only briefly introduced with estimation tasks based on hands-on material, before they use simulations to identify the related quantities. In the following experimental phase, the students observe the variation and covariation of the quantities in the simulations and verbally describe the relationships discovered. Subsequently graphs are generated within the simulations and in the post-processing phase students are asked to analyse the form of the graphs and connect their insights with the relationship described in the previous phase, before they observe individual values of quantities to check their estimations and answer the overarching task like in setting 1. The last two contexts are again briefly introduced with hands-on material and estimation tasks, followed by the request for verbal descriptions of the relationships. Based on these descriptions and on their insights from the previous contexts, students are asked to group the contexts by their kind of covariation. The students then continue with the experimentation phase and take measurements with the hands-on material and then proceed with simulations as in setting 1. In the post-processing phase students are now asked to verify their hypotheses on the relationships and their grouping with the graphs and tables from the experimental phase. Finally, students check their estimations and answer the overarching task like in setting 1.

CONTEXTS

Both settings use a treehouse building story with identical overarching tasks. The contexts are implemented with the same hands-on material (see figure 1 and 2) and simulations, but different components of the simulations are visible in the settings.

Figure 1: Hands-on material of the first three contexts in setting 1 and 2

The contexts are chosen to represent a linear and a quadratic relationship and one with varying change rate: the perimeter of a circular disc determined by its diameter, the number of cubes needed for a "staircase" determined by the number of steps and the fill height of a vessel determined by the volume of water filled into.

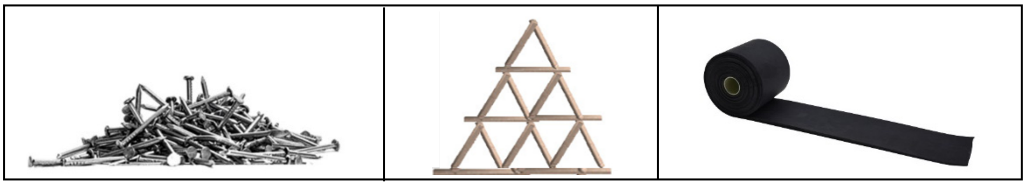

Figure 2: Hands-on material of contexts 4, 5 and bonus context 6

Contexts four and five (linear and quadratic) are the weight of a package of nails determined by the number of nails and number of beams needed for a woodwork determined by the number of floors. A bonus context for quick learners depicts the diameter of a unrolling tape determined by the length of tape that has been unrolled.

The simulations can be accessed in the digital classrooms (www.geogebra.org/classroom – for Setting 1, Team Engineers: Code HQX7 UZRQ – for Setting 2, Team Architects: Code D3XM DDSB).

STUDY DESIGN

A comparative intervention study (pre-post design) will contrast the two approaches to answer the following research question:

Is it possible to recognize differences between setting 1 and setting 2 in the processing of the tasks regarding the aspects of functional thinking and the conceptualization of function?

The intervention will take place at the University of Koblenz-Landau, as part of the mathematics laboratory program, where school classes work in groups of four in half-day projects with hands-on material and computer simulations. The intervention is preceded by a short test on functional thinking (FT-short, adapted from Lichti and Roth 2018), to compare the learning outcomes in both settings, and a three-minute intelligence screening (Baudson and Preckel 2016). Both tests take place approximately one week before. The intervention is designed for three 90-minutes-lessons including the post test of FT-short. A follow-up of the test is planned 4-8 weeks after the intervention. Two focus groups (low-/high-performer in FT-short) per school class will be videotaped. All student products and videos from the intervention are evaluated regarding the presence of the aspects of functional thinking (qualitative content analysis, validated category system from Lichti 2019) and the students' function conceptualization is assessed using the indicators from Dubinsky and Wilson (2013).

A pilot study intends to verify the comparability of the two approaches in terms of processing time and difficulty. Due to the corona shutdown and the ongoing rules for physical distancing the study is adapted to an online classroom supplemented with a "math box" containing the hands-on material. The dyadic approach and the videotaping in the pilot are replaced by an expert rating (questionnaire with four-point Likert scale N = 4 / open answers N = 9) and a reflective analysis based on the ALACT model (Korthagen 2017) with student teachers (N = 12, masters course in mathematics). The student pilot took place in two sessions with students from a high-school course held by the first author. They were assigned to the settings so that results in the FT pretest, overall math skills (half-term grade) and reading skills (half-term grade; two dyslexics) were equally represented in each setting.

PRELIMINARY RESULTS AND DISCUSSION

Here we present preliminary results of the student pilot study. One participant in each setting completed the whole program including the bonus context, two in setting 1 and three in setting 2 completed the 5th context (hence only bonus left) and the other students were still working on the 4th/5th context when time elapsed, so that regarding time both settings seem to be comparable. One identical task for both settings will be discussed in detail and compared with a related task of the FT pretest, indicating differences in the conceptual development.

In task no.48 students are asked to describe how the fill height of the liquid rises in the curved vessel using the graph and a given word list (slow, fast, steep, flat, rise, broad, narrow). In setting 2 all students were able to combine the fill height at least with the form of the vessel in their description, while in setting 1 only two students did connect their description of the fill height to the graph or the vessel. One student in setting 2 wrote:

> “The liquid rises slow first and from the value 1 on it gets faster because the vessel is broad at the beginning. From the value 2 on it becomes slower, because the vessel has a curve in the middle and was narrow but now becomes broader. Until value 6 the vessel keeps even, but from 6 on it rises up fast because the vessel becomes narrower and narrower. At point 10 there is a curve again, since the vessel is broader again.”

Although this student is not capable of interpreting the slope (or the form) of the graph, the references to the values indicate a connection between the measurement points in the graph and the fill height in the vessel. The description of the varying fill height shows a dynamic perspective and the concurrent statements about the variation in the vessel form show a covariational conception. One could argue that this conception does not include the fill volume itself, but the variation is given in the form of the vessel since liquid is filled with a constant rate.

Another student in setting 2 wrote:

> “When the line is steep in the graph, the water rises fast and the vessel is narrow. When the line is flat in the graph, the water rises slow and the vessel is broad.”

From a semantic view this student is arguing with conditions rather than changes, which reminds of a grading in intervals or chunky thinking as described by Castillo-Garsow et al. (2013). At the same time, he interprets the slope of the graph and connects it to the change in the fill height (representational switch), revealing a dynamic perspective, but the covariational conception does not include a dynamic perspective on the change of the form of the vessel (or at least it is not expressed).

The most elaborate description in setting 1 was:

> “At first it rises slow because the glass is broad, then it becomes narrower and narrower and it rises faster.”

Although this student grasps a variation in the fill height, the first argument points to the correspondence aspect and is based on simplification (constant diameter). The second statement shows a more dynamic view of the form of the vessel (“narrower and narrower”), but the related fill height is not described accordingly. Hence the covariational conception is only displayed in a preliminary stage.

These analyses only show an extract of the student documents, but they already indicate different stages of dynamic view and covariational conception in the two settings. To get an idea of the development towards these stages, we conclude with statements form a related task in the pretest. In this task students have to assign two out of four given fill curves to two vessels (cylinder and frustum of a cone). They provided the following explanations (comparably in both settings):

> “the vessel will fill faster and faster; the vessel is even, and the graph is straight; it becomes constantly more; at the bottom it fits more; first it takes a while because it is broad at the bottom”

Compared to the statements discussed above, one can detect a development in a) the connection between vessel, fill height and/or graph, b) in the dynamic view of one or more quantities and the graph and c) from a correspondence conception towards the variation of quantities and covariation.

Hence both settings are capable to foster functional thinking and the key aspect of covariation is addressed since students in both settings improved in the dynamic view and/or covariational thinking. The students in setting 2 seem to benefit more regarding the interpretation of the graph, which might be caused by the intensified usage of the multi-representational simulations. The qualitative approach of setting 2 might have set the focus on (co-)variation as intended, at least the improved dynamic perspective, better connection between quantities and rudiment covariational conceptions lead to this assumption. The main study will give more detailed insights to the development of the aspect of functional thinking and the conceptualization.

REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245–274.

Balacheff, N., & Kaput, J. J. (1997). Computer-based learning environments in mathematics. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 469–501). Dordrecht: Springer.

Baudson, T. G. & Preckel, F. (2016). mini-q: Intelligenzscreening in drei Minuten [mini-q: A Three-Minute Intelligence Screening]. *Diagnostica, 62*(3), 182–197.

Bossé, M., Adu-Gyamfi, K., & Cheetham, M. (2011). Assessing the Difficulty of Mathematical Translations: Synthesizing the Literature and Novel Findings. *International Electronic Journal of Mathematics Education, 6*(3), 113–133.

Castillo-Garsow, C., Johnson, H. L. & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243–1267.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y. & Maschietto, M. (2016). Uses of Technology in Lower Secondary Mathematics Education. *ICME-13 Topical Surveys*. Dordrecht: Springer

Drijvers, P. (2020). Embodied instrumentation: Combining different views on using digital technology in mathematics education. *Eleventh Congress of the European Society for Research in Mathematics Education*, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02436279

Dubinsky, E. & Wilson, R.T. (2013): High school students' understanding of the function concept. In: *The Journal of Mathematical Behavior, 32*(1), 83–101.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992): Dynamic representation and the development of an understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. (pp. 235–260). Washington, DC: Mathematical Association of America.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Lichti, M. (2019). *Funktionales Denken fördern: Experimentieren mit gegenständlichen Materialien oder Computer-Simulationen* [Fostering functional thinking: Experimenting with real materials or computer-based simulations]. Wiesbaden: Springer Spektrum.

Lichti, M., & Roth, J. (2018): How to Foster Functional Thinking in Learning Environments: Using Computer-Based Simulations or Real Materials. *Journal for STEM Education Research 1*(1-2), 148–172.

Monk, G. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–194). Washington, DC: Mathematical Association of America.

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments.* University of Paris 8, <hal-01020705>.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Vollrath, H.-J. (1986). Search strategies as indicators of functional thinking. *Educational Studies in Mathematics, 17*(4), 387–400.

Vollrath, H.-J. (1989). Funktionales Denken [Functional thinking]. *Journal für Mathematikdidaktik, 10*(1), 3–37.

Fostering heuristic strategies in mathematical problem solving with virtual and tangible manipulatives

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Potentials of a variety of technologies to foster problem solving have been explored in relation to dynamic geometry software extensively. This paper takes a step forward and looks beyond the 'well-known' DGS application, at the aptitudes and challenges of manipulatives created with 3D print-technologies for development of heuristic strategies in problem solving. It presents two specific designs of virtual and haptic manipulatives inspired from outdoor contexts. The designs can stimulate the interplay between heuristic strategies in problem solving in a narrow sense as "guess – check – revise", "look for a pattern", "problem reformulation" and "solution drawing", but also in a wider sense as "problem finding" and "problem solving further". A specific task for the designs finalizes the paper and opens new aspects for research.

Keywords: heuristic problem solving, virtual and tangible manipulatives, dynamic geometry software (DGS), 3D print-technology, design research.

INTRODUCTION

„Attempts to build problem-solving programs based on heuristics like those characterized by Pólya, have generally been unsuccessful", yet teaching heuristics in school mathematics is valuable for plenty of reasons (Schoenfeld, 1985, p. 72). Despite the huge amount of research on mathematical problem solving and heuristics, adequate details about heuristic strategies whose development may be influenced by the use of novel digital and physical media are still lacking. Considering personal resources as „mathematical knowledge possessed by an individual", Schoenfeld (1985, p. 15) argued that the selection and implementation of strategies in problem-solving is dependent on the type of the resource that is accessible to the individual (Schoenfeld, 1985, p. 70). This paper examines heuristic strategies in connection to the usage of external resources for learning such as manipulatives. It tackles questions as, what heuristic strategies can be stimulated and how do they interplay in problem solving by the use of a particular virtual or physical material such as DGS and 3D prints? The focus is set on contents related to geometry at high school.

The paper reports about part of the project „New possibilities for differentiation in mathematics education through digitalization" at the Goethe University Frankfurt am Main. The project involves collaboration with schools in and out of the city and moreover, international joint research works with the Institute Federal de Educação in Brasil.

THEORETICAL GROUNDING

Problem solving is one of the key competences in mathematics education emphasized in many national curricula (e.g. Carpenter & Gorg, 2000; KMK, 2012; BNCC, 2018). Problems requiring application of mathematical theorems and their proofs in geometry, e.g. in construction problems at secondary level, have been discussed in literature for a long time (e.g. Zech, 1998). In this section, problem solving and related heuristic strategies are discussed from two perspectives, i.e. in a narrow and wider sense.

Problem Solving in a Narrow Sense

Well established Pólya's (1973) phases in problem solving: understanding the problem, deriving a plan, carrying out the plan and verification, have inspired rich research debates. Later literature has extended the list and the diversity of the initially suggested heuristic strategies in problem solving to a significant amount. For example, strategies as *"draw a diagram", "guess and check", "look for a pattern", "make a systematic list", "use before-after conception"* are discussed by Fan and Zhu (2007). Novotná et al. (2014) have examined the improvement of pupils' abilities to solve problems when using the strategies: *"guess – check – revise", "systematic experimentation", "problem reformulation"* and *"solution drawing"* . Further, the German literature differentiates between *general* and *content-specific heuristic strategies*. Among the general strategies, Wittmann (2014) places the *"use of representations and translations between them", "working forward", "working backward", "analogizing", "principle of invariance", "specialization"* and *"generalization"*. Exemplified content-specific heuristic strategies for two-dimensional geometry in secondary school involve: drawing of suitable auxiliary lines, a search for equally long distances and angles of equal or complementary sizes (isosceles or equilateral triangles, right-angled triangle, sides of a parallelogram, circle-radii, etc.), sum of interior angles in polygons, angles at an intersecting line of two parallel lines, a search for areas that are identical in terms of patterning or puzzling and so on.

Problem Solving in a Wider Sense

In a wider sense, problem solving is defined through three phases: "*finding a problem", "solving a problem"* (in a narrow sense) and "*further developing the problem"* (Leuders, 2003). "*Finding a problem"* refers to detecting, identifying and describing problems in outdoor contexts with reduced complexity, e.g. noticing geometric polygons and their properties in buildings. Further, *problem solving in a narrow sense*, involves application of mathematical competences for analyzing and verifying solutions in new ways or combinations, e.g. 2D constructions of geometrical figures with traditional Euclidean tools and novel digital technologies, which enables consolidation and flexible usage of previous knowledge. For instance, a virtual manipulative that has originated in a real outdoor context, e.g. architecture of buildings (*finding a problem*) can further be developed with DGS. "*Problem solving further"* leads to development of new ideas and strategies for solving the initial problem, but moreover, its variations or creations of new problems (e.g. creating figures to be used in problems of patterning and tessellations, puzzles or tangrams). This opens new perspectives for further learning of other mathematical concepts in a network (e.g. tessellations obtained through congruence transformations or similarities). In this phase, a DGS creation can further be transformed into a physical manipulative, e.g. mosaic pieces of a palpable geometric shape or a pattern by using 3D print-technology. It is also possible that a physical exploration can trigger students in designing new patterns digitally with a DGS.

It is worth mentioning that these three phases are not disjoint and necessarily sequenced in a static order, rather exchange and complement each other. They are all tidily related to the well-known heuristic strategies in problem solving in a narrow sense, but we reconstruct them for problem solving in a wider sense and identify them as *spot, sketch* and *create.* Further, we describe each of these heuristics in relation to the interplay of virtual and physical manipulatives.

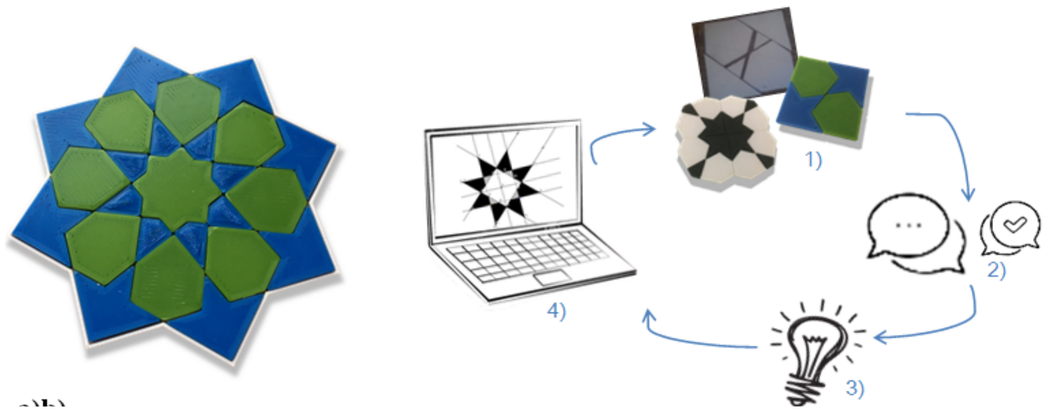
USE OF PHYSICAL AND VIRTUAL MANIPULATIVES IN MATHEMATICS EDUCATION

Literature has by now recognized the benefits of using both physical and virtual manipulatives in mathematics classroom. Effects of balanced use of physical and virtual manipulatives have been studied regarding development of students' representations in algebra (e.g. Suh & Moyer-Packenham, 2007) or development of middle school students' visualization and spatial reasoning skills (Drickey, 2000). Sarama & Clements have concluded that "manipulatives are meaningful for learning only with respect to learners' activities and that both physical and virtual manipulatives can be useful" (Sarama & Clements, 2016, p. 71). When applied in comprehensive, well planned, structured and goal-oriented settings, both types of manipulatives can inspire students to make their knowledge explicit, which helps them build "Integrated-Concrete knowledge" (Sarama & Clements, 2016) or develop heuristic strategies as spotting/identifying the problem; sketching, drawing, constructing and creating the virtual and the tangible manipulative. Likely, guided participation in activities that can be organized in the three 'steps' might be appropriate to integrate DGS and 3D print technologies.

RESEARCH METHODOLOGY

Our methodological approach relays on the work of Kolb (1984) about experiential education that depicts learning as a cyclic process involving four modes: 1) concrete experience (i.e., doing stage; engaging in a hands-on activity), 2) reflective observation (i.e., thinking, recording, discussing the experience), 3) abstract conceptualization (i.e., concluding stage; generate new understandings about the practices), and 4) active experimentation (i.e., adapting stage; trying out new ideas as part of the learning process; testing hypotheses). Figure 1 illustrates how these phases meet our experiments. Within this cycle students are exposed to different sets of physical tiles of a given particular pattern which students have to discover first (colourful shapes in

Figure 1.a and 1.b.1). Students are asked to figure out the geometrical properties and relationships regarding angles and sides of an initial mosaic in which they apply different strategies as *“guess – check – revise“ and “look for a pattern”*. This process opens conjecturing debates (Figure 1.b.2) to generate ideas (Figure 1.b.3) and encourage students to develop their own dynamic digital representations by using heuristic strategies (Figure 1.b.4) that may wary from those used on the beginning (Figure 1.b.1).



a) **b)**

Figure 1: a) 3D print, b) Kolb’s cycle about experimental learning

After that and mediated by a teacher, students work further with physical manipulatives 1) to check their strategies and refine their thoughts, restarting a new discovering cycle. The transfer from 4) to 1) requires strategies as *“problem reformulation“ and “working backward”* that can be stimulated by questioning what should the construction be like in order to produce new 3D printed tiles. This coincides with *“problem solving further”* in the wider sense.

We have undertaken a case study with three students in upper secondary school participating in a mathematics training program. The empirical findings of these cases, each of them undergoing the above Kolb’s cycle will be reported separately from this contribution.

PRESENTATION OF THE DESIGNS

Linking open spaces and mathematics classroom can initiate engaging and creative learning environments. By *spotting* and capturing shapes through sketches in an intentionally chosen out-of-school environment offered by the teacher, students can be curious and inspired to bring in and experiment with own discoveries in math classroom or media lab. In continuation, we show such environments in two European cities, Wien and Berlin (Fig. 2, a) and Fig. 3, a), correspondingly). In this first step students are expected to discover, identify and describe geometrical shapes in objects suggested by the teacher - *finding a problem in the wider sense,* as described above. Once they spot a particular geometrical shape, they can complete a *sketch* which will be taken for further usage in a math classroom or a digital lab. In order to reduce difficulties that may occur, students should be offered solid organisation, guidance and a paper-pencil worksheet for the required sketch.

a) Spot (in reduced context)

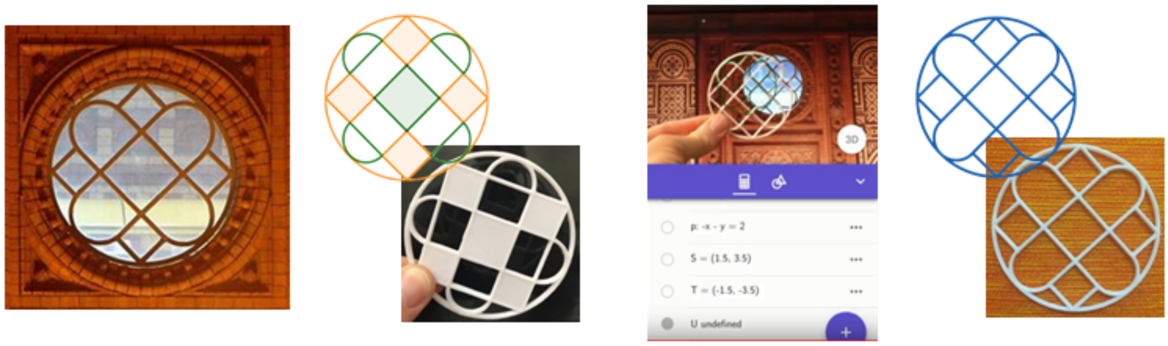
b) Sketch (paper-pencil)

c) Create (with DGS)

Figure 2: Problem solving in a wide sense with virtual and tangible manipulatives inspired by a rosette on a restaurant window in Wien

Further on, such spotted and documented mathematical object can be *created* with interactive dynamic software in addition to the traditional construction with straightedge and compass – problem solving related to constructions (Zech, 1998) (Fig. 2, b and Fig. 3, b). A traditional paper-pencil construction with straightedge and compass of this famous rosette tiling using different methods can be motivated and instructed by online educational videos [3]. This is an algorithmic phase containing an execution of the construction and documentation of the solution (construction description). For example, for the suggested constructions (Fig. 2, b and Fig. 3, b) students need to know and use different properties of plane geometric figures, inscribed and circumscribed circle of polygons, the interior angle sum theorem for polygons, congruent figures and congruence mappings like rotation. An analytical phase may follow. This phase comprises of justification of the correctness of the undertaken construction and considerations regarding the uniqueness of the solution.

Finally, the produced virtual manipulative, e.g. a dynamic geometry file can further be used for turning it out in a 3D printed tangible manipulative, inspiring the re-design and creation of new mosaic patterns. This step coincides with the Leuder's statement – *developing further problem* (Leuders, 2003) and it may be undertaken in a variety of ways. We illustrate the step creation through an example about tessellation (Fig. 2) and the re-design through different creation steps (Fig. 3). Such new situation may enthuse students to continue beyond the virtual manipulation and again re-design through overlapping three models: a real one through sketching or photographing, a virtual one by the use of Augmented Reality (AR) (Fig. 3.c) and the physical one obtained by the 3D printing (Fig. 3.d).



a) Spot (in reduced context) **b) Create** (DGS and 3D print) **c) Re-create** (3D print and AR) **d) Re-create again** (DGS and 3D print)

Figure 3: Problem solving in a wide sense with virtual and tangible manipulatives inspired by the rosette at Hachischer Markt in Berlin

All of these suggested designs require diverse level of mathematical knowledge and integrated digital skills and are therefore appropriate for differentiation in mathematics education. Simple variations of posing the problems may lead to requirements with significantly different level of difficulty. Therefore, our expectations are related to offering possibilities for students' individual trials, learning trajectories and results, rather than achieving homogeneity in a learning group. This diversity includes application of different heuristic strategies. Having in mind the affordances that may appear by increased number of learning materials, a systematical development of each teaching unit and activity is a necessity. Several task designs about symmetry of plane geometric figures and tessellation including 3D printed artifacts are suggested to sustenance pragmatically understanding of abstract concepts in math school curricula at different levels of education (Donevska-Todorova, 2020; Leung & Donevska-Todorova, 2020; Lieban, Lavicza, & Reichenberger, 2020). In continuation, we suggest further possibilities for problem solving and task design with these manipulatives.

Heuristic strategies in specific problem solving related to the manipulatives

This sub-section offers suggestions for continuing and completing the activities during the creation of the manipulatives and their application along the Kolb's cycle (Fig. 1.b). Tasks like the following one may stimulate different use of heuristic strategise when choosing unlike manipulatives.

> Task: Since the pattern in Figure 2 has an axial symmetry, which angles can be determined in each of the puzzle tiles?

Strategies that may apply for this problem are *"guess – check – revise"* and *"look for a pattern"* in a paper-pencil environment. However, *"problem reformulation"* by selecting subsets of the tiles which are congruent to each other and searching the angles only within one subset seems a more appropriate strategy when using the palpable manipulatives. The strategy *"systematic experimentation"* seems also appropriate with the same manipulatives. This also relates to *"problem solving further"* in the wider sense. However, when changing the manipulatives to digital ones such as DGS, the content-specific strategies as *“drawing of suitable auxiliary lines”, “searching for angles at an intersecting line of two parallel lines”* and *“using sums of interior angles in polygons”* may be more suitable.

CONCLUSION AND FURTHER PERSPECTIVES

In this paper, we have theoretically considered problem solving and the use of heuristic strategies which may diverge depending on the nature of the manipulatives used. We have suggested designs and explorations of virtual manipulatives with DGS and physical manipulatives with 3D print-technologies (Fig. 2 and 3) which directly refer to *“guess – check – revise“, “look for a pattern”, “problem reformulation“* and *“solution drawing“* strategies, but also *“problem finding”* and *“problem solving further”* in a wider sense. This directly addresses the posed research question which we have also illustrated with a specific task design related to the created manipulatives. Possible efficacious of the designs for fostering the strategies are in the process of empirical investigation according to the methodological approach (Fig. 1.b). Finally, this work has lightened new ideas for scientific observations related to support of creative mathematical thinking by the design of similar virtual and 3D printed manipulatives (Donevska-Todorova & Lieban, 2020).

NOTES

1. The used DGS in this paper is GeoGebra. The designed resources by the authors are available here: [1] https://www.geogebra.org/m/avhrrnde and [2] https://www.geogebra.org/m/hbfmucrs and [3] https://www.youtube.com/watch?v=ILD0Hvf5Xbo.

REFERENCES

BNCC (2018). Base Nacional Comum Curricular. Ministério da Educação. Brasil. Retrieved from http://basenacionalcomum.mec.gov.br in 23.09.2019.

Carpenter, J., & Gorg, S. (2000). Principles and Standards for School Mathematics. NCTM. https://en.calameo.com/read/0006786711df6b77d3cd0, last retrieved on 14.03.2020.

Donevska-Todorova, A. (2020). Problemorientiertes Lernen zur Leitidee „Raum und Form“ in der Sekundarstufe mit Hilfe von DGS und 3D Druck Technologie. *Beiträge zum Mathematikunterricht. Vorträge auf den 54. Jahrestagung der Gesellschaft für Didaktik der Mathematik* vom 09. bis 13. März 2020 in Würzburg.

Donevska-Todorova, A. & Lieban, D. (2020). Fostering Creativity Through Design of Virtual and Tangible Manipulatives. In the Proceedings of the tenth ERME Topic Conference Mathematics Education in the Digital Age (MEDA) 2020.

Drickey, N. A. (2000). *A comparison of virtual and physical manipulatives in teaching visualization and spatial reasoning to middle school mathematics students* (pp. 1-138). Utah State University.

Fan, L. & Zhu,Y. (2007). Representation of Problem-Solving Procedures: A Comparative Look at China, Singapore, and US mathematics textbooks, *Educational Studies in Mathematics,* 66(1), pp. 61-75.

KMK (2012). Bildungstandards Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife (Beschluss der Kultusministerkonferenz vom 18.10.2012).https://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2012/2012_10_18-Bildungsstandards-Mathe-Abi.pdf.

Kolb, D. A. (1984). Experiential Learning: Experience as a Source of Learning and Development. Englewood Cliffs, NJ: Prentice Hall.

Lieban, D., Lavicza, Z., & Reichenberger, S. (2020). 3D tesselation triggering the design of open-ended tasks combining physical and digital resources. 14th ICME, Shanghai (submitted).

Leuders, T. (2003). Problemlösen im weiteren Sinn (Hrsg.): Mathematik-Didaktik, Cornelsen Scriptor.

Leung, A. & Donevska-Todorova, A. (2020). The Role of Technology in the Pragmatic-Abstract Continuum in Mathematics Curriculum Development and Task Design. In: Clark-Wilson, A., Donevska-Todorova, A., Fagianno, E., Trgalova, J. & Weigand, H.-G. (Eds.) *Mathematics Education in the Digital Age: Learning, Practice and Theory.* ERME Series Routledge (2020, accepted).

Novotná, J. et al. (2014). “Problem Solving in School Mathematics Based on Heuristic Strategies”, *Journal on Efficiency and Responsibility in Education and Science,* 7(1), pp. 1-6, doi: 10.7160/eriesj.2013.070101.

Pólya, G. (1973). *How to solve it. A new aspect of mathematical methods* (2nd ed.). Princeton University Press

Sarama, J. & Clements, D.H. (2016). Physical and Virtual Manipulatives: What Is “Concrete”?. In: Moyer-Packenham P. (eds). International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives. Mathematics Education in the Digital Era, vol 7. Springer, Cham.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Suh, J., & Moyer-Packenham, P. (2007). Developing students’ representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching, 26*(2), 155-173.

Wittmann, G. (2014). Problemlösen In Weigand et al. *Didaktik der Geometrie für die Sekundarstufe I,* Springer. Spektrum

Zech, F. (1998). *Grundkurs Mathematikdidaktik. Theoretische und praktische Anleitungen für das Lehren und Lernen von Mathematik.* Beltz Verlag.

Student responses as a basis for whole-class discussions in technology-rich environments

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This paper reports a study of four upper secondary school teachers' use of Connected Classroom Technology to select student responses to computer-based activities, and to use these responses to launch successive stages of a planned whole-class discussion. Although the preparation for the class discussion was quite successful, it was a challenge for the teachers to conduct the whole-class discussion, particularly in posing specific questions based on appropriate student responses.

Keywords: connected classroom technology, mathematics education, whole-class discussion.

In a previous study (Fahlgren & Brunström, 2018) we examined students' written explanations of an observation made in a dynamic mathematics software (DMS) environment. We found that few students offered a complete mathematical explanation. However, most of them provided elements of explanation, many of which that could be useful as starting points for a whole-class discussion. This highlighted a need to provide support for teachers in surveying students' computer-based work, preferably in real time, to inform such a discussion. For example, the participating teachers requested technological support to monitor all the students' work and to easily choose different student solutions for whole-class discussion. Nowadays, there is a type of technology available that can support teachers to achieve this which we refer to as *Connected Classroom Technology* (CCT).

This led us to conduct a case study in a Swedish upper secondary school, working with four teachers and their classes. The overarching aim was to identify critical aspects when using CCT to prepare and conduct a whole-class discussion based on students' responses to computer-based activities. A teaching unit consisting of three stages − *introduction*, *pair work*, and *whole-class discussion* − was designed and trialled with the four classes. In an earlier paper, we have provided a detailed description of the design of the teaching unit (Fahlgren & Brunström, 2019). In another paper we have reported on the teachers' utilization of the CCT during the pair work stage (Fahlgren & Brunström, 2020). The focus in this paper is on the last stage of the teaching unit − the whole-class discussion. The research question is: What are the challenges for teachers when using CCT to select student responses, and use these responses to launch successive stages of a planned whole-class discussion?

WHOLE-CLASS DISCUSSION BASED ON STUDENT RESPONSES

The importance of following up students' previous work in pairs or small groups and using it as a basis for a whole-class discussion is well established in the mathematics education research literature (e.g. Franke, Kazemi, & Battey, 2007; Stein, Engle, Smith, & Hughes, 2008). At the same time, this literature highlights the challenge for teachers to orchestrate student- active classroom dialogues (Ruthven & Hofmann, 2013; Stein et al., 2008). To address this, Stein et al. developed a model consisting of five practices to support teachers in their planning and implementation of whole-class discussion. The first practice, ”Anticipating likely student responses…” (p. 321) relates to the planning of the lesson. The second, third and fourth practices: monitoring, selecting and sequencing student responses, all relate to the stage of the lesson where students are working on activities. Finally, the fifth practice concerns the collective stage of the lesson where different student responses are displayed and discussed in the whole class. Although the model is primarily intended to serve as a road map for teachers, it provides a theoretical frame which researchers can use "…as a way of conceptualizing investigations of classroom discourse" (Stein et al., 2008, p. 314) as well. For example, Cusi, Morselli and Sabena (2017) used this model when investigating how CCT could be used to facilitate whole-class activities.

Kieran et al. (2012), demonstrate the challenge for teachers to orchestrate follow-up discussions which take students' computer-based work into account. In their study, only one of the (three) participating teachers really inquired into students' thinking and utilized it as a point of departure for a whole-class discussion, although such discussions were an expected part of the researcher-designed lessons. However, this was not made explicit in the accompanied teacher guidance since it did not specify how to perform the discussion, although it included suggestions for mathematical content to discuss. Similarly, Ruthven and Hofmann (2013), in a design study, identified situations where disappointing classroom mathematical discussion arose from teachers not capitalising on promising student contributions. To address this, they suggest, there is a need to sensitise teachers, typically through making the potential of such contributions more explicit in the teacher guidance.

METHOD

The fieldwork for this study was conducted in spring 2019 with four upper secondary school teachers and their classes undertaking the 3-stage teaching unit already referred to. In undertaking the unit, two types of technology were used − a dynamic mathematics software (DMS), in this case *GeoGebra*, and a specific CCT, *Desmos Classroom Activities*. During the pair-work stage, the students used two computers; one with *GeoGebra* and one with an e-worksheet in *Desmos*. In contrast to the DMS, the CCT was a novel teaching resource for the participating teachers.

When planning the teaching unit, then, we gave particular attention to providing teachers with guidance on making use of the CCT to examine students' work during the central pair-work stage of the lesson, and to prepare examples from this work for use in the concluding whole-class discussion stage. This guidance was informed primarily by the Stein et al. model, with the necessary mathematical-conceptual detail derived through analysis of student responses (gathered from eight classes in a previous study (Fahlgren & Brunström, 2018)) to the explanation task featured in the lesson (Task 1c in Figure 1). This analysis provided information about answers likely to be produced by students during the pair-work stage, i.e. relating to the first practice in the Stein et al. (2008) model – Anticipating likely student responses.

Task 1 Quadratic functions can always be written in the form $f(x) = ax^2 + bx + c$ where *a, b* and *c* are real numbers and $a \neq 0$.

(a) Investigate, by dragging the slider *c*, in what way the value of *c* alters the graph. Describe in your own words.

(b) The value of the constant *c* can be found in the coordinate system. How?

(c) Give a mathematical explanation why the value of *c* can be found in this way.

Figure 1. The first tasks including a request for an explanation (Task 1c).

Detailed step-by-step guidance, based on the Stein et al. model (2008) and exploiting particular functionalities of CCT, was developed and discussed with the teachers. During the pair-work stage, teachers are encouraged to use two different CCT views to monitor the students' work. In the *Summary* view, the teacher can survey all the students' progression across the whole activity, and in the *Specific item* view, the teacher can monitor all students' answers to a particular task (relating to the second practice in the Stein et al. model – Monitoring student responses). In the latter view, the teacher also can select appropriate student responses to display and use as a basis for the whole-class discussion (relating to the third practice in the Stein et al. model – Selecting student responses). In the view that we denote *Presentation preparation* view, the teachers can sequence the selected student responses (relating to the fourth practice in the Stein et al. model – Sequencing student responses). To support the selection and sequencing, the guidance included response categories to search for as well as a suggested sequencing of the responses (see Figure 2).

Identify and select one or two appropriate student responses from the different categories

(a) Repeating the answer to Task 1b, i.e. only indicating that it is where the graph **intersects the *y*-axis**
(b) Providing **example** (e.g. "if c=3, it intersects the *y*-axis at 3" or referring to GeoGebra)
(c) Comparing with the standard **linear equation**, e.g. "*c* corresponds to *m*"
(d) Indicating that **"*c* is independent of *x*"** or that **"*c* is the constant term"**
(e) ***x* = 0 gives *y* = *c***

Figure 2. Excerpt from the teacher guidance illustrating the response categories.

Moreover, the guidance includes a probing question to pose in relation to each of the different response categories (see Figure 3) during the whole-class discussion. The detailed thinking behind the recommended sequencing and the corresponding questions is reported in Fahlgren and Brunström (2019), but the gist should be clear from inspection of the two Tables. At a planning meeting, the guidance was discussed and the researchers and the teachers agreed that it was appropriate.

(a) *What is the distinction between Task 1b and Task 1c?* (i.e. what is the distinction between a description and an explanation in mathematics?
(b) *Can examples be used as an explanation? Is it enough to refer to GeoGebra (as a mathematical explanation)?*
(c) *What do m in f(x) = kx + m and c in f(x)=ax² + bx + c have in common?*
(d) *Could the explanation be strengthened further?, i.e. Why does this mean that the graph intersects the y-axis when y=c?*
(e) These discussions should lead to a class agreement on what constitutes an approprate explanation in this particular case (Task 1c).

Figure 3. Excerpt from the teacher guidance showing the questions suggested.

DATA COLLECTION AND ANALYSIS

We sought, then, to examine how teachers made use of CCT in preparing for, and implementing, the whole-class-discussion stage of the teaching unit. The main data consist of screen recordings of each teacher's computer as well as audio recordings and field notes from the whole-class-discussion. In addition, a joint meeting with the teachers afterwards was audio recorded. The focus in this paper is on the student responses selected and on teachers' utilization of these during the whole-class discussion, guided by the questions suggested in the teacher-support materials.

All student responses were categorized based on the anticipated categories in Figure 2. This analysis provided information about the occurrence of student responses in each class as well as responses selected by the teachers. This enabled us to ascertain whether each teacher managed to select responses from all of the categories available in their class. Screen recordings of the *Presentation preparation* view provided information about the sequencing (of the selected responses) made by the teachers in preparation for the whole-class discussion.

Next, the audio recordings were transcribed and screen shots from the screen recordings were inserted to indicate which response the teacher displayed when posing different questions. These transcripts were analysed to indicate the number of questions posed by each teacher in relation to different response categories. We also compared these questions with the suggested ones, categorizing the teacher's action as Suggested question (or equivalent), Some other form of question, or No question.

RESULTS

This section starts by presenting the findings related to the teachers' preparation in terms of selection and sequencing of student responses. Then the findings concerning their conduct of the whole-class discussions is reported.

Preparation for whole-class discussion

The responses making up our category system are idealized in the sense that each appeals to a single distinctive idea. However, the empirical responses that we received from students could make reference to more than one of theses idealized responses and/or to further ideas absent from the category system. Consequently, these empirical responses were mainly of three types. First, there were those that (in our interpretation) refer only to a single category. Second, some refer to more than one category or combine one category with other irrelevant information. Finally, there were responses that did not relate at all to the anticipated answer categories (categorized as "Other"). These responses were irrelevant, e.g. "Because the *c* variable is the slope in this case", or not informative enough, e.g. "$y=c$".

Although the teachers found the selection process challenging, they all managed to select responses from all categories available (in their class). There were no category (b) answers available in any of the classes, and no category (e) answer in T4's class. As indicated in Table 1, some of the selected responses did not only consist of one category or were categorized as "Other". As will be demonstrated later, this caused trouble in the subsequent stage of the lesson.

Table 1 shows the sequencing made by the teachers. The letters within brackets indicate our categorization of the different student responses. In cases when a student response includes more than one category (or further irrelevant information), both categories are indicated within the brackets. For example, in T1's first presentation view, there are two student responses, one categorized as both (c) and (a) and the other as (a) only. Table 1 indicates that the teachers more or less followed the suggested sequencing, and that three teachers utilized the opportunity to display more than one response on the same presentation view.

Presentation view	T1	T2	T3	T4
P1	(c+a), (a)	(a)	(a)	(a)
P2	Other	(c)	(c)	(c)
P3	(c)	(d+irr)	(c), (c)	(d), (c+e)
P4	(c+d)	(d)	(a)	(d+a+irr), (c+e) as in P3
P5	(e)	(d+irr) as in P3	(d)	
P6		(e)	Other	

Table 1. The sequencing of different student responses in the four classes.

Challenges in launching the stages of the planned whole-class discussion

The times devoted for the whole-class discussion were 18 min. (T1), 10 min. (T2), 14 min. (T3), and 5 min. (T4). The presentation of the results in this section is organized according to two identified critical aspects: *Challenge in using the suggested questions* and *Challenges due to student response features.*

Challenge in using the suggested questions

Table 2 shows the type of question posed by the teachers in relation to response categories (a), (c), and (d). In cases where several questions were posed, the number of questions is indicated within brackets.

Type of student response under discussion	Type of question used by teacher		
	Suggested question (or equivalent)	Some other form of question	No question
(a)	T3	T1, T2	T4
(c)	T3	T1(4), T2(2), T4(2)	
(d)		T2	T1, T3, T4

Table 2. The number and types of question posed by the teachers.

One of the teachers (T3) used the question suggested for category (a), while two of the teachers posed questions without referring to the distinction between a description and an explanation. Instead, T1 focused on assessing the quality of the explanation, "What do you think about that explanation?", while T2 just asked whether the response could be regarded as an explanation. T4 made a point about the differences between "what you can see in the graph" (i.e. a description) and an explanation, although without posing any question.

Questions related to category (c) responses were posed several times in three of the classes. For example, T1 posed questions in relation to three different student responses. However, the question closest to the suggested one, "What do these two have in common?", was posed when pointing to the two formulas (written) on the board ($y=kx+m$ and $y=ax^2+bx+c$), i.e. not based on a student response. This was also observed in T2's class, although at the end of the class discussion. One of the teachers (T3) used the question suggested for category (c); however, the teacher added some further questions that might have been confusing for the students. Most of the questions posed on this category are vague and of a more general character.

In relation to category (d), only one teacher (T2) posed a question, "Is this a mathematical explanation?" The reason why T4 did not pose any question might be that a student provided a satisfactory explanation without any request from the teacher.

Challenges due to student response features

Student responses that include one category plus either one more category or some irrelevant information caused trouble during this stage of the lesson. Two of the teachers (T1 and T4) displayed responses including more than one category. Below are descriptions of these instances.

Unfortunately, T1 initially missed to utilize the Presentation view for displaying the selected (and ordered) responses. Instead, the teacher utilized the *Specific item* view, and displayed the first answer in this view, which happened to be one of the answers in P1. The teacher read aloud the response categorized as (c+a):

Because the value of c is m where the line intersects the y axis $y = kx + m \quad m = c$

Then, the teacher asked "Comparing *m* and *c*, is there anyone who can explain this, the thinking behind it?" Since nobody responded to this question, the teacher shifted the focus towards the part of the response that belongs to category (a), and asked "What do you think about that explanation?" In this way, the teacher did not follow the planned (and suggested) sequencing, probably because the displayed response include two response categories.

When displaying the response categorized as (c+e), T4 directed the focus to the category (c) part of the response, despite that category (c) already had been displayed and discussed. S/he asked "But how do we, then, see that c is m?". One student answered by providing a proper category (e) answer, i.e. focusing on the other part of the response. This illustrates how an answer that includes two response categories might influence the discussion in two ways. First, the same kind of response was discussed several times, and second, there was a mismatch between the question posed by the teacher and the student response.

To summarize, the findings indicate that the teachers quite successfully followed the suggested selection and sequencing. However, challenges during the conduct of the whole-class discussion appeared when student responses including more than one category were displayed. Moreover, it was challenging overall for the teachers to follow the guidance in terms of the suggested questions.

DISCUSSION

Since this is a case study, the intention is not to provide generalizable results, but to identify some challenges appearing when teachers utilize CCT to orchestrate a whole-class discussion based on students' computer-based work. In this way, the findings can provide some guidance for future practice and research.

Although the teachers found the CCT useful, the findings indicate that it was a challenge for them to follow the agreed teacher guidance. Particularly, the suggested question to pose in relation to different types of student response were used to a small extent, and when used, they most often were posed in a quite different way. Of course, there could be several reasons for this, not least the teachers' own beliefs and knowledge as well as the classroom norms (Kieran et al., 2012). However, one reason probably was the need for teachers to make in-moment decisions in the classroom. One way to facilitate for the teachers to follow the teacher guidance would be to embed the planned questions into the presentation view together with the student responses. This could also address the challenge for teachers to base the discussion on proper student responses, a challenge also observed by Ruthven and Hofmann (2013).

It was also a challenge for the teachers to follow the teacher guidance, in cases when they selected responses including more than one category. Two problems were observed during the whole-class discussions. First, the planned sequencing became disturbed, and second it became unclear what part of the response that was discussed. One way to deal with this would be to select only "clean" responses, if possible. This raises the question whether technology can support teachers with the challenging task of categorising student responses on the spot. This is an important issue for future research.

In this study, teachers were supposed to implement an agreed lesson design based on a systematic analysis (done by the researchers). In reflection, the study illustrates that this type of implementation is not straightforward. Some reasons for this and what could be done to support teachers have been discussed.

REFERENCES

Cusi, A., Morselli, F., & Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: Design and implementation of digital resources. *ZDM, 49*(5), 755-767. doi:10.1007/s11858-017-0878-0

Fahlgren, M., & Brunström, M. (2018). How the word 'mathematical'influences students' responses to explanation tasks in a dynamic mathematics software environment. In H. G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, N. Faggiano, N. Grønbæk, & J. Trgalová (Eds.), *Proceedings of the 5th ERME Topic Conference MEDA 2018* (pp. 83-90). Denmark: University of Copenhagen.

Fahlgren, M., & Brunström, M. (2019). Orchestrating whole-class discussions in mathematics using connected classroom technology. In B. Barzel, R. Bebernik, L. Göbel, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *Proceedings of the 14th International Conference on Technology in Mathematics Teaching - ICTMT 14* (pp. 173-182). Essen, Germany.

Fahlgren, M., & Brunström, M. (2020). *Connected Classroom Technology to monitor, select and sequence student responses*. Paper presented at the twelfth research seminar of the Swedish Society for Research in Mathematics Education, Växjö, Sweden.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 225-256). Charlotte, NC: Information Age Publishing.

Kieran, C., Tanguay, D., & Solares, A. (2012). Researcher-designed resources and their adaptation within classroom teaching practice: Shaping both the implicit and the explicit. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From Text to 'Lived' Resources* (pp. 189-213). New York, NY: Springer.

Ruthven, K., & Hofmann, R. (2013). Chance by design: devising an introductory probability module for implementation at scale in English early-secondary education. *ZDM, 45*(3), 409-423. doi:10.1007/s11858-012-0470-6

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340. doi:10.1080/10986060802229675

The design principles of an online professional development short course for mentors of mathematics teachers

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This paper describes the design principles and content of an online asynchronous short course contributing to the professional development of prospective Mentors of Mathematics Teachers. We aimed at bringing together the participating teachers' expertise and wisdom of practice, and the evidence from relevant research and professional literature in mathematics education through carefully designed online activities and 'lightly' orchestrated peer collaborations. We expect our course to develop the participating teachers' appreciation of how their gained knowledge from research and literature empowers them to critically reflect on their own teaching practices and on how they support the practices of teachers they mentor.

Keywords: professional development, research-informed mentoring, online community, online learning, subject specific mentoring.

INTRODUCTION

Practicing mathematics teachers in schools in England, UK and all over the world (e.g. Australia, Ireland, Germany) are expected to contribute to the professional development (PD) of less-experienced colleagues. Repeated government calls in the UK require that all school-based mentors are experienced in delivering high quality PD of colleagues, have a deep understanding of the specialist subject required for high quality teaching of the subject and understanding of how teachers develop this knowledge (Cordingley, Greany, Crisp, Seleznyov, Bradbury & Perry, 2018).

However, nationally and internationally support for mentors is sparse and rather generic, at the expense of subject specific support (ACME 2015, Barrera-Pedemonte, 2016). Currently in England, in schools that work in partnership with teacher training institutions, mentors are offered subject specific PD support. This training though is limited to one or two twilight sessions in a year. Furthermore, the mentoring issues discussed and reported in the literature are mostly of a generic nature, i.e. more concerned with general teaching situations rather than with subject-specific teaching or teaching after initial training (e.g. Martin, 1996). According to the findings of the *Developing Great Teachers* review (2018), subject-specific Continuous Professional Development (CPD) that focuses on enhancing teachers' understanding of the subjects they teach; how pupils learn in those subjects; and how to teach them, is more effective in terms of its impact on pupil outcomes, than generic pedagogic CPD.

Additional factors such as time constraints, workload issues, caring responsibilities, costs of PD courses, prevent teachers accessing PD support. In particular, when the PD support is run at specific times in a year and physical attendance is required, teachers may be unable to take advantage of such support. Similar factors also account for a lack of engagement of teachers with the research. Despite an increasing recognition in the UK of the need for teaching to be a research-literate profession, teachers repeatedly indicate that their working conditions do not enable them to spend time reading research to improve their understanding or to determine how to use it to adapt their practice *(*Royal Society and British Academy, 2018).

To address some of the above factors and in particular accessibility issues, using modern technologies to support distance, life-long and online learning has become a common trend (e.g. Chen, 2007). Massive Open Online Courses (MOOCs) and Short Online Courses have made their appearance in the educational field worldwide (e.g. Laurillard, 2014). Laurillard and her colleagues have offered great insights into how best to design such online courses that “provide access to key materials and resources, and the opportunity to exchange ideas and experiences from [participants’] own institutional and national contexts” (p.5) and subsequently enable participants to co-construct knowledge. This opportunity to exchange ideas and experiences and network with peers is of great importance in online learning environments, especially those relying on asynchronous mode of delivery, since it is viewed as a way to compensate for the lack of teacher presence (e.g. Liyanagunawardena, Kennedy & Cuffe, 2015). Murphy and Laferrière (2003) argued that the success of online teacher communities for PD happens when the course materials are high quality interactive instructional materials, valuing what participants experience and contribute to the learning of others. As expected, though, online learning brings a number of challenges. For example, participants’ online learning depends on the quantity as well as the quality of their peers’ postings (e.g. Geraniou & Crisan, 2019). Building a mutual trust between participants and working together to reach a successful learning outcome takes longer in an online asynchronous course compared to face-to-face learning opportunities (e.g. Haythornthwaite, 2002). Such challenges need to be carefully considered when designing online courses and deciding upon the pedagogical strategies for providing an effective learning experience.

Our aim in this paper is to share our learning design and facilitation strategies adopted for an online asynchronous CPD course that offers a functional learning space with appropriate activities for critical self-reflection, meaningful discussions and where appropriate, co-construction of knowledge to take place. We begin by presenting the design of our ‘Key Ideas in Mentoring Mathematics Teachers’ (KIMMT) course that allows participants to learn from the course content and from each other while enrolled on an online asynchronous CPD course.

THE RESEARCH-INFORMED COURSE DESIGN

As mentioned above, our review of the mentor provision in England, but also worldwide, highlighted that there is an ever increased demand of new mentors in schools, a demand for support for mentors that is subject specific, and according to the recommendation of the *Harnessing educational research report (*Royal Society and British Academy, 2018*),* a need of support for *all* teachers to use evidence and insights from research to develop their practice.

Our aim was thus to design a PD course, with the following learning goals for prospective mentors: to be informed by the subject specific maths education research; to engage with the evidence from such research in their own teaching; and to consistently draw on such knowledge in their conversations with the mentees, hence promoting a research-informed teaching practice of their mentees.

To orchestrate learning in such ways, we engaged with a research-informed framework for implementation of active learning practices into an asynchronous online environment, referred to in literature as 'An Architecture of Engagement' (Riggs & Linder, 2016) consisting of: Element 1: Syllabus Communication and Engagement Policy, Element 2: Course Orientation and Element 3: Modular Course Structure. Below we explain how this framework influenced the design of our course.

According to Riggs and Linder (2016), a modular course structure helps to frame the architecture of engagement throughout the course. *As such,* our KIMMT course is designed as a short online course, with activities that spread over five weeks. An orientation for an online asynchronous course introduces participants to the structure of the course (Element 2: Course Orientation, Riggs & Linder 2016), with each week requiring on average about four hours study time, as this is an amount of time manageable by teaching professionals, an argument supported by Laurillard (2014).

The first week focuses on welcoming participants by sharing expectations for online meaningful engagement on this course, informing participants of communication policies and the course schedule (Element 1: Syllabus Communication and Engagement Policy, Riggs & Linder, 2016) and asking them to introduce themselves to the course's online community. The other four weeks focus on powerful pedagogical inter-connected mathematics themes titled as "Fostering Algebraic/Geometric/Numerical/Functional Reasoning". Every themed week consists of an 'Introduction' to the week and the learning goals, followed by three main activities relevant to the respective theme, and finally a 'Concluding' section focusing on "Reflections, Learning Live and Concluding Remarks" (Element 3: Modular Course Structure, Riggs & Linder, 2016).

Our short five-week course (https://www.futurelearn.com/courses/key-ideas-in-mentoring-mathematics-teachers) is available on the FutureLearn platform and was first launched in January 2020. FutureLearn is a MOOC learning digital education platform jointly owned by the Open University, UK and SEEK Ltd, extending thus the access to our course resources to teachers in schools located in the UK and abroad, hence promoting principles of inclusivity in education.

The mode of delivery of this course is online and asynchronous, with ongoing online forum discussions between the participants and supported by us, aimed at promoting an **O**nline **Co**mmunity of practice (Goos, 2014) for prospective **M**athematics **M**entors (OCoMM). This delivery mode facilitates self-paced studying that accommodates more flexibly the various needs of practicing mathematics teachers.

As Laurillard (2014) reported about the design of a MOOC for Teacher CPD, such a course "has an audience who can benefit from each other's knowledge and experience, in addition to the information, ideas and research evidence the team could provide. The approach therefore was to 'orchestrate peer collaborative learning' as well as 'curate the key resources'" (p.12). As such, creating opportunities for participants to share insights from their own teaching practices, to learn about research evidence, and how to apply the newly acquired knowledge to their own practices, were of paramount importance in the design of our course. We achieved these by organizing the content of the course in two main strands, which we refer to as 'The Pedagogical strand' and 'The Research strand', while 'The OCoMM' provided the online space for learning throughout the course.

COURSE CONTENT

The two main interweaving strands of this course, namely the pedagogical and the research strands, are not context specific and as such the course offers learning opportunities to mathematics teachers who hold a variety of views about mathematics, its teaching and learning. Moreover, the research strand of this course draws from international mathematics education sources (handbooks, journals, conference proceedings, book chapters, etc.). We have selected descriptive and experimental research from relevant quantitative and qualitative studies recognised as influential in the mathematics education community worldwide, hence not adhering to a particular theoretical stance or view of mathematics teaching and learning.

The Pedagogical strand

For the *main activities* within each theme, we needed to choose which three pedagogical aspects of the teaching and learning of the maths topics to focus on. Considering the 'Fostering Geometric Reasoning' theme for example, our maths education research background enabled us to choose the most salient aspects of research in the teaching and learning of geometry, namely that geometric thinking and reasoning involve developing, attending to, and learning how to work with geometric images. So, one of the three activities is 'Visualising' and is designed to support participants to learn about the role visualisation plays in one's geometric reasoning and the importance of being pedagogically aware of what pupils 'see' when they 'look' at diagrams.

On the other hand, our experience as classroom teachers and teacher educators informed our design of the activities, each consisting of a number of tasks, called 'steps' in FutureLearn. The first step is **'A Mathematical Problem'**, usually related to a concept or a challenging topic to teach and is presented as a fictional scenario inspired from real life classroom situations that we experience ourselves as teachers, teacher educators, or read about in mathematics education literature. Biza, Nardi and Zachariades (2007) refer to such scenarios they used in the teacher education contexts "as tools for the identification and exploration of mathematically, didactically and pedagogically specific issues regarding teacher knowledge" (p. 308). In this first step we ask the participants to 'think about and suggest a solution to the 'problem' posed' and share their thoughts in the Comments section of each step. We included prompts such as '*What difficulties do you envisage pupils might have in tackling this question and communicating their solution both orally and in writing? When you have posted your response, consider that of another learner and offer your views and opinions on their answer'*. The next step of each activity focuses on reviewing and synthesising maths education research related to the specific maths topic under consideration.

The Research strand

One of our main goals for developing this course was to promote and empower prospective mentors with ideas, suggestions, advice, etc., which are informed by relevant mathematics education specific research. We wanted to offer them the means for reflecting on how such ideas can be applied to their own practice. In this respect, each mathematics-specific situation introduced in the activities of this course is either preceded or followed by a step titled **'What does the research say?'**. In each such step, we provide a selective summary of the research insights and results related to the specific mathematics topics under consideration. This summary consists of a very concise review of the research, where important details of the research studies themselves are left out (a deliberate decision), while references were included for participants to investigate deeper and further.

The review of research is then followed by a task that requires participants to engage with the research step and in the Comments section they are encouraged to reflect on how the reading could possibly help them gain an insight into the presented scenario. This step is important, as it precedes the step where we model for the participants how engagement with research could potentially support teachers in teaching the topic in a way that supports pupils' understanding of the particular concept or topic.

We have done this in a variety of ways. For example, participants are presented with a scenario in which a beginner teacher seeks advice from their mentor about how to address a particular misconception, or cognitive difficulty, or mistake, or flawed reasoning pupils propose. The participants are encouraged to act the role of a research-informed mentor: '*Reflect on your reading so far this week and imagine you are the mentor of the beginner teacher. In the Comments area, share your views on how you would advise them in this situation'*. We want the participants to make sense for themselves of the research and start thinking about ways in which such newly gained knowledge could be applicable to their real-life classroom situations.

In a final step, '**Using research to support pupils' learning'**, we model how the research reviewed could be used in mentor-mentee conversations. These are usually videos of mentors-mentees in conversations, where the mentor makes explicit references to the research reviewed when offering explanations or suggestions to her mentee. This step provides a 'solution' to the 'problem' given to participants in the previous step.

The OCoMM

When designing our course, one of the aims was to establish an online community to complement the learning on the course and ensure sustainability for our course goals. Our long-term goals were: (a) to empower mentors with research informed practice and instil in them a welcoming stance towards mathematics education research, and (b) to encourage peer collaborative learning and sharing good practice through a sustainable online mentors' community of practice.

The literature indicates that there is a need to facilitate and cultivate conditions that will nurture the development of the online community by community members. Learning in such communities does not just happen. As designers, we knew we had to create opportunities to purposefully foster the growth of our online community. Guided by Murphy and Laferrière (2003)'s finding that the success of online teacher communities for PD happens when there are opportunities for teachers to engage systematically and formally in this very process, we built in the design of our course such opportunities. Each activity has a number of online spaces for sharing and discussing ideas, allowing participants to dip into their wisdom of practice and feel that the experiences they bring to the learning on the course are valued.

Similarly, Laurillard (2014) suggested that they promoted engagement with their MOOC's resources by proposing discussions around "'to what extent they could implement a teaching idea shown in a video', and 'how they would overcome the barriers within their own school'" (p.13). The activities are designed such that the participants are regularly prompted to share their thoughts and ideas. Even though reviewing the output of their peers was not a requirement as per the advice by Laurillard (2014), we relied on the participants' own motivation in sharing their ideas and viewing this as an opportunity to reflect upon their own views and potentially reconsider and improve their current teaching and mentoring practice. It is the partnerships and interactions among the participants that define the learning community, and not the digital media, that are used (Riel, 1996). FutureLearn indeed provides the online space where participants 'come together' and interact as and when prompted by the activities of our course. Such interaction fosters the "process of building and rebuilding interpersonal relationships" (Di Petta, 1998, p. 62). We envisaged that our participants not only interact, but also "learn from each other's work, and provide knowledge and information resources to the group related to certain agree-upon topics of shared interest" (Hunter, 2002, p. 96).

CONCLUDING REMARKS IN LIGHT OF THE FIRST COURSE PRESENTATION

One of our main goals for developing this course was to promote and empower mentors with ideas, suggestions, advice on their mentoring and teaching practice, which are informed by relevant mathematics education research.

Reflecting upon the first course presentation and the participant feedback received via an end of course evaluation survey, we recognised that while FutureLearn provided the online platform where participants 'came together', it was the OCoMM that provided the online space where participants started building relationships and learned together.

In 2014 Laurillard claimed that engaging participants fully in online collaborative learning activities aimed at developing a research informed practice was still a challenge. In 2020, we found that the research-informed architecture of engagement framework supported us in establishing an OCoMM, where prospective mentors in particular, but also mathematics teachers in general from around the world, would network, contribute to each other's reflective comments, share experiences, seek advice, co-construct knowledge, and discuss research informed mathematics teaching practice and how it can be applied.

REFERENCES

ACME Advisory Committee on Mathematics Education. (2015). Beginning teaching: Best in class? High-quality initial teacher education for all teachers of mathematics in England.

Barrera-Pedemonte, F. (2016). "High-Quality Teacher Professional Development and Classroom Teaching Practices: Evidence from Talis 2013", *OECD Education Working Papers*, No. 141, OECD Publishing, Paris, https://doi.org/10.1787/5jlpszw26rvd-en.

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education*, *10*(4-6), 301-309.

Cordingley, P., Greany, T., Crisp, B., Seleznyov, S., Bradbury, M., & Perry, T. (2018). Developing Great Subject Teaching: Rapid evidence review of subject-specific continuing professional development in the UK. *London: Wellcome Trust.* Retrieved 11 March, 2020 from https://wellcome.ac.uk/sites/default/files/developing-great-subject-teaching.pdf

Di Petta, T. (1998). Community on-line: New professional environments for higher education. *New directions for teaching and learning*, No. 76 (pp. 53-66). San Francisco: Jossey-Bass.

Geraniou, E., & Crisan, C. (2019). University students' engagement with an asynchronous online course on digital technologies for mathematical learning. In van den Heuvel-Panhuizen, Marja and Veldhuis, Michiel, (eds.) *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (CERME 11). Freudenthal Group, Utrecht University and ERME: Utrecht, the Netherlands.

Goos M. (2014) Communities of Practice in Mathematics Teacher Education. In Lerman S. (ed.) *Encyclopedia of Mathematics Education*. Springer, Dordrecht.

Haythornthwaite, C. (2002). Building social networks via computer networks: creating and sustaining distributed learning communities. In K.A. Renninger & W. Shumar

(Eds.), *Building Virtual Communities: Learning and Change in Cyberspace*, (pp.159–190). Cambridge: Cambridge University Press.

Hunter, B. (2002). Learning in the virtual community depends upon changes in local communities. In K. A. Renninger & W. Shumar (Eds.), *Building virtual communities: Learning and change in cyberspace* (pp. 96-126). Cambridge, UK: Cambridge University Press.

Laurillard, D. (2014): Anatomy of a MOOC for Teacher CPD, Technical report. UCL Institute of Education. Retrieved March 1, 2020 from: https://iite.unesco.org/files/news/639194/Anatomy_of_a_MOOC.pdf

Liyanagunawardena, T. R., Kennedy, E. and Cuffe, P. (2015). Design patterns for promoting peer interaction in discussion forums in MOOCs. *eLearning Papers, 42*. 7. ISSN 18871542

Martin, S. (1996) Support and Challenge: conflicting or complementary aspects of mentoring novice teachers? *Teachers and Teaching: theory and practice*, 2(1), 41–56.

Murphy, E. & Laferrière, T. (2003). Virtual communities for professional development: Helping teachers map the territory in landscapes without bearings. *Alberta Journal of Educational Research, 49*(1), 70-82.

Riel, M. (1996). The Internet: A land to settle rather than an ocean to surf and a new" place" for school reform through community development. ISTE SIG/Tel Technology in Education Newsletter.

Riggs, S. A. and Linder, K. E. (2016). Actively Engaging Students in Asynchronous Online Classes. *IDEA Paper* No.64.

Royal Society and British Academy. (2018). *Harnessing educational research,* Joint report of the Royal Society and British Academy, retrieved from https://royalsociety.org/~/media/policy/projects/rs-ba-educational-research/educational-research-report.pdf

Fostering process skills with the educational technology software MathemaTIC in elementary schools

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This study reports the use of automated tutoring and scaffolding implemented in the module "arithmetic word problem" in the educational technology software MathemaTIC in grade 3 (age 8 to 10). We examined 246 students with access to MathemaTIC and receiving tutoring and scaffolding through a one-to-one learning setting with this technology. The control group (n=226) had access to the same learning tasks and worked with paper-and-pencil without MathemaTIC but with their teachers. Results showed that the experimental group finished with higher outcome scores than the control group. This paper will outline the study and attempts to explain these results.

Keywords: educational technology, process skills, elementary school, mathematics, problem-solving

INTRODUCTION

Teaching arithmetic word problems in grade 3 (age 8 to 10) is reported by the teachers in Luxemburg, as one of the most challenging topics in mathematics. Teachers in our study suggested that students struggle in class to solve arithmetic word problems, due to a lack of comprehension in reading, understanding of the wording and identifying the arithmetic operations to execute. In these tasks, students required process skills such as arguing, communicating, representing, and problem-solving (Selter & Zannetin, 2018). Moreover, based on the result of the national school monitoring EpStan in mathematics and language, students with low reading skills are also those who are low performing in mathematics (Sonnleitner et al., 2018). Similar to the findings of LeBlanc and Weber-Russel (1996), there is a connection between well-developed skills in reading and understanding of the mathematics course language and mastering process skills. Therefore, many students need continuous assistants from a teacher while learning to solve arithmetic word problems. In class, however, the group of students is heterogeneous, and a close follow-up is challenging to realize.

In 2016, the Ministry of Education in Luxemburg developed, jointly with the Canadian company Vretta, an educational technology software for mathematics learning in elementary schools called MathemaTIC. A multidisciplinary team created a module inside MathemaTIC with an automated tutoring system to foster process skills in arithmetic word problems in grade 3 in order to create new learning possibilities and to address the low performances. The instructional design of the module aimed for one-to-one learning in class and at home for students without additional guidance from teachers or parents. We carried out a quantitative study to obtain findings on the use of this arithmetic word problem module by addressing the following two research questions:

RQ1: Are students who learn process skills in arithmetic word problems with MathemaTIC likely to improve at the same degree compared to a traditional paper-and-pencil setting with the guidance of the teachers?

RQ2: What are the limitations and opportunities of a one-to-one setting with MathemaTIC?

Hence, we will present the design of the automated tutoring system and some results of the quantitative study in which we examined 246 students with access to MathemaTIC in a one-to-one learning setting without their teachers compared to a control group (n=226), that had access to the same learning items using paper-and-pencil, but worked with their teachers.

THEORETICAL FRAMEWORK

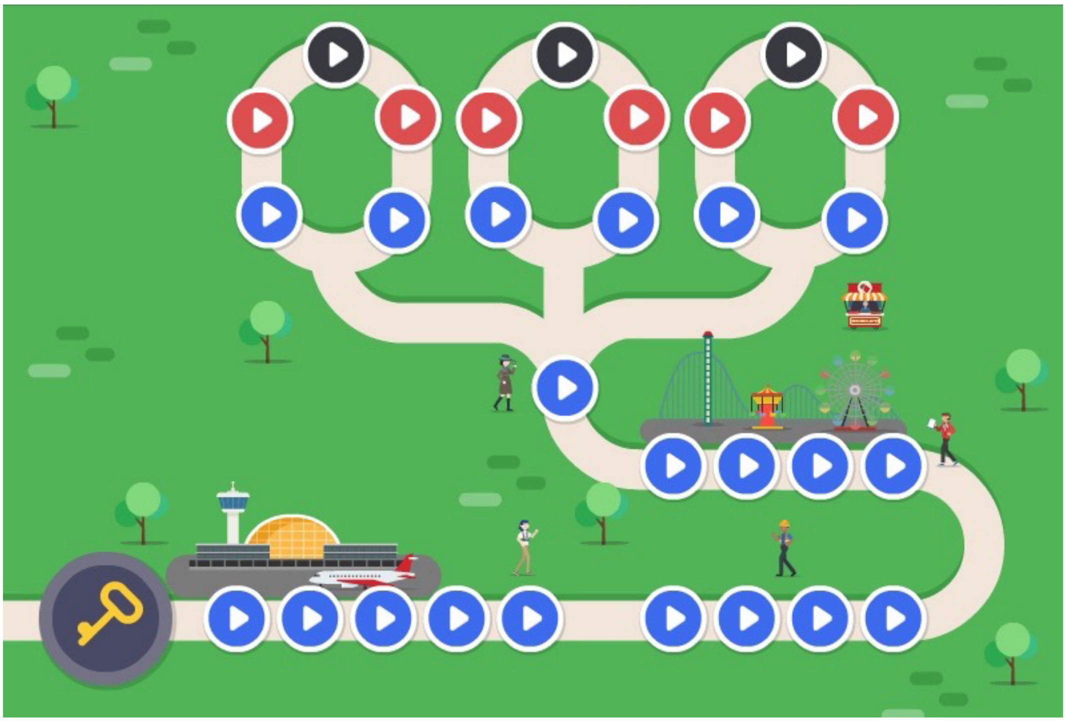

Figure 1: Main view of the module "arithmetic word problem"

The structure of the automated tutoring in the arithmetic word problem module was based on the Competence-Learning-Intervention-Assessment model by De Corte et al. (2004) and the Four-Component Instructional Design (4C-ID) model by van Merriënboer& Kester (2005) for learning complex skills. Both models suggested that students should learn through guided learning tasks and then apply the skills in tasks that are gaining in complexity and lowering in guidance. Furthermore, students should develop a cognitive structure applicable for these complex skills with meta-tools (Trouche, 2004) in the new tasks. Hence, in the module arithmetic word problems (compare figure 1), students started working on guided learning tasks (blue), followed by semi-guided tasks (red) and finally complex tasks without guidance (black). The scaffolding system, based on the multimedia learning theory (Mayer, 2005), consisted in listening to the wordings, interacting with the images and arithmetic operations, and self-regulating their solving process through (non-adaptive) guidance from a fictitious character (one for each of the four solving steps). The tasks were autocorrected, and direct feedback was given to the student.

The different arithmetic tasks were addition and subtraction word problems as recommended in the curriculum for grade 3 (MENFP, 2011), based on the criteria for constructing and solving arithmetic word problems (Franke & Ruwisch, 2010) and structured through the semantic classification of Vergnaud(1982) in transformative, compositional and comparative problems. The tasks were related to situations and places from the students' living environment: "Luc does a bike tour from Luxembourg to Echternach with his 3 friends. The odometer on his bike is at 125 km at the start of his trip. The tour is 42 km long. What will the odometer show when they get to Echternach?".

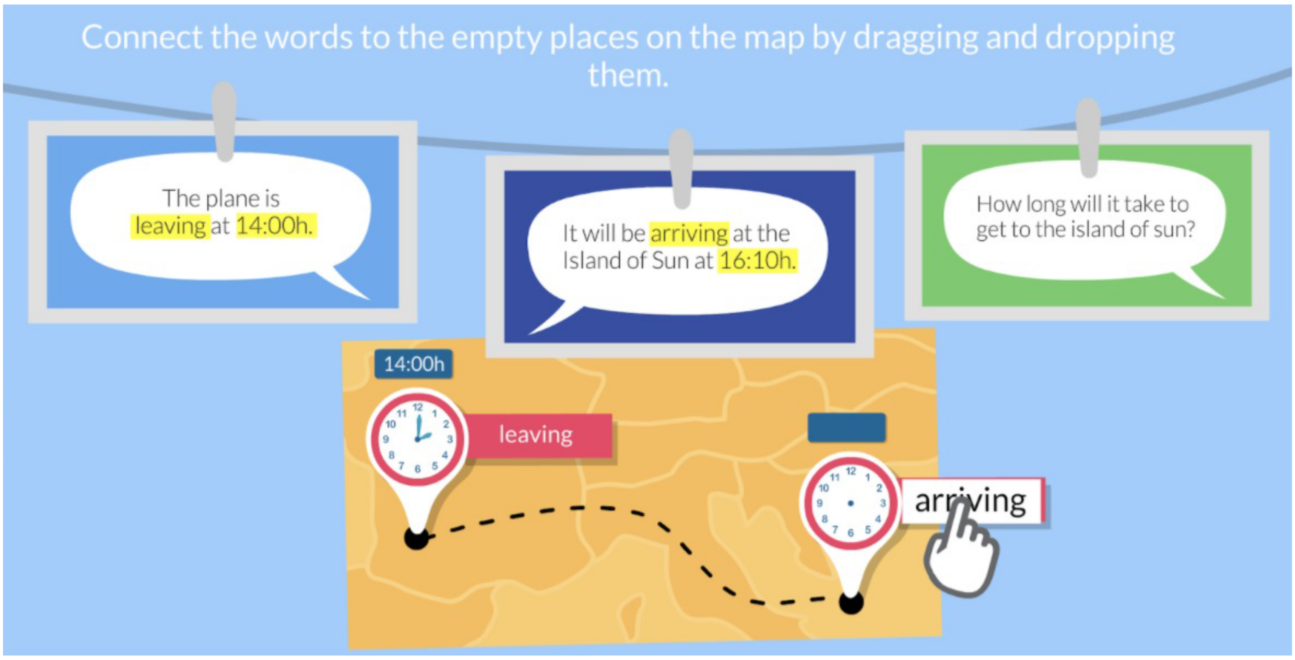

Figure 2: Guided use of a meta-tool: highlighting information and creating a scheme

The first part of the module was dedicated to learning process skills (first 14 "blue" tasks after the key in figure 1). Students practised different process skills separately in guided learning tasks. These tasks then lead to discovery and manipulation of meta-tools supporting the different process skills (i.e. identifying relevant information in the wording with a highlighter tool and creating a resolution scheme, compare figure 2) to make it easier for students to solve the problem.

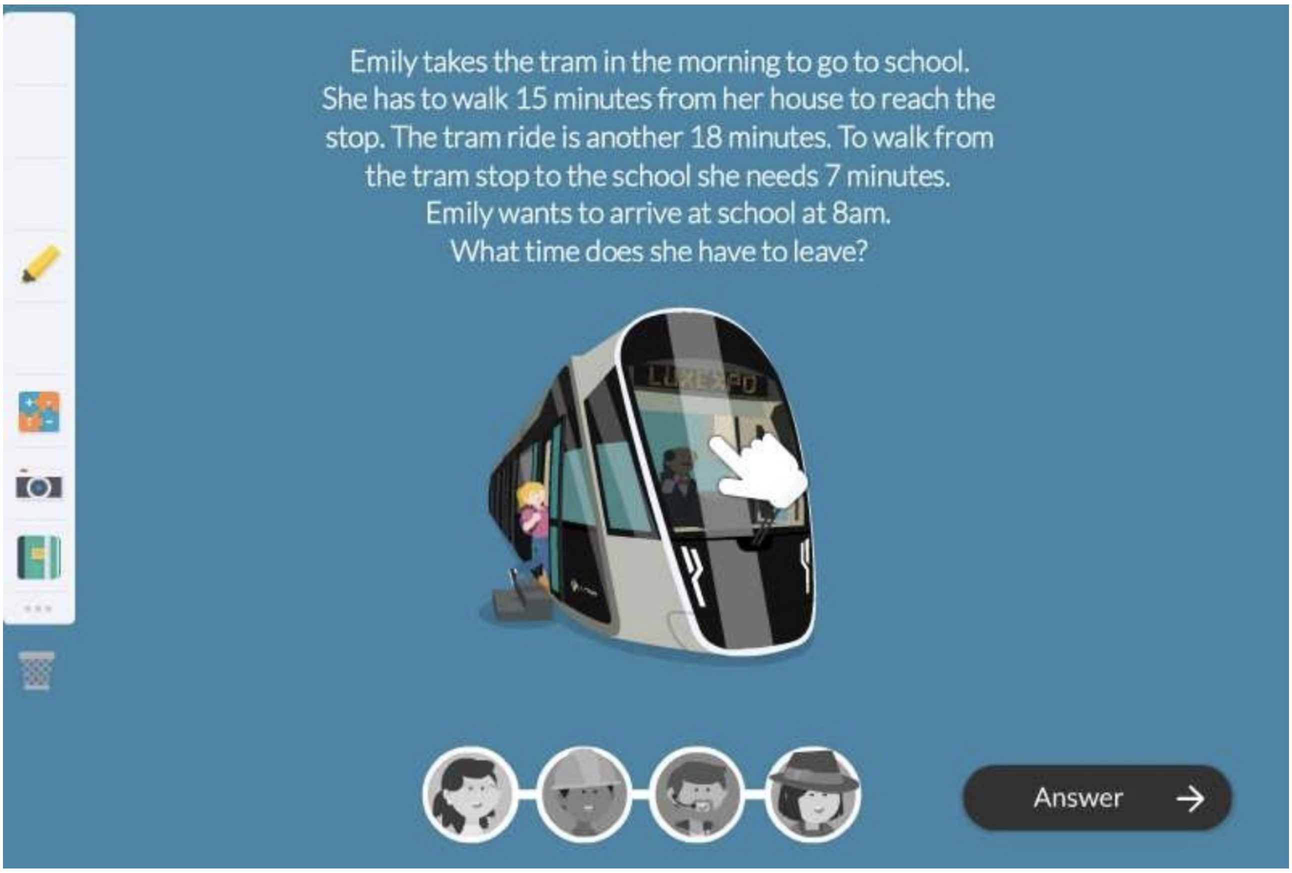

Figure 3: "Black" item without guidance

The second part focused on using the learned process skills in the identified arithmetic word problems (15 tasks organised as 3 ovals on the top of figure 1). Thus, students solved the different typologies of arithmetic word problems (combination, transformation and comparison) with the help of the learned meta-tools (compare menu bar on the left of figure 3). Each typology was presented in a set of three levels from guided tasks (blue), semi-guided tasks (red) to complex tasks (black). In the guided tasks (blue) in each typology, students needed to follow four steps solving procedure using the learned meta-tools: they analysed the wording, modulated the content into a resolution plan, executed the arithmetic operations and verified their results. In the semi-guided tasks (red), students were asked if they wanted to use the meta-tools, but could choose not to utilise them. In the complex tasks (black), they solved tasks with multiple arithmetic operations and no scaffolding was offered. They could use the meta-tools, but without additional guidance.

METHODOLOGY

In this section, we describe the quantitative pre-/post-test approach we utilised to measure if students in grade 3 (age 8 to 10) who learn arithmetic word problems with MathemaTIC in a one-to-one setting are likely to improve at the same degree compared to a traditional paper-and-pencil setting with teachers. The experimental group worked with the arithmetic word problem module in MathemaTIC in a one-to-one setting, without a specific teacher guidance. Their teachers did only ensure access to

MathemaTIC and helped with technical issues. The control group did the same tasks using paper-and-pencil, however with the guidance and assistance of their teachers. Both groups worked for 20 hours (2 hours per week over a period of 10 weeks) on process skills in arithmetic word problems. During the study, we observed three different moments in the learning behaviour of the students within the experimental and control groups. Besides, we interviewed their teachers on their perception of the students' learning with or without MathemaTIC based on the research questions RQ1 and RQ2.

Participants of this quantitative study were 48 randomly selected classes with 667 students in grade 3 (age 8 to 10) in elementary schools in Luxemburg. We used the variables age, gender, and performance in mathematics of EpStan to identify matched pairs and assigned classes to experimental and control groups. We allocated 278 students to the control group, working using paper-and-pencil, and 389 students to the experimental group, working with the arithmetic word problems module in MathemaTIC. At the end of the study, 34 classes with 472 (8 with a missing post-test) students remained: 246 (2) students in the experimental group and 226 (6) students in the control group. The dropout was due to local technical errors (low WiFi signal or non-working hardware) while using MathemaTIC or simply because of a missing post-test for the whole class. Students in both groups performed an identical pre-test and post-test with 15 items based on the different typologies of arithmetic word problems (combination, transformation and comparison) with one or two operations and one item with a combinatorial problem. This combinatorial item allowed us to observe if students would transfer their learned skills into another typology of problem. Both tests have been created based on the experiences from author groups from the national school monitoring and based on the skills from the curriculum.

RESULTS AND DISCUSSION

Results from the experiment suggest that the use of the module "arithmetic word problem" in MathemaTIC is a promising approach to foster process skills in mathematics in a one-to-one setting. The statistical analysis below was carried out using the software R (R Core Team, 2020).

Cronbach's alpha (Revelle, 2019) indicates a good reliability for the pre-test (α=0.774) and the post-test (α=0.787). The detailed analysis shows that dropping one of the 16 items will only slightly increase the reliability for question 1 of the pre-test (α=0.777) and that there are no reverse-scored items. Thus, from this point of view, all items are to be kept in the tests. However, several questions in the pre-test (Q1: 0.16, Q3: 0.27, Q11: 0.24) as well as in the post-test (Q1: 0.26, Q3: 0.26) have an item-rest correlation below 0.3. Hence, they do not correlate very well with the scale overall. The success rates are 81% vs. 76% for question 1 (low difficulty level), 18% vs. 38% for question 3 (high difficulty level) and 3% vs. 9% for question 11 (very high difficulty level). Although extremely easy or difficult items only poorly allow to discriminate, they were needed to sample content and objectives adequately. Thus, we kept all items for further analysis.

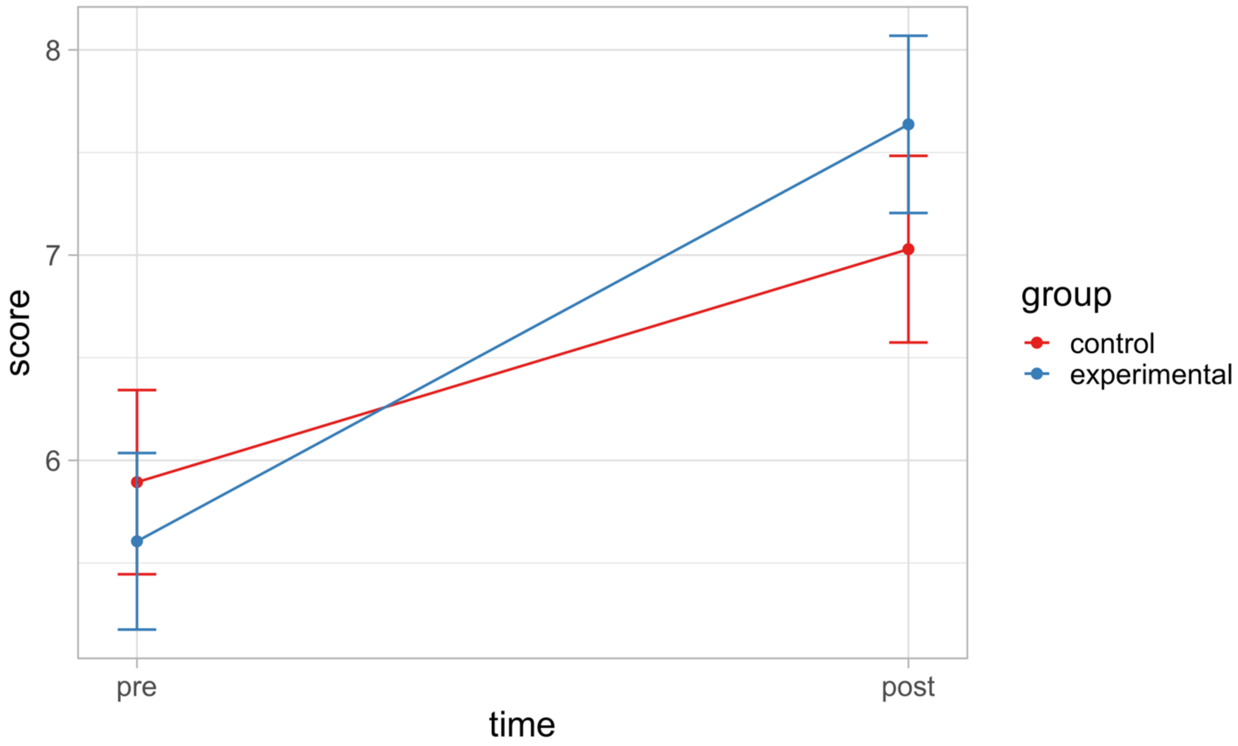

Figure 4: Increase in *score* over *time* for the *group*

We used *lme4*(Bates et al., 2015) to perform a linear mixed-effect analysis of the test result *score* predicted by the fixed effects *time* (pre-/post-test), control/experimental *group* and their interaction as well as the random effect *student*. Visual inspection of residuals plots revealed minor deviations from homoscedasticity and normality, which we accounted for by using bootstrapped confidence intervals. The main effect *time* has an estimate of 1.14 points (95% CI [0.56, 1.72]) for the control group. Thus, post-test scores of students from the control group were significantly higher than those in the pre-test. The main effect *group* has an estimate of -0.29 points (95% CI [-0.89, 0.28]) in the pre-test. On the one hand, both groups were comparable at the beginning of the study, because the confidence interval contains 0, and on the other hand, the experimental group probably had, in the pre-test, slightly lower test results than in the control group. Finally, the interaction effect *time x group* had an estimate of 0.90 points (95% CI [0.15, 1.68]). This effect underscores the fact that the performance gains of the experimental group, working with the educational software MathemaTIC, were significantly larger than those in the control group, resulting in somewhat better post-test performance although starting with a somewhat lower pre-test performance (compare figure 4).

During the classroom observations and the interviews, we were able to collect data in both groups on the motivation, participation and transfer of skills. Thus, in the experimental group, teachers reported that students' motivations to solve and discuss arithmetic word problems were higher than during the regular course (without MathemaTIC) and that they voluntarily exchanged on the tasks after the resolution. Teachers attributed the increase of motivation to the gamification aspect of MathemaTIC as well as the guidance and direct feedback given by the educational technology. According to teachers' reports, some students suggested in other teaching hours (without MathemaTIC) to use the learned process skills to solve mathematical tasks (i.e.: Calculating the area of the classroom floor). In the control group, teachers stated that there was no change in motivation and some students had significant difficulties (i.e. understanding wording or findings of the arithmetic operation) to solve all the given tasks on paper.

CONCLUSION AND OUTLOOK

Our findings highlighted that students in the experimental group improved their performances in arithmetic word problems significantly using the educational technology software MathemaTIC in one-to-one setting. Students learned meta-tools on process skills and successfully solved addition and subtraction word problems in all topologies without the direct guidance of a teacher or a parent. Teachers reported a high acceptance in class and an overall increase in motivation and participation of the students in mathematics courses. Thus, the module on arithmetic word problems in the educational technology software MathemaTIC is a viable alternative to the traditional paper-and-pencil course. Over time it could be a valuable asset to support students individually or in groups or even the entire class with MathemaTIC within traditional courses.

We will perform further investigation on the fixed effects of *gender*, *age*, *nationality*, spoken *language* (L1) and the random effect *school*. Additionally, we will investigate all process skills in detail by performing a qualitative comparison of the pre-test and the post-test in our future analyses. Hopefully, we can further narrow the origin of the observed significant performance gains of the experimental group and we will report these analyses in future publications.

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REFERENCES

Bates, D., Mächler, M., Bolker, B., & Walker, S. (2015). Fitting Linear Mixed-Effects Models Using lme4. *Journal of Statistical Software*, *67*(1), 1–48.

De Corte, E., Verschaffel, L., &Masui, C. (2004). The CLIA-model: A framework for designing powerful learning environments for thinking and problem solving. *European Journal of Psychology of Education*, *19*(4), 365–384.

Franke, M., &Ruwisch, S. (2010). *Didaktik des Sachrechnens in der Grundschule* (F. Padberg, Ed.; 2nd ed.). Springer Spektrum.

LeBlanc, M. D., & Weber-Russell, S. (1996). Text integration and mathematical connections: A computer model of arithmetic word problem solving. *Cognitive Science*, *20*(3), 357–407.

Mayer, R. (2005). *The Cambridge Handbook of Multimedia Learning* (R. Mayer, Ed.). Cambridge UniversityPress.

MENFP. (2011). *Plan d'études: École fondamentale*. Ministère de l'Éducation nationale et de la Formation professionnelle. http://www.men.public.lu/catalogue-publications/themes-transversaux/cen/cens/plan-etudes/fr.pdf

R Core Team. (2020). *R: A Language and Environment for Statistical Computing* (4.0.1) [Computer software]. R Foundation for Statistical Computing. https://www.R-project.org/

Revelle, W. (2019). *psych: Procedures for Psychological, Psychometric, and Personality Research* (1.9.12) [Computer software]. Northwestern University. https://CRAN.R-project.org/package=psych

Selter, C., &Zannetin, E. (2018). *Mathematik unterrichten in der Grundschule: Inhalte—Leitideen—Beispiele*. Friedrich Verlag GmbH.

Sonnleitner, P., Krämer, C., Gamo, S., Reichert, M., Muller, C., Keller, U., &Ugen, S. (2018). *Schülerkompetenzen im Längsschnitt—Die Entwicklung von Deutsch-Leseverstehen und Mathematik in Luxemburg zwischen der 3. Und 9. Klasse*. Luxembourg Centre for EducationalTesting, UniversitätLuxemburg; Service de Coordination de la Recherche et de l'Innovation pédagogiques et technologiques (SCRIPT).

Trouche, L. (2004). Environnements Informatisés et Mathématiques: Quels usages pour quels apprentissages? *Educational Studies in Mathematics*, *55*(1), 181–197.

van Merriënboer, J. J. G., & Kester, L. (2005). The Four-Component Instructional Design Model: Multimedia Principles in Environments for Complex Learning. In R. Mayer (Ed.), *The Cambridge Handbook of Multimedia Learning* (pp. 71–94). Cambridge University Press; Cambridge Core.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. *Addition and Subtraction : A Cognitive Perspective*, 39–59.

New curricular goals and new digital learning tools: conflicting or mutually reinforcing developments?

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Ideas on a new national Dutch curriculum include a shift in the type of aims. Additional to the traditional aims, aims described in terms of active verbs are formulated, like 'to abstract', 'to model' and 'to reason'. At the same time there's a shift going on from books to digital learning tools. In this paper we explore whether both developments conflict with each other and if so, what might be done to turn this into mutually reinforcing one another.

Keywords: curriculum, learning goals, mathematical thinking, digital tools.

DEVELOPMENTS IN CURRICULA: CONTENT AND/OR COMPETENCES?

The international group of experts on science and mathematics education (SME) came to the conclusion that new perspectives for evidence-based policy for SME were needed (UNESCO, 2012). In a report it is argued that additional to traditional emphasis on content in curriculum description, acquisition of competencies like

> (1) thinking mathematically, (2) posing and solving mathematical problem, (3) modelling mathematically, (4) reasoning mathematically, (5) representing mathematical entities, (6) handling mathematical symbols and formalisms, (7) communicating in, with and about mathematics, and (8) making use of aids and tools,

should be described. Curriculum developers are called to strike a balance between learning goals in terms of content and learning goals in terms of competencies. In recent publications both on research and on national curricula the call of UNESCO seems to be acknowledged.

In PISA's definition of mathematical literacy (OECD, 2018) the focus is on active engagement in mathematics. This definition encompasses verbs like formulate, employ, interpret and reason (mathematically). In a model of mathematical literacy in practice which is used by PISA, a set of fundamental mathematical capabilities is mentioned, which consists again of a combination of (nouns derived from) verbs, such as communication, representation, devising strategies, mathematization, reasoning and argument. Speaking about mathematics as an activity has a long tradition (Clarkson, 1968). From the onset in 1990 the Connected Mathematics Project at Michigan State University (Edson, Phillips & Bieda, 2018) wants to design learning environments in which pupils go through the process of exploring, conjecturing, reasoning, communicating and reflecting and need higher-level thinking, reasoning and problem solving. Österman and Bråting (2019) indicate a similar shift in the vocabulary in curriculum documents from nouns like notions, concepts, theories, methods and results to generic competences described in verbs like conceptual understanding, problem solving, reasoning and communication skills. In short there is a tendency to speak in terms of verbs when thinking about mathematics education. Though different authors use different words, some tend to be used by all, like reasoning. Others differ from each other but are somehow connected to each other, like problem solving and mathematization, which is part of problem solving. Verbs like communicating are mentioned often.

LEARNING GOALS: IMITATING OR CREATIVE REASONING

Lithner (2017) shows that in many countries learning materials, teaching and assessment promote rote learning. Pupils are most often confronted with a learning environment in which they can learn facts and simple procedures but not how to find solution methods themselves. Tasks can be solved by imitating given procedures. These kind of tasks do not meet the requirements for learning competencies like reasoning abilities, problem solving and modelling. In a report of the European Commission (Rocard, Csermely, Jorde, Lenzen, Walber-Hendriksson, & Hemmo, 2007) a similar conclusion is drawn: in most European countries pupils often merely reproduce activities and have few modelling activities. In education focus seems to be on memorizing and retaining information, instead of (conceptual) understanding. Since mathematics teaching methods are essentially deductive, the presentation of concepts and intellectual frameworks comes first and is followed by the search for operational consequences. The common teaching in European countries lacks scaffolding the development of before mentioned competencies. In order to make a change other types of tasks are needed. In line with the need for different types of tasks, the research programme on learning by imitative and creative reasoning (Lithner, 2017), might trigger this change. The goal of this research programme is to design tasks that do meet the requirements for the acquisition of competences. Lithner concludes that for teaching it depends on the goals whether one should teach through algorithmic reasoning or through creative mathematically founded reasoning. Learning goals like problem solving need teaching through the latter.

Limiting the focus only to tasks which support the development of competencies, is insufficient. Teachers capability of teaching while using such tasks, is also important. The OECD (2018) notices that teacher seem to lack the needed requirements, as in they don't adopt more active, co-operative and project-based strategies in their teaching. UNESCO (2012) describes several requirements teachers need to have: (1) capability of dealing with the unexpected, (2) identifying the mathematical potential of pupils' ideas and work that have not been anticipated, (3) capability of helping pupils to link their results in particular context to targeted more general learning goals on both content and way of presenting. Instead of knowing in advance what pupils will say, draw or write down, the teacher needs to react on what pupils bring forward. This might be a new insight, new way of saying, or new form of presenting.

Another major recent change in education are digital tools and technology entering education more and more. This change enlarges the number of needed teaching capabilities. The framework that focuses on Mathematics Knowledge for Teaching

(MKT) has evolved into a new theoretical framework Mathematical – technological pedagogical content knowledge (M-TPACK) (Guo & Cao, 2015), in which teachers' knowledge of integrating technology into the mathematics classroom is integrated (Thomas & Palmer, 2014). In the next paragraphs we will elaborate on the combination of pedagogical technological knowledge (PTK) and the scaffolding of the development of mathematical thinking skills.

NEW DIGITAL LEARNING ENVIRONMENTS

Digitalisation is a worldwide development in many aspects of live, not restricted to education. However in education this development goes beyond the digitalisation of learning materials. It also influences the content of education. Time needs to be spent on how to use the digital systems (Gravemeijer, Stephan, Julie, Lin & Ohtani, 2017) and learning goals might need to change in order to prepare pupils for a digital society. The OECD (2018) predicts that the demand for people with non-routine high level cognitive skills will increase; another plead to add learning goals like competences.

New technology tends to result in a call for integrating it into education. Again and again expectation is that the new technology will improve learning. Nowadays ICT is labelled promising and seem to have replaced for example television. In the 20th century expectations on the use of television were high, among others that it would help to clarify mathematics and illustrate it attractively (Ficken, 1958). Recently developers and researchers articulate similar expectations on technology, for example that due to technology pupils learn mathematics better or in a more smooth way (Edson, Phillips & Bieda, 2018), that computer simulations are the basis for attractive learning environments (Vreman-de Olde, 2006) or that they will result in deeper learning due to the possibility of multiple (dynamic) representations (van der Meij, 2007). Since television didn't change mathematics education fundamentally in the past 50 years, we might be a bit more cautious about the change technology will bring. Drijvers, Doorman, Boon, Reed and Gravemeijer (2010) already remarked that integration of technology in mathematics education lacked behind expectations. Hence let us consider in advance more closely what technology can and should bring. In this paper we focus on (1) offering of possibilities that otherwise would be impossible or take too much time, (2) offering of feedback, and (3) presenting information in another way than a one dimensional text.

Offer possibilities that otherwise would be impossible or take too much time

New technology that widens our world always has been a reason to integrate this technology into our lives. Motorised vehicles make it possible to travel larger distances in less time. Airplanes even made it possible to fly on high altitudes. In what way can technology widen pupils' world? Sins (2006) states that computer models can overcome the lack of mathematical skills needed for describing and predicting complex phenomena and enables discussing complex real world phenomena at a higher level. Calculators have already fulfilled several possibilities to take away the calculation from the pupils, for example the calculation of roots, the sine, etc. Computer programs like

Excel make it possible to calculate the mean of large amounts of numbers. What might technologies like Virtual Reality and Augmented Reality add?

Offered feedback

Getting feedback is essential for learning. Output of tools is in itself (implicit) feedback that needs to be interpreted. There's much to discuss on this issue, but we would like to focus on explicit feedback generated by the tool. Feedback can be given on many different aspects like the correctness of the final answer, the chosen solution strategy, the diversity of given possible solution strategies or how the thinking process is presented. It can also have different goals, for example to help pupils discern whether they understand or to enlarge pupils' self-confidence. Teachers are limited in the number of pupils they can provide with feedback in a certain amount of time. In order to have technology support teachers, UNESCO's demands are actual, e.g. technology should (1) have capability of dealing with the unexpected, (2) identify the mathematical potential of pupils' unanticipated ideas, (3) have capability of helping pupils to link their results in particular context to more general learning goals.

Presenting information

In textbooks information is presented in one dimension: from the top side of the page downwards. As a result connections between tasks, definitions, etc. must be made by the pupils or the teacher. It is desirable that information can be presented in such a way that these connections become more visible.

To summarize technology can have large impact on learning if it (1) enables to experience mathematics like in real life, (2) offers feedback needed for all learning goals, and (3) presents information in such a way that connections between units are more explicit.

STATE OF THE ART IN DUTCH MATHEMATICS EDUCATION AND FUTURE WISHES

In 2017 the Dutch government decided that the whole Dutch national curriculum, all subjects at both primary and secondary level, should be updated. At date this curriculum revision is still ongoing. For mathematics this is a chance to make the curricula of primary education and lower secondary education more coherent with each other. In 2015, learning goals for upper secondary education were revised, based on ideas from a report titled "thinking and acting" (vernieuwingscommissie wiskunde cTWO, 2013) which are in line with the tendency of formulating goals in terms of verbs. A major change between the new and previous curricula was the addition of six mathematical thinking skills (Curriculum.nu, 2019): (1) modelling and formulating algebraic, (2) ordering and structuring, (3) analytical thinking and problem solving, (4) manipulating formulas, (5) abstracting, and (6) logical reasoning and proving. To date this addition is restricted to upper secondary education and has not yet been added to the description of the learning goals of primary and lower secondary education. In the proposals for the new mathematics curriculum, which are still under construction, mathematical problem solving, abstraction, logical reasoning, representing and communicating, modelling and algorithmic thinking are added to the curriculum. Another change is the addition of a learning goal on using tools and technology to the traditional content learning goals. The proposal combines traditional content, mathematical activities and technology.

A parallel recent development in Dutch education is the digitalisation of learning materials. Remark the use of the expression ‘digitalisation’ instead of ‘creation of digital learning environments’. Digitalised textbook assignments are only a small part of the many different and diverse possibilities in digital environments. An example of the result of digitalising textbooks is given in figure 1. The digital environment hardly differs from a textbook and doesn’t really offer new possibilities. All animals are drawn the same size, the ruler is already placed, an assisting line is already drawn.

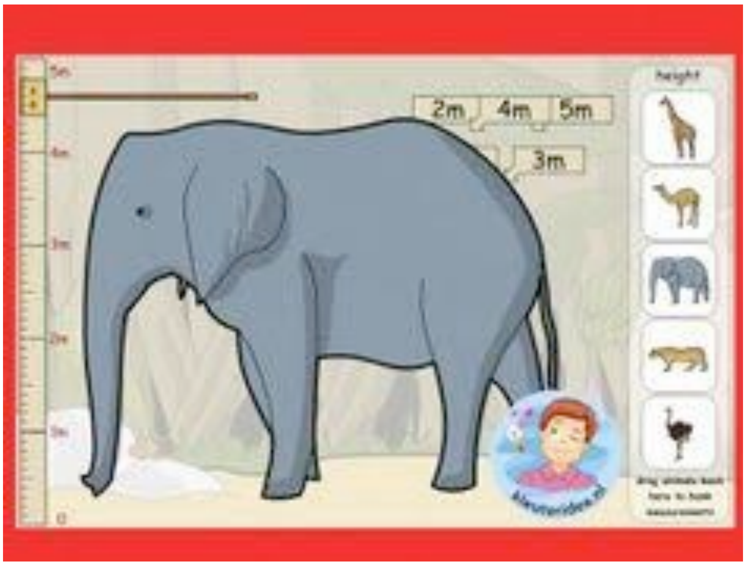

Figure 1: A ‘real-life’ problem in a digital learning tool.

Technology does offer new possibilities in this example, since it is not possible to measure an elephant in a classroom in real-life. Virtual reality has great potential: for large animals pupils would have to reach high, for small animals they would have to bend down in order to measure. It would enrich the learning environment if pupils would have to choose the tool with which they would like to measure. In order to have a more open task, pupils could be asked to choose several ways to order the animals (for example length of tail or length of ear) and draw a conclusion (for example ordering is dependant/independent of the subject chosen to be measured). A reason for current formulation of problems is the frequent absence of possibilities to check such endless variety of possible answers by technology. Digital tools for the most part lack the three capabilities formulated by UNESCO (2012). It is questionable (if possible at all) if it benefits to develop digital tools that anticipate on all possible objects of ordering, since for example the length of tales might result in a different order than height. And what about the criterium ‘identifying the mathematical potential of pupils’ ideas and work that have not been anticipated’?

Technology checking only the final numerical answer doesn't provide feedback on the chosen solving strategy. It is comparable to a teacher asking pupils only their final answer. If a pupil for example answers 8 to the question '5+3=', it is undesirable if this correct answer was reached by counting fingers (if the child is not just mastering the concept). The same is valid for the answer 8 to the question '2x + 3= 19' if it was reached by trial and error. For this reason in the Netherlands at secondary level math tests pupils are asked to write down their elaboration next to the answer most of the time. They are used to do so while working on problems in class on paper. However, many digital tools don't offer the possibility to add elaborations. If adding is possible, it quite often has to be done in a strictly prescribed protocol, e.g. the tool forces pupils to a fixed number of steps via a fixed strategy. These limitations in freedom of elaboration hinder pupils in following their own preferred strategy in their own preferred number of steps. Digital tools would have a great surplus if they would offer pupils freedom in manner of writing their elaboration and if they would give feedback on the elaboration. With the new curriculum in mind, feedback should also be given on aspects like abstraction and reasoning.

Why is there such a prominent demand for digital tools checking answers? Dutch education aims to teach each pupil at its own level. To provide an assignment at the suited level, either the teacher has to select a set of assignments for each individual, which is time consuming, or the digital tool selects assignments. The latest can be done on the basis of the correctness of the answer of the preceding assignment (as argued before it is desirable if this would be on the basis of the correctness and efficiency of the answer and the elaboration). In the Netherlands a consequence of each pupil working on an individual set of assignments has led to classes in which pupils simultaneously work on different topics. This has muted the discourse (which is rare in many Dutch classes anyway), which is amongst others undesirable in the light of the (social) duty schools fulfil to have pupils learn to deal with ambiguity regarding other people's opinion, conceptions, learning methods and speeds. Digital environments might support discourse differently, by having pupils discuss with pupils from other schools working simultaneously on the same assignment or have a discourse over time with their classmates. Digital environments might connect two pupils who both solved assignments correctly but used different strategies. These pupils can discuss the efficiency and effectivity of each strategy. Pupils with different social, economic and cultural backgrounds can be connected to each other. This leads to the need to learn to communicate about ones work and to present owns learning in an understandable way, skills aimed for in the new Dutch curriculum. Tools might also provide the teacher with an overview of which strategy was used by which pupils. Additional the environment can send examples of different solution strategies which the teacher can show on the digital blackboard so all pupils can see them while discussing. Ordering and selecting pupils work by technology, relieves the overload for the teacher. These are examples of new possibilities which could be offered by technology in order to support and strengthen the discourse.

Another consequence of giving pupils assignments on the basis of their performance in the current digital tools is pupils seeing solely the assignment they are working on. Thinking about relations between assignments, is hard. Questions like ‘what is the difference between the assignments in this paragraph from the latter’ or ‘what solution strategy did you use on what type of assignments’ are powerful questions. Digital environments would have a great surplus if they could provide an overview of the assignments. It would, for example, be nice if a pupil could order the environment to show all the assignment in which a certain solution strategy was used. But maybe a first step would be to show a conceptual map and see to which ‘archetype’ the current assignment belongs. Some digital environments in the Netherlands already provide pupils some information about the concept and level an assignment belongs to and what their mastery level of this concept is, but this can be improved.

CONCLUSION

In the Netherlands developments in digitalising learning materials and curricular changes are conflicting at quite some aspects, like the focus on the final numerical answer in digital environments and the new curricular call to focus on capabilities like reasoning. However there are many possibilities that both developments can reinforce each other in future. There’s work to be done!

REFERENCES

Clarkson, D. (1968). Mathematical activity. *The Arithmetic Teacher, 15(*6), 493-498. www.jstor.org/stable/41187429

Curriculum.nu. (2019). *Leergebied rekenen en wiskunde. Voorstel voor de basis van de herziening van de kerndoelen en eindtermen van de leraren en schoolleiders uit het ontwikkelteam rekenen & wiskunde.* https://www.rijksoverheid.nl/documenten/rapporten/2019/10/10/leergebied-rekenen-en-wiskunde.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75,* 213-234.

Edson, A.J., Phillips, E.D., & Bieda, K. (2018). Transitioning a problem-based curriculum from print to digital: New Considerations for task design. In H.-G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, N. Gronbaek, & J. Trgalova (Eds), *Proceedings of the 5th ERME Topic Conference MEDA 2018* (pp. 59-66). Copenhagen, Denmark.

Ficken, F. (1958). The Use of Films and Television in Mathematics Education. *The American Mathematical Monthly, 65*(6), 393-403.

Gravemeijer, K., Stephan, M., Julie, C., Lin, F., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education, 15,* 105-123.

Guo, K., & Coa, Y. (2015). Survey of mathematics teachers' technological pedagogical content knowledge and analysis of the influence factors. *Educational Science Research, 3,* 41-48.

Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *Mathematics Education, 49,* 937-949.

OECD (2018). *The future of skills: as prepared by the OECD for the G20 education working group.* Argentina. http://oecd.org/g20/topics/employment-education-and-social-policies/OECD-Note-on-the-Future-of-Skills.pdf

OECD (2019). *PISA 2018 Assessment and Analytical Framework*, PISA, OECD Publishing, Paris.

Österman, T. & Bråting, K. (2019). Dewey and mathematical practice: revisiting the distinction between procedural and conceptual knowledge. *Journal of Curriculum Studies, 51(*4), 457-470.

Sins, P.H.M. (2006). *Students' reasoning during scientific computer-based modelling. The impact of epistemology, motivation and communication mode.* Amsterdam, the Netherlands: University of Amsterdam.

Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walber-Hendriksson, H., & Hemmo, V. (2007). *Science education now: A renewed pedagogy for the future of Europe.* Belgium: Office for Official Publications of the European Communities. https://ec.europa.eu/research/science-society/document_library/pdf_06/report-rocard-on-science-education_en.pdf

Thomas, M.O.J. & Palmer, J.M. (2014). Teaching with digital technology: obstacles and opportunities. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era. An International Perspective on Technology Focused Professional Development* (pp. 71-89). Dordrecht: Springer

UNESCO. (2012). *Challenges in basic mathematics education*. Parijs: UNESCO.

Van der Meij, J. (2007). *Support for learning with multiple representations. Designing simulation-based learning environments.* Enschede, The Netherlands: University of Twente.

Vernieuwingscommissie wiskunde cTWO. (2013). *Denken & doen: wiskunde op de havo en vwo per 2015*. Utrecht: Commissie Toekomst Wiskunde Onderwijs.

Vreman-de Olde, C. (2006). *Look, experiment, design. Learning by designing instruction.* Enschede, the Netherlands: University of Twente.

A “toolbox puzzle” approach to bridge the gap between conjectures and proof in dynamic geometry [1]

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The paper presents the findings of the analysis of two Danish grade 8 students working together to prove conjectures, which they formulated based on guided explorations in a dynamic geometry environment, in the frame of a design based research project. The case indicates that the designed task can bridge a connection between conjecturing activities in dynamic geometry environments and deductive reasoning. The students manage to explain theoretically, what is initially empirically evident for them in their exploration in the dynamic geometry environment. The proving activity seems to make sense for the students, as a way of explaining "why" the conjecture is true. Certain findings coming from other groups are also presented.

Keywords: Dynamic Geometry Environments, Conjectures, Proof, Toolbox puzzle approach.

INTRODUCTION AND THEORETICAL BACKGROUND

An ongoing issue in the mathematics education research field concerns the role of dynamic geometry environments (DGE hereinafter) in relation to proof. Several studies highlight the potentials of DGE in relation to development of mathematical reasoning, abilities in generalization and in conjecturing (e.g. Arzarello, Olivero, Paola & Robutti, 2002; Laborde, 2001; Leung, 2015; Baccaglini-Frank & Mariotti, 2010; Edwards et al., 2014). However, it is not clear whether such activities in DGE can support students' development of abilities in deductive argumentation. Some studies indicate that the empirical nature of the DGE investigations may impede the progression of deductive reasoning (e.g. Marrades & Gutiérrez, 2000; Connor, Moss, & Grover, 2007). That is to say, once the students have explored a construction in the DGE and discovered some relationship, they may become so convinced by the empirical experience that it does not make sense for them to prove (again) what they "know". However, other researchers suggest that students' explorative work in DGE does not have to risk development of deductive reasoning (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Seemingly, the didactic design surrounding the work in the DGE and the role of the teacher is of utmost importance (e.g. Mariotti, 2012). De Villiers (2007) argues against a common method, which is for the teacher to devalue the result of the students' empirical investigation as a means of motivating students to undertake theoretical validation. Instead, he suggests highlighting the role of proof as an explanation. The teacher may turn the theoretical validation into a meaningful activity for the students as a challenge to explain "why" their DGE investigations are true (de Villiers, 2007). Trocki (2014) suggests that motivating the students to theoretically justify their empirical explorations may also be incorporated into the task design itself.

In light of the ongoing discussion in the field on the role of DGE in conjecturing and proof, the following research question arises: *How can students' conjecturing activities in DGE be combined with theoretical validation, to make theoretical validation a meaningful activity for the students?*

This research question is investigated as a part of a larger design-based research project, in which the overarching mathematical aim is to utilize potentials of DGE in order to support students' development of mathematical reasoning competency, which is a notion from the Danish KOM framework (Niss & Højgaard, 2019). The KOM framework is a competency-based approach to describe what mathematical mastery entails, and it is integrated into lower secondary school curriculum as well as most other educational levels in Denmark. The mathematical reasoning competency includes abilities concerning reasoning, conjecturing and proving (Niss & Højgaard, 2019, p. 16). The specific designed task that is reported upon in this paper aims at bridging a connection between students' conjecturing activities in the popular DGE software, GeoGebra, and proving. However, diverging understandings exist regarding the meaning of the notion of proof in a teaching and learning context (Mariotti, 2012; Balacheff, 2008). Therefore, I will briefly impart what is implied by the notion of proof in the context of school mathematics, both in the research field and in this project.

Mariotti (2012) elaborates on different understandings of proof in school context and unfolds two extremes; 1) proof as the product of theoretical validation of already stated theorems, and 2) proof as the product of a proving process, which includes exploration and conjecturing as well as proving conjectures. Sinclair and Robutti (2013) state that the view on proof in the context of school mathematics has largely shifted to comprise proof as a process, and that this may in part be attributed to the facilitation of experimentation provided by digital technologies. The KOM framework does not address proof using the same terminology, however, proof as a process resonates with the emphasis stated in the KOM framework concerning the ability to investigate and do mathematics (Niss & Højgaard, 2019). Therefore, in this project, proof is understood as a process that includes exploration, conjecturing and deductive reasoning.

In the following sections, the method and educational context of the study is explained, followed by a description of the task design principles. Then a case is presented of two students working together on the task, followed by an analysis of the data. Finally, some conclusions are made concerning the specific case, but also referring to results coming from other groups and to research aims going forward.

METHOD

The research project is anchored in the frame of design-based research methodology (Bakker & van Eerde, 2015). Based on analysis of DGE literature, a hypothetical learning trajectory was proposed (see more in Højsted, 2019; 2020a), leading to the development of a didactic sequence that included 15 tasks. The sequence design was also influenced by results from a survey (Højsted, 2020b). The didactic sequence was tested and redesigned in three design cycles in three different schools that each lasted approximately three weeks (14-16 lessons). The data presented in this paper is from the second design cycle. To investigate the research question in this paper, a “toolbox puzzle” task was designed with the aim of supporting the students to first formulate conjectures based on guided investigations in GeoGebra, and then to undertake theoretical validation of the conjectures.

Data from each design cycle was acquired in the form of screencast recordings of the students’ work in GeoGebra; external video of certain groups (chosen in collaboration with the teacher to comprise a spectrum of high-low achieving students); and written reports that were collected from the students.

In this paper, data is analysed from one pair of students, Ida and Sif, in order to investigate to what extent the toolbox puzzle design supports them in proving their conjectures and if the activity seems meaningful to them. Some results coming from other groups is also mentioned in the conclusion

EDUCATIONAL CONTEXT

The study took place in an 8th grade (age 13-14) mathematics classroom in Denmark during a period of three weeks. The students had some previous experience using the geometry part of GeoGebra, which is common in Denmark, since ability in relation to dynamic geometry programs are highlighted in the curriculum *mathematics common aims* already from grade 3 (BUVM, 2019). However, the students had no experience related to theoretical validation of conjectures or theorems, which is not surprising since it is almost non-existent in lower secondary school in Denmark, which is evident at curriculum level, in textbooks and in practice. In that light, it is no shock that Jessen, Holm and Winsløw (2015) found that Danish upper lower secondary school students lack in reasoning abilities.

TASK DESIGN

The initial tasks in the didactic sequence were designed to highlight the theoretical properties of figures, and how they are mediated by DGE in the form of invariants, e.g. by constructing robust figures in “construction tasks” (Mariotti, 2012). In subsequent tasks, the students were engaged in constructing and investigating the constructions in order to make conjectures. Generally, the design heuristic of Predict-Observe-Explain (White & Gunstone, 2014, p. 44-65) was applied to some extent in most tasks. The students were required to make a prediction concerning some geometrical properties, and to justify their prediction. Afterwards, they were to report what they observed and explain in case there were differences between prediction and observation, leading to conjectures about the geometrical constructions. Afterwards, the students were expected to explain why their conjectures were true. They were provided with a toolbox (on the right in figure 1) that contained theorems to be used in their argumentation. In the design, theoretical validation was portrayed to the students as an activity of finding out and explaining why conjectures are true, as suggested by de Villiers (2007). The toolbox was introduced to the students as a helping hand of already established truths comprising the necessary clues to find out why their conjectures were true, much like pieces to solve a puzzle.

The task [2] reported upon in this paper consisted of an initial construction part, followed by questions (Predict-Observe-Explain) to guide the students to discover and make a conjecture about the relationship of an exterior angle of a triangle with its interior angles. Finally, the students were encouraged to explain/prove the conjecture in a proof sheet (on the left in Figure 1), using a toolbox, which contains a support drawing as well as information (angle over a line is 180°, and the angle sum of a triangle is 180°) to be used in the argumentation.

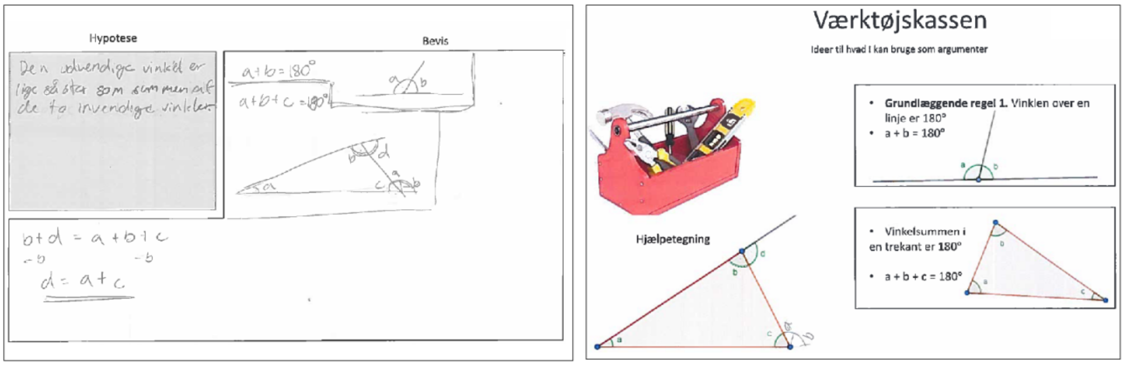

Figure 1: The proof sheet and toolbox. Solved by Ida and Sif

THE CASE OF IDA AND SIF

Ida and Sif were described by their teacher as medium to high achieving students. In the previous task, they found the proving activity and the toolbox to be confusing. The following excerpt ensues after Ida and Sif have constructed the figure from task 9, they have guessed, investigated and put forward the correct conjecture (9a-9f) and are about to try to explain/prove why it is true (9g):

516	Ida	The sum of the two interior angles… [*Writes the conjecture in the proof sheet (Figure 1) translated: "The external angle is as large as the sum of the two internal angles"*]
517	Sif	Beautiful! Okay, now we have to prove it. Oh no…
519	Sif	Now that again…
520	Ida	*a* plus *c* equals *b*, and see. Basic Rule 1: The angle over a line is. The angle sum of a triangle is. [*reading from the tool box*]
521	Sif	Yes! I understand. Look… [*points to the support drawing in the tool box*]
522	Ida	Ohh.
523	Sif	Super! In here, that's what's missing. [*points to angle b in the support drawing (see figure 2)*]

Figure 2: Using the toolbox to explain

…

533	Sif	And add this one here, to here. [*pointing to angle b being added to a+c and to d respectively*]
534	Ida	That's right, so it makes 180. AND it makes sense. Is there more to say?
535	Sif	That is… just how it is.
536	Ida	We know that the sum in a triangle is 180 degrees and that the sum… the angle sum of a line is 180 degrees. Therefore, when we are missing an angle here…

In the events that follow, they write their answer (Figure 1), but express difficulty in doing so, because they expect that they must use algebra in their answer:

574	Ida	How do we write that in mathematical language?
…		
595	Ida	Ah okay! And *a* plus *b* and *c* yes. And *b* plus *d* it also gives 180
597	Sif	This one plus this one, is the same as these three. [*pointing to b+d and a+b+c*]
598	Ida	That's right. It's actually right. Oh, *b* plus *d* equals *a* plus *b* plus *c* because this makes 180, and this makes 180.

Analysis

We can notice from lines 517-519 that Sif is not excited about the prospect of having to prove the hypothesis. In fact, it was observed in several groups, that the activity of theoretical validation was not enthusiastically undertaken immediately. It was also evident, that the proving part was the most challenging part of the task, which may partly explain the lack of enthusiasm. However, the mood towards the proving activity changed in the case of Ida and Sif, and in some other groups as well, when they had worked on 2-3 tasks of this type, which indicates that they had to get accustomed to the task design. Some of the difficulty may be attributed to the openness and unfamiliarity of the answer format, since several students could put forward their reasoning verbally, but struggled to write down their argumentation. Ida and Sif also struggle with this issue (line 574-590). However, they find it easier to write the answer in subsequent tasks, after the teacher explained that they could write their arguments using natural language narratives.

In line 520, we see that Ida immediately turns to the toolbox information, reading aloud the two pieces of information provided, which indicates that she has realised the usefulness of the toolbox. Sif listens and seems to recognize that adding angle *b* to *a*+*c* and *d* respectively in both cases gives 180° (line 521-533), which she manages to support Ida to grasp and elaborate as well (line 534-536). They manage to reason deductively that their conjecture is valid, and after some struggle, write their answer algebraically (Figure 1). The sequence of utterances from the students indicate that it is a sense making activity for them, and that there seems to be intellectual satisfaction attached to their experience (line 534-536).

CONCLUSION AND FORTHCOMING REPORTS

The study indicates that the "toolbox puzzle" approach can bridge a connection between conjecturing activities in DGE and deductive reasoning. The students explained theoretically, what they initially guessed purely visually and secondly investigated empirically in DGE. Importantly, the activity of conducting the theoretical validation seemed to make sense to them.

It was apparent that Ida and Sif had to become acquainted with the structure of the toolbox puzzle approach, before it became a sense making activity for them. This point was also evident in other groups. Additionally, several groups found it difficult to write down their arguments even though they could convince each other verbally and with the help of gestures.

Most groups of students succeeded and seemed to enjoy the exploration and conjecturing part of the tasks in the sequence. However, medium-low achieving students struggled to string together coherent deductive reasoning, and some never managed to overcome the toolbox puzzle part of the task on their own.

Other aspects of interest in this study is to what degree the students use DGE as they are trying to make a deductive argument, and what role the DGE plays in this regard. There are some indications that the students go back to the DGE in order to exemplify arguments to each other. Notably, early analyses also show that some students return to DGE in order to verify what they have proven(!). In that case, even though proof as an explanation makes sense to the students, it does not highlight the status of their product. I.e. the value of theoretical validation is not yet appreciated. This interplay between the theoretical validation and ensuing DGE actions will be the focus of attention in the ongoing project, which I hope to report on in future publications.

NOTES

1. A short earlier version of this paper was accepted for presentation at the 14th International Congress on Mathematical Education (ICME-14).

2. Task 9 **a**. Construct an arbitrary triangle and extend one of the sides. **b**. What is the relationship between the exterior angle c and the interior angles a and b? Guess first before you measure! [*There is a figure with the mentioned angles on the task sheet*]. **c**. Measure the angles and find the

relationship. **d**. Drag to investigate which situations the relationship applies to. **e**. Discuss with your partner and make a conjecture about the relationship between the exterior angle and the interior angles. **f**. write the conjecture in the proof sheet. g. You can see in GeoGebra that it is true, but can you explain why it is true? Use the information from the TOOLBOX to argue.

REFERENCES

Arzarello, F., Olivero, F., Paola, D. & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, 34: 66. https://doi.org/10.1007/BF02655708

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM - The International Journal on Mathematics Education* 40(3), 501-12.

Baccaglini-Frank, A. & Mariotti, M. A. (2010). Generating conjectures in Dynamic Geometry: The maintaining-dragging model. *International Journal of Computers for Mathematical Learning*, 15(3), 225–253.

BUVM (Børne- og Undervisningsministeriet) (2019) *Matematik Fælles Mål* (Mathematics common aims). https://emu.dk/sites/default/files/2019-08/GSK%20-%20F%C3%A6lles%20M%C3%A5l%20-%20Matematik.pdf

Connor, J., Moss, L., & Grover, B. (2007). Student evaluation of mathematical statements using dynamic geometry software. *International Journal of Mathematical Education in Science and Technology*, 38(1), 55–63.

De Villiers, M. (2007). Proof in Dynamic geometry: More than Verification. *9th International Conference in Mathematics Education in the Global Community*

Edwards, M. T., Harper, S. R., Cox, D. C., Quinlan, J. & Phelps, S. (2014). Cultivating deductive thinking with angle chasing. *The mathematics Teacher*, 107(6), 426-431.

Højsted, I. H. (2020a) Utilizing affordances of dynamic geometry environments to support students' development of reasoning competency. *Nordic Studies in Mathematics Education*, 25(2), 71–98.

Højsted, I.H. (2020b) Teachers Reporting on Dynamic Geometry Utilization Related to Reasoning Competency in Danish Lower Secondary School. *Digital Experiences in Mathematics Education* 6, 91–105. https://doi.org/10.1007/s40751-020-00059-3

Højsted, I. H. (2019). Guidelines for the design of didactic sequences utilizing dynamic geometry affordances related to mathematical reasoning competency. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). (2019). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6 – 10, 2019)*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Jessen, B. E., Holm, C. & Winsløw, C. (2015). *Matematikudredningen – udredning af den gymnasiale matematiks rolle og udviklingsbehov.* København: Institut for Naturfagenes Didaktik, Københavns Universitet.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning,* 6, 283–317.

Lachmy, R. & Koichu, B. (2014). The interplay of empirical and deductive reasoning in proving "if" and "only if" statements in a dynamic geometry environment. *The Journal of Mathematical Behavior*, 36, 150-165

Leung, A. (2015). Discernment and Reasoning in Dynamic Geometry Environments. In: Cho S. (Eds.) *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 451-469). Springer, Cham.

Mariotti, M. A. (2012). Proof and proving in the classroom: Dynamic Geometry Systems as tools of semiotic mediation. *Research in Mathematics Education*, 14(2), 163-185, DOI: 10.1080/14794802.2012.694282

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87–125.

Niss, M. & Højgaard, T. (2019) Mathematical competencies revisited. *Educational Studies in Mathematics*, 102(1), 9-28. https://doi.org/10.1007/s10649-019-09903-9

Sinclair, N. & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In M. Clements, A. Bishop, C. Keitel-Kreidt, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 571–596). New York: Springer.

Trocki, A. (2014). Evaluating and Writing Dynamic Geometry Tasks. *The Mathematics Teacher*, 107(9), 701-705. doi:10.5951/mathteacher.107.9.0701

White, R.T. & Gunstone, R. F. (2014). *Probing Understanding.* London: Routledge.

Designing periodic logos: a programming approach to understand trigonometric functions

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This paper addresses the problem of students' understanding of trigonometry and analyses students' meaning making process while engaging in an alternative learning approach centered on periodicity. The research studies the potential of a programmable 3D modeller called MaLT2 for exploring and understanding trigonometric functions through their periodic nature and its artistic quality. It describes an empirical study investigating students' meanings while using the digital medium in order to construct periodically animated logos. It has only been implemented in a pre-pilot level to a small group of 9th and 10th grade students in order to gain important feedback regarding students' produced meanings for the design of the main research.

Keywords: trigonometric functions, periodicity, digital media, artistic design.

INTRODUCTION

Trigonometry can be seen as a mathematics field where diverse mathematical concepts reside. The fundamental trigonometric components, sine and cosine, can be perceived through different representations, namely the right-angled triangle, the unit circle and the graphical and functional representation. They can be examined through different mathematical domains like algebra, geometry and mathematical analysis. With regards to students' learning, the complexity of these concepts and the diversity of their approaches led to many difficulties in their understanding. However, the existing literature on learning and teaching trigonometry is sparse compared to other mathematics fields and is mainly focused on addressing these difficulties (Chin, 2013).

There is a range of research emphasizing the meaning making processes of students while engaging with tasks outside the traditional school curriculum structure. Adopting an approach not necessarily bound by the ways mathematical concepts are organised in traditional curricula raises worthwhile questions regarding the understanding of trigonometric functions through them being put to use by students while engaging with expressive and explorative digital tools. Existing literature suggests that digital technology has enabled the access to hitherto obscure and inaccessible properties of many mathematical concepts and the creative engagement of students with rich meaning making processes on them (DiSessa, 2001, Hoyles & Noss, 2003). This particular study places periodicity in the centre of students' exploration, forming an alternative approach for investigating trigonometric functions. Approaching periodicity as a main property of trigonometric functions can also bring to light other elusive algebraic, geometric or analytical properties of these functions, as well as the links between them. Moreover, applications of periodicity possess an artistic quality which is able to reinforce students' creativity and imagination. Periodic behaviour can be rich in artistic features like symmetry, uniformity and harmony which can be externalized through digital tools within an animated artefact.

In this study, we examine the potential of a digital medium, called MaLT2, which introduces the integration of three tools: 3D graphics, a Logo programming language and dynamic manipulation of variable procedure values. The integration of these three tools in this particular way provides an environment interesting enough to consider as a distinct representation. Programming to create figural animated models can be employed to represent the periodic nature of trigonometric functions and also to provoke students' creativity and imagination. That way, MaLT2 can play a twofold role; for expressing both mathematical and artistic ideas. Thus, a question emerges: what kind of mathematical meanings can be produced by students while engaging with MaLT2 for expressing an artistic idea around periodicity?

In order to answer this question, we designed an experimental activity with MaLT2, which was implemented to a pilot level to a small group of 9th and 10th graders. Then we searched for students' instances that indicate meaning making on mathematical concepts used. These instances are briefly presented in the results in order to provide elaboration of this process - which will hopefully be further developed in the future.

THEORETICAL FRAMEWORK

Our theoretical approach concerning task design and the ensuing study of student activity is socio-constructionist (Harel & Papert, 1991; Kafai & Resnick,1996, Kynigos, 2015). We give particular attention to learning through the construction and tinkering of personally meaningful artefacts (Papert, 1980). Constructionist design aims at fostering students' creativity and meaning making on mathematical ideas used in the creation of artefacts (Healy & Kynigos, 2010). These terms, which play a pivotal role in this study, are both defined in a constructionist perspective. Mathematical meaning making is approached as the way that a student understands, uses and thinks of a certain mathematical concept, forming a unique dimension for every student (Kynigos et al, 2020). Under this scope, creativity is conceived as construction of math ideas or objects which can be expressed through exploration, modification and creation of digital artefacts.

The social aspect is also an important component for the analysis of students' meaning making in our study. Our aim is for students to embody the role not only of the creator and the designer, but also that of the publisher, who is encouraged to externalize his tacit ideas (Kafai, 2006). According to Kafai (2006), when artifacts are published intensively and densely in a learning collective, meaning making process happens naturally.

DIGITAL MEDIUM AND TASKS

MaLT2 (Machine Lab Turtle-sphere, http://etl.ppp.uoa.gr/malt2/) is an online environment of our lab's design which integrates a UCB inspised Logo textual programming (Harvey, 1985) with the affordances of dynamic manipulation and 3D graphics (Reggini, 2985, Kynigos & Grizioti, 2018). Dynamic change of figural models by manipulating procedure variable values enriches the opportunity for constantly exploring new properties, formulating assumptions in order to create a personal expectations, getting instant feedback on them. In this way students engage in a never-ending loop of interaction with the artefact. In this circle of exploration, mathematics becomes a tool for expressing ideas. But its most important feature is the potentiality of the artefact being personally meaningful for a student; as meaningful as a painting is meaningful to its painter.

The tasks were divided into two phases of an “artistic challenge”. The first phase included decoding and reconstructing a given 2D animated logo specially designed by us. It represented a periodically reshaping right-angled triangle, as its perpendicular sides included sine and cosine functions of a variable (t), which corresponded to one of its acute angles. The only accessible sources were the virtual outcome in motion while changing the values of (t) by dragging a cursor, as well as the corresponding length of each side at every instance (Figure 1). The second phase involved free construction of a new animated logo with the added requirement for it to be 3D. The last part of the activity included presentation of the constructed artefacts and voting for the most impressive one, according to students’ criteria. The main hypothesis beneath this challenge is that by designing a geometric logo and its motion by themselves through programming, students would physically grasp the essence of the trigonometric functions and experience their inner properties in a meaningful and creative way.

PRE-PILOT IMPLEMENTATION OF THE RESEARCH

The methodological tool being used for the main research is that of “*design experiments*” (Collins et al., 2004). It is found on designing and implementing an educational intervention in classroom and searching for relations between the learning process and the use of digital media by students during the implementation phase. Up to this time, the idea described above is formed in an initial experimental stage and only involved five students of 9th and 10th grade, working together in two groups (three 9th graders at the first group and two 10th graders at the second) in an out-of-school context. Students engaged in a 3-hour-activity, during which they were encouraged to use the digital tools as both explorative and expressive mean. They were already familiarized with the environment of MaLT2 and its functionalities.

During the first phase, students explored the changes of the periodically moving triangle and managed to uncover the -as mentioned by one of them- “bizarre functions” which caused the fluctuation of the triangle’s perpendicular sides; sine and cosine. This “uncovering” was reinforced by students’ intimacy of trigonometry in the triangle model, but they extended its borders; they handled sine and cosine as functions, emphasizing to their dynamic features. They collaborated in order to produce the code that constructs the requested shape (Figure 1).

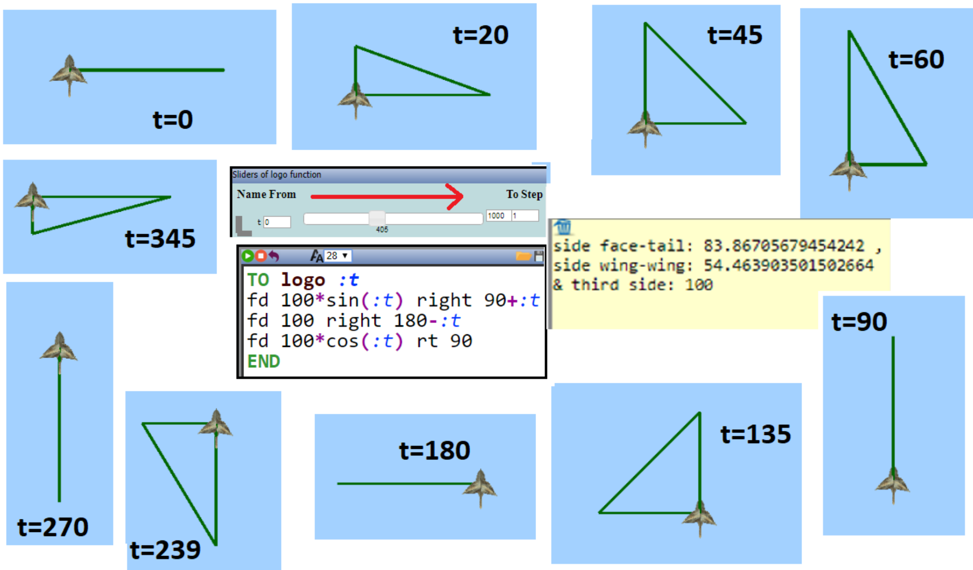

Figure 1. Instances of the animated 2D-logo in Phase 1.

Student 4:	So we have only one variable and we need to express both angles and sides of the triangle? Apart from the hypotenuses which always equals 100.
Researcher:	Exactly. Can you recall any way to connect them?
Student 5:	(…) What if we use sine or cosine? (…) If “t” is one of the angles, then everything can be expressed by it!
Student 1:	(…) That’s it! Oh my, this is so cool! How can sine and cosine do that?
Student 4:	If we just use the command “sine t” alone and change the value of t, we get this up and down movement. I hadn’t realized this about sine before!

By dragging the slider which controls the values of the variable (t), they realized the dynamic power of sine(t) and cosine(t) on their own construction. The revealing periodicity impressed them and consisted of a strong motive for making more complicated constructions.

During the second phase, both groups used the trigonometric functions in their designs. Even though students had only engaged with the trigonometric concepts as ratios (in terms of a triangle or the Cartesian coordinate system), they naturally adjusted to their functional aspect and its properties. They perceived these concepts as functions that cause the periodical fluctuation of a segment and exploited them to formulate geometric 3D figures whose sides move (change length) periodically through the changes of a variable. They experimented with various mathematical ideas and each time they were improving their previous construction in order to achieve a desirable result. Their engagement with the digital medium awakened their creativity as they were exploring, modifying and creating new artefacts filled with mathematical features.

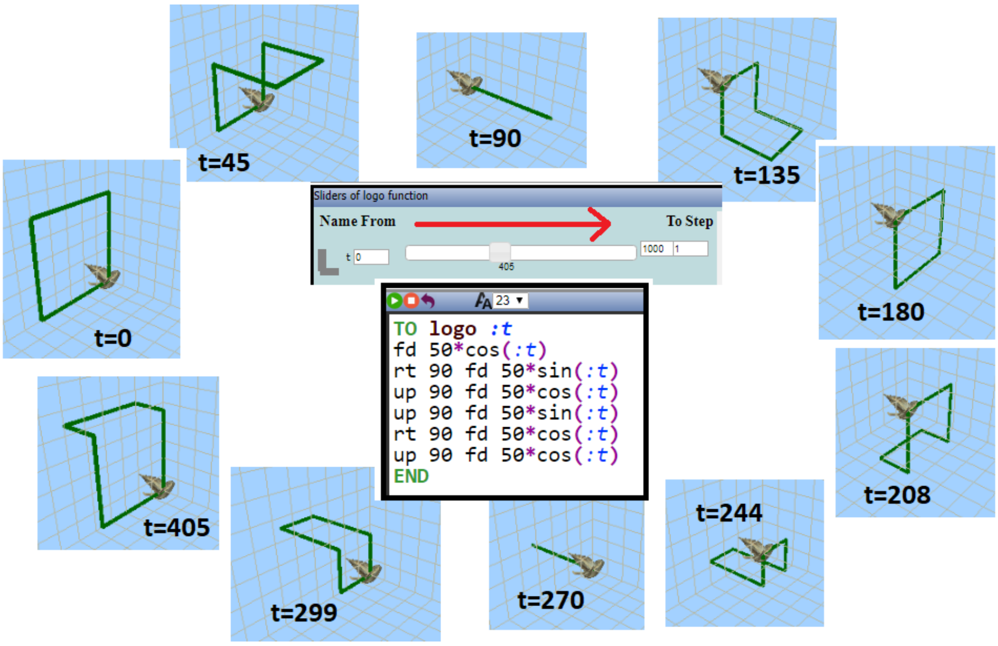

Figure 2. Instances of the animated 3D logo as constructed by the first group

The first group ended up constructing the animated logo presented in Figure 2. They experimented a lot with the complementary relation between sine and cosine and created a harmonically moving artefact based on the discovered trigonometric properties. The following part of their conversation reveals some steps of their experimentation:

Student 1:	We could also use cosine apart from sine.
Student 2:	Won't it be exactly the same?
Student 1:	(Adds the command "fd 50*cos(:t)" in their code.) It's almost the same. Only some seconds late. (…) When the cosine is the biggest the sine disappears. It becomes zero.
Student 3:	(…) We can use cosine as height and sine as length to make a folding rectangle.

Students made the "folding rectangle" and observed its motion while changing the values of the variable t. This observation led to further exploration and meaning making on trigonometric properties, such as the domain and codomain of trigonometric functions and their rate of change:

Researcher:	How would you describe this motion?
Student 3:	The rectangle is like folding and unfolding periodically.
Student 1:	When sine(t) takes the biggest length, cosine(t) takes the lowest; zero. (…) The highest sine(t) can be is 1 since 50·sine(t) is 50.
Researcher:	For which values of t does this happen?
Student 1:	When t=90, 270, 450, 630... (…) So basically all the odd multiples of 90.
Researcher:	What about the speed of the fluctuation?
Student 3:	(While dragging the slider of t steadily.) It seems it isn't steady. I think it's faster when it goes from the biggest to the smallest length.

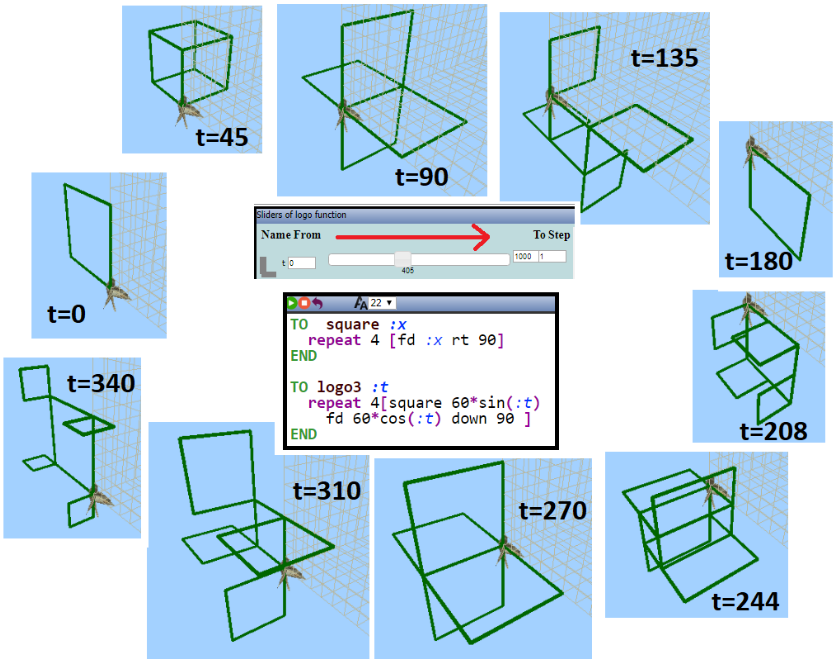

Figure 3. Instances of the animated 3D logo as constructed by the second group

The second group constructed a different 3D artefact (Figure 3). After trying many possible ideas based on random use of the trigonometric functions in their code, they decided to make a “periodically moving cube”:

Student 4: (…) We can focus on making a known geometric shape. Such as a cube!

Student 5: Yes, we have made a cube before! We can make a procedure that constructs a square whose sides change periodically. We can use sine as sides!

Student 4: (…) Do you want to see what will happen if we add cosine to one of its sides? It may look better like before! (After trying it) Wow! I told you! I can’t believe we make this!

Student 5: It’s like... When the squares whose sides have sine get bigger, the square whose sides have cosine gets smaller and vice versa! The result is very satisfying and relaxing!

Student 5: (…)Yes, (there is a difference) between the velocity that each side grows and shrinks. When the side is the biggest, it kind of slows down.

Students in this group, after some time of experimentation, had the idea of constructing a cube whose sides correspond to the trigonometric functions (60·sint and 60·cost) in order to create a periodical variation by changing the values of the variable t. The last comment was made after moving the cursor of the variable t with a steady rhythm. It indicates the production of meaning on the rate of change of sine and cosine, as well as on the difference between them.

Both groups discovered the functional relation between sine and cosine after exploration through the digital medium and exploited it in their constructions. During the presentation of their final logo, the second group referred to this relation as “harmonically counterbalancing”. The “periodically moving cube” logo was chosen as the most “impressive” one by both groups:

Researcher:	Why did you vote for the logo of the other group?
Student 1:	Because they used the trigonometric functions more cleverly! It could be even better if we made some adjustments!

While presenting their final constructions, they adjusted the role of the publisher and shared their thinking process followed in order to reach to a desirable result. During this last part of the activity, they externalized the meanings constructed on the mathematical concepts around trigonometric functions, which involved many properties from the algebraic, the geometric and the analytical domains.

CONCLUSION

MaLT2 worked as a programmable “digital canvas” for students, as they explored and expressed both their mathematical and their artistic ideas through them. It hosted the notion of periodicity which created an artistic moving effect on geometric constructs that impressed the students. Even though the research was implemented to a small group of students, the results were enlightening enough to compose a valuable feedback on the way students used mathematical notions in order to create a periodically animated artefact. The instances described in the results indicate that this learning situation provokes naturally meaning generation on various mathematical ideas from different fields such as the connection of squares sides to the cosine function.

Except for the creating process, students also experienced this activity as a social process with elements like collaboration, expression, competiveness, social evaluation and appreciation starring in it. Students’ meanings on these properties can be analyzed at a profound level since the process of their generation is externalized through the construction and constant re-modifying of their artefacts as well as their argumentation over the most “impressive” one. After reflecting on this small scale study, we aim at designing more tasks that would involve creative creation with MaLT2 in order to further elaborate the way students produce meanings in this learning situation.

REFERENCES

Chin, K. E. (2013). *Making sense of mathematics: supportive and problematic conceptions with special reference to trigonometry* (Doctoral dissertation, University of Warwick).

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the learning sciences*, 13(1), 15-42.

Kynigos, C., Essonnier, N., & Trgalova J. (2020). Social creativity in the education sector: The case of collaborative design of resources in mathematics, *Creativity Research Journal,* 32(1), 17-29.

diSessa, A. (2001). *Changing Minds: Computers, Learning and Literacy*. Cambridge: MIT Press.

Harel, I. E., & Papert, S. E. (1991). *Constructionism*. Ablex Publishing.

Harvey, B. (1997). *Computer Science Logo Style*. Cambridge: MIT Press.

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: design and construction in mathematics education. *ZDM*, 42(1), 63-76.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. Bishop, M.A.K. Clements, C. Keitel-Kreidt, J. Kilpatrick, F.K.-S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 323-349). Dordrecht: Springer.

Kafai, Y. B. (2006). Playing and making games for learning: Instructionist and constructionist perspectives for game studies. *Games and culture*, 1(1), 36-40.

Kafai, Y., & Resnick, M. (1996). *Constructionism in practice: designing, thinking and learning in a digital world*. New Jersey: Lawrence Erlbaum Associates, Inc.

Kynigos, C. (2015). Constructionism: Theory of learning or theory of design? In J. C. Sung (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 417-438). Cham: Springer.

Kynigos, C., & Grizioti, M. (2018). Programming Approaches to Computational Thinking: Integrating Turtle Geometry, Dynamic Manipulation and 3D Space. *Informatics in Education*, 17(2), 321-340.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books, Inc., Publishers.

Reggini, H. C. (1985). *Ideas y formas: Explorando el espacio con Logo*. Buenos Aires: Galápago.

Impact of place value chart app on students' understanding of bundling and unbundling

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Place value understanding is recognised as a critical component in the development of number understanding for young children. In this paper we investigate the usefulness of a purpose built app in supporting children's understanding in German and Australian contexts. In particular, we investigate whether the app supports the development of bundling and unbundling. Our findings indicate positive and negative aspects of the use, which have implications for both teachers and app designers.

Keywords: apps, place value, curriculum, professional development

In addition to the development of an ordinal and a cardinal concept of numbers, and the part-whole concept, the concept of place value is an important component of children's early learning in mathematics. Our number system is based on five principles: the principle of bundling and unbundling; the decimal system; the principle of place value; the multiplicative principle; and the additive principle (Kortenkamp & Ladel, 2014; Ross, 1989). Teachers require a complete understanding of the didactical concept of place value, as well as compatible working materials (e.g. place value apps) to use, to assist children in understanding place value. Whilst there are other apps that are designed to teach place value (Number Pieces; PV MAB), here we focus only on the Place Value Chart app as it was the one used in the research.

The importance of place value

The decimal number system is a powerful tool for writing mathematics and doing arithmetic as any rational number can be written using only ten different digits in a unique way (Larkin et al., 2019). The starting point of place value is the notion of unit – in our system ten ones form a new unit. This underlying process of repeated bundling means that we can represent whole numbers greater than nine by bundling in tens, and tens of tens, and tens of tens of tens… until no further bundling is possible (Houdement & Tempier, 2019; Kortenkamp & Ladel, 2014). Researching the teaching of place value is necessary. Fuson (1990, p. 345) reports that "less than 50% of third graders in the National Assessment of Educational Progress (NAEP) could do items identifying the hundreds digit, and only 65% identified the tens digit correctly". Rogers (2012, p. 648) notes that "despite the unchanging and recursive nature of our base-ten system, it seems some students never manage to fully unravel the hidden code that underlies place value". Kortenkamp and Ladel (2014, p.35) indicate the prevalence and persistence of misconceptions identifying that place value "is difficult to understand and to teach".

Place value in the German and Australian curriculums

The nationwide German curriculum is substantiated in special curriculums of the separate federal states. As an example, in the curriculum of Baden-Württemberg place value is mentioned at several junctures. One instance indicates that students should be able to use the decimal place value system and to recognise its structure (ones, tens, hundreds, bundling, unbundling). In grade 1/2 the number range is up to 100. The number range in grade 3/4 is up to 1000. For grade 3/4 the aim in the sub-strand "numbers and operations" is that the student should be able to use the building of the decimal place value system and to recognise and understand its structure (ones, tens, hundreds – as group of three, thousands, ten thousands, hundred thousands, million; bundling, unbundling) (Ministerium für Kultus, Jugend und Sport, 2016, p. 24). Similar content is incorporated in the curriculum documents of Brandenburg (LISUM, 2015). Likewise, the Australian curriculum: Mathematics (ACARA, 2018) is prescriptive in relation to place value, indicating the range of numbers that children work with. In year 1 children work with numbers up to 100; year 2 up to 1 000; year 3 up to 10 000; year 4 up to 100 000. In effect, one "place" is added each year.

Place value chart app

The app "Place Value Chart" (Kortenkamp, 2012-2018) shows a virtual place value chart, with tokens that can be moved between columns, to represent different amounts. In the app, numbers are represented by touching the screen and thus creating tokens. An example: to represent the number five, the user has to tap five times (or simultaneously once with five fingers), in the corresponding column, in this case the Ones-column. Tokens can be deleted by moving them out of the chart or by shaking the iPad. The app focusses on bundling and unbundling: by moving one token to the adjacent column to the right, the token will be unbundled into ten tokens. Moving tokens to the adjacent left column bundles ten tokens into one token in the adjacent column (N.B. only if more than ten tokens of the smaller value are available). In this way moving tokens in the app creates a change in representation, e.g. 234 can be represented as 2H 3T 4O or as 1H 12T 14O. The app offers various settings including as language, number of columns, column labels, counting base, and word/symbolic representations of totals.

Limitations of current curriculum approaches to PV and research questions

Both German and Australian curricula suggest that students learn about place value in a rigid way – with prescribed upper limits to the numbers with which children should work at each year level. We argue that this approach has detrimental impact on how children develop generalised understanding of place value, rather than an understanding determined by (and in our view limited by) the number of places prescribed in the curricula. This likely results in piecemeal, context specific knowledge (10 ones = 1 ten; 10 tens = 1 hundred) rather than generalisable knowledge (10 units always equals one unit in the adjacent column to the left). Two questions guided this research: 1. *Were children in year 1 and 2 capable of generating new columns for* *standard partitioning?* and 2. *In what ways did the app support children in understanding bundling and unbundling?*

METHODOLOGY

School 1 is a rural school in Baden-Württemberg, Germany. 43 students from two different classes at the end of year 2 took part. One classroom teacher taught both classes. School 2 is a rural school in Brandenburg, Germany. 61 students from three different classes at the start of year 3 took part and each class was taught by their own teacher. School 3 is an urban school in Queensland, Australia. 130 students in eight classes took part, four from year 1 (59) and four from year 2 (71). Each class was taught by their own teacher. The schools selected were a convenience sample located near each respective university. Teachers at the schools self-selected to be part of the research. The teaching consisted of three lessons of approximately 45-60 minutes each. The content of the lessons included bundling and unbundling, standard and non-standard representations of numbers, the need to add a column (hundreds) when representing three digit numbers in standard representation, and using the app.

Research design

The researchers met with teachers and school leaders at each of the schools. The researchers and teachers collaboratively planned a series of place value lessons that incorporated the use of the app. At this stage teachers also received a one-hour professional development session on place value, delivered by a member of the research team. All students with permission to participate completed a post intervention test (this test replicated the activities that students had been completing on the app e.g. they were required to represent various numbers on paper, using physical circular stamps that “mimicked" the use of tokens on the app). These tests were then marked by the researchers (correct response, incorrect response, no response). A range of codes were assigned to the incorrect responses indicating a typology of errors. RQ1 addresses the issue of whether or not children added an additional column when required for standard representation. RQ2 addresses the impact of the app on student performance in the post intervention test and is answered from test data and device data logs from the school in Brandenburg.

Data collected

Data collected include: Four sets of post-intervention tests that assessed whether children added an additional column when required (e.g. 97 + 5 where only the tens and ones columns are shown; Two sets of video data children completing the lessons; and one set of matched iPad data to posttests (Brandenburg school). The different data collected - i.e. video or data logs - is a consequence of differences in ethical clearance from the different schools.

The findings

The project generated a number of very interesting findings in relation to the children’s understanding of place value. However, we focus here only on whether the affordances of the app assisted (or limited) children's understanding of the general structure of place value (bundle and create a new column) and standard partitioning (i.e. use the least number of tokens to represent an amount). In this article we focus on the results of the component of the post-intervention test that required children to add an additional column to represent a number in its standard representation. The items 2a, 2b, 2c and 2d were structurally similar: In an initial place value chart with two (T,O) or three (H,T,O) columns a number x was represented by tokens. Then the children were asked to represent the number $x+y$ for a given y in a second, empty chart with the same number of columns *with the least possible number of tokens*. The students answered the questions by using a stamp that created tokens.

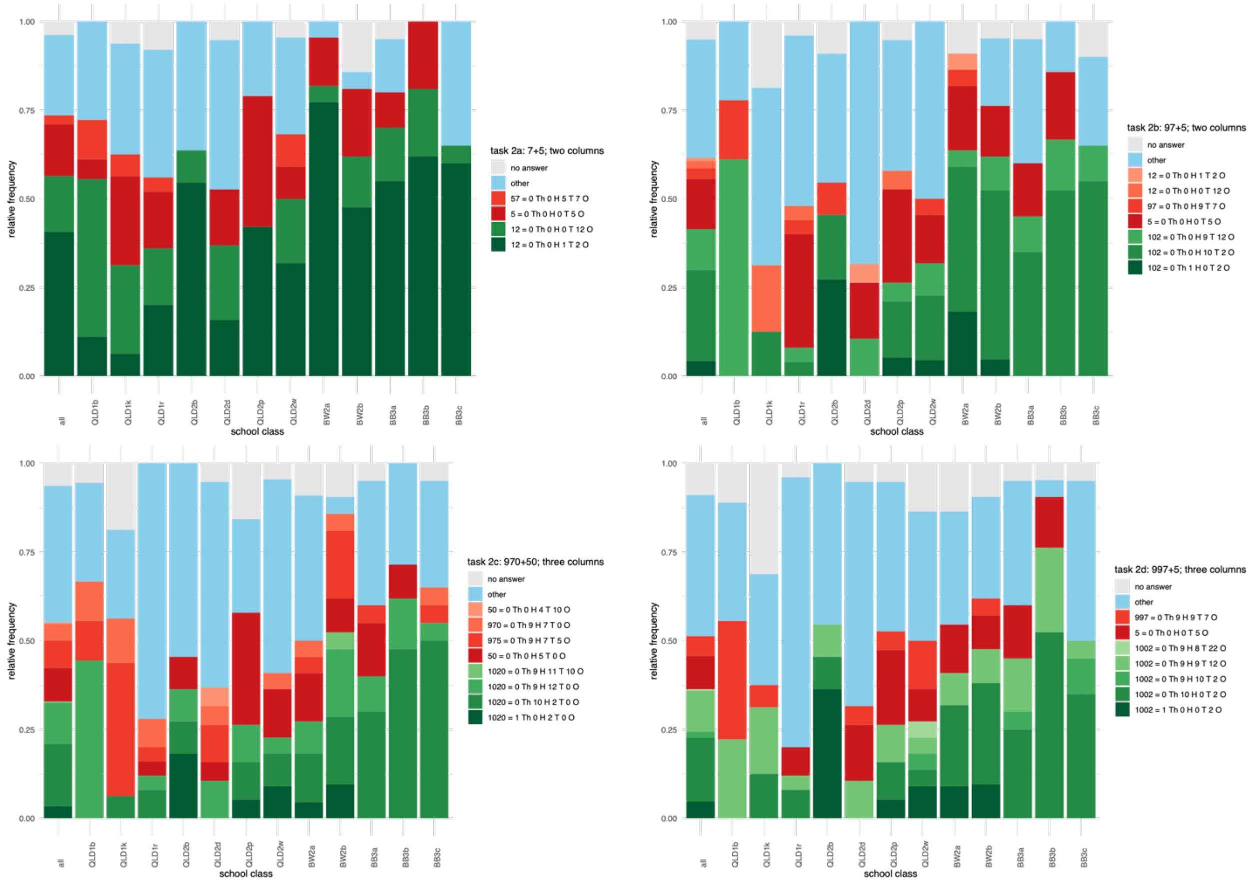

Figure 1: Students' answers to task 2a-d

Tasks 2b–d were constructed in a way that it was only possible to create a standard representation (and thus the representation with the least number of tokens overall) by adding a column (space was provided for children to do so). In Figure 1 we show the students' results grouped by school (QLD1 = Queensland school, end of year 1; QLD2 = Queensland school, end of year 2; BW = Baden-Württemberg school, end of year 2; BB = Brandenburg school, start of year 3) and class id. The bars show the frequency of answers in relation to the number of children in a class. We only distinguish answers that result in a represented number that was stamped by more than 2.5% (at least 6 children) of the overall number of children (n = 234). The green bottom bars depict the number of correct results, where an answer is "correct" if the number represented matches the result of the addition (e.g. 7+5 = 1T2O or 12O). The "best" answer is the one with the least number of tokens, which could only be obtained in tasks 2b–d by adding a column, and is the bottommost bar in dark green.

The most apparent and relevant results were: Adding a column was done rarely, apart from class QLD2b (No student at BB did it, but many created a maximum bundling with the available columns); many wrong answers were created across almost all classes by stamping the second summand only (dark red bars); and there was a great deal of variety amongst the classes in relation to their bundling.

In a second step, we connected the log data from the iPad to individual children and their test results for students in Brandenburg. In this way, we can analyse how individual children worked with the app and connect it to their test results. In order to better understand the cross-task behaviour, we grouped the correct answers by partition type: *Standard* is an answer where at most 9 tokens are used for each place, i.e., the standard representation of a number; *Max bundling* is an answer where at most 9 tokens are used for each place except the highest one, i.e. the representation closest to a standard representation of a number if there are not enough columns (e.g. 102 is 10T2O); *Incomplete bundling* is an answer where there are more than 9 tokens in any place except the highest one, i.e. the representation could be transformed into a representation that uses less tokens without adding a column (e.g. 102 is 9T12O). As no student in Brandenburg added any columns, the only task where they could achieve a standard partition was task 2a. The second-best answer they could give in 2b-2d was a max bundling. Incomplete bundling, on the other hand, is an indication of a missing component of place value understanding (rather than potentially a consequence of being unsure as to whether adding another column is permissible).

The log files of the app contain detailed and time-stamped information about all user interaction and configuration data of the app. In particular, they contain information about student's actions for: changing the number by *adding* or *removing* tokens (CREATE, REMOVE); *rearranging* tokens in a column without changing the value (REARRANGE); *successful* and *unsuccessful* bundling by moving a token to a column on the left (BUNDLING, REJECTED); *unbundling* by moving a token to a column on the right (UNBUNDLING); and *clearing all tokens* by shaking the iPad (SHAKE).

While the detailed information can give deep insights into what students actually did, we wanted to quantitatively measure the degree of meaningful interaction. We achieved this by counting the frequency of certain actions. Children who worked more with the app had the potential to achieve more interactions, so this measure is dependent on the teaching and thus the classroom under consideration. However, as we are investigating a possible connection of overall interactions in the app to the achievements in the test, it is feasible to combine all three groups.

We compare the distribution of interactivity ratings (that is, the frequency count of actions over all three lessons for a student) with the partition types in correct answers of these students. In the case of the SHAKE action, which is the least intentional action, we expect to see no difference in the distributions for the three types. Figure 2 shows the boxplots of the distributions and confirms this ad-hoc hypothesis.

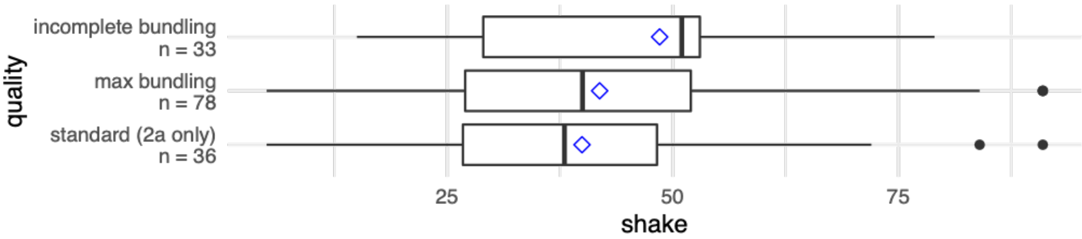

Figure 2: Distribution of SHAKE actions

For the SHAKE action, we see similar distributions of its frequency in all three conditions. On the other hand, the other actions listed above showed a different picture, similar to Figure 3, where we can see that the activity frequency distribution of students who bundled incompletely is lower compared the ones of those who found *standard* partitions or *max bundlings*.

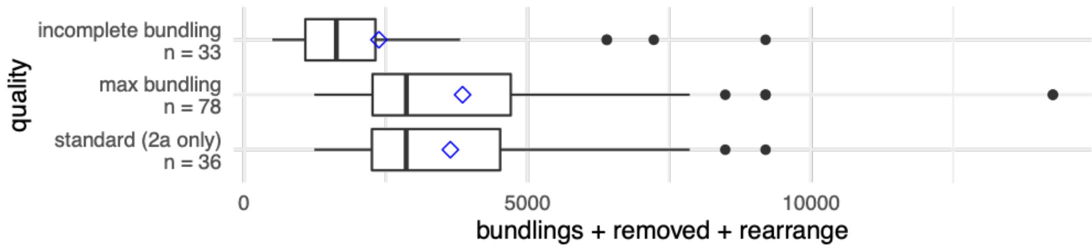

Figure 3: Distribution of intentional actions

The interaction measure we used in Figure 3 is the sum of the frequencies of (successful) bundling, removal of single tokens, and rearranging within a column. We chose these three actions as they are most likely to be intentional. Of the actions that are logged, these appear to be the most intentional ones to us. The lower quartile $Q_1 = 2267.75$ of the *max bundling* group and the lower quartile $Q_1 = 2256.75$ of the *standard* group are close to the *upper* quartile $Q_3 = 2318$ of the *incomplete bundling* group, showing that the interaction of those who performed better on the test is usually higher than those who did not perform as well.

DISCUSSION

Classifying the answers, we see a variety of representations of correct (numerical) and incorrect results. Most of the incorrect results occurred only a few times, often just once. This is unsurprising as we included tasks that are beyond the expected place value students' competencies. We can, however, still identify some typical mistakes. The most striking one is stamping only the summand (5 or 50) instead of the sum. This can be caused by misreading or misunderstanding the task. In most classes, about 10-15 % of the students made this mistake. However, two of the classes did stand out: In QLD2b the mistake occurred only with task 2c, where the two-digit number 50 had to be added, and in BB3c this mistake did not occur at all. The reason for this can be either that the students had less trouble reading and understanding the task text, perhaps caused by a classroom culture that pays more attention to reading exactly, or they might have been given additional instructions by the teacher when the test was administered. As we have no video data from the test sessions, we cannot identify the true reason.

In the first task, 2a, most students successfully did a *max bundling* if they obtained correct numerical results. With the year 1 classes, many children answered 57: This shows that they did not have a place value understanding yet, but stamped both summands individually in separate columns. This is an expected problem if place value understanding is still developing. From the data we see that the German classes performed better than the average on all tasks. In Australia, both the year 1 class QLD1b and the year 2 class QLD2b performed at or better than the average, while the other classes performed below average on all tasks. So, while we would expect German classes perform better due to their longer school experience, we also observe that younger children, even year 1 students, can show the same or better performance.

Our initial question investigated whether children would add an additional column to represent a total with the least number of tokens (often requiring the addition of a column). We do not see this behaviour in general. Out of the 234 students, only 17 added a column in one of the tasks; 10 of them did this in all tasks where necessary, 5 did it just once, and 2 did it in two out of three tasks. It is not possible to trace this to either teaching or the use of the Place Value app as, although no students in Brandenburg added a column, they performed better than other groups if we just look at the numerical result without taking the representation into account.

In addition to the possibility that some children were unsure whether they were ‘allowed’ to add a column in the paper test version, we also see that the app itself imposes a constraint on students’ actions. As the number of columns can only be changed by accessing system preferences, separate from the app, the action of adding a column is not within easy reach. It is not a common action to “just add a column”, even though the students know that it can be done. This result might imply that the app is helpful for developing a proper place value understanding, as this should encourage students to do *max bundling* of a number until they arrive at the *standard* representation. If there are not enough columns, then the idea of repeated bundling would cause adding columns to the left until every column is holding less than 10 tokens. On the other hand, inspecting the data more carefully we see that many students did a *max bundling*, i.e., they tried to be as close to a standard representation as they could without adding a column. And, when asked, the students knew that another column to the left will enable them to continue the bundling process.

The data collected from iPad logs, only available for BB3a-c, supports the theory that students who did more intentional actions within the app exhibited a better place value understanding in the test. Comparing the distributions of activity for students who did a *max bundling* in all tasks 2b–d with the distributions of activity for students who found the *standard* representation in 2a or those who did not bundle completely, we see that the distributions matches only in the first case. This means, that the activity of students who tried to bundle as much as possible matches the activity of those who found the *standard* representation in task 2a, while the group of lower-performing students (in terms of bundling) showed lower activity.

CONCLUSIONS AND IMPLICATIONS

Our findings show that merely using the Place Value Chart app in the classroom is not a guarantee for the development of place value understanding. Some of the findings might be influenced by side effects of test administration. In future research it will be necessary to standardise how the test is administered to the students. We also found that there could be a design issue of the app, in combination with a pedagogical issue: Students did not add columns to the charts in the test, although from video data we saw that they knew that it is necessary. On the positive side, the group of students who did *max bundlings*, was the group that had a higher frequency of targeted activity in the app than those who failed to bundle.

REFERENCES

Australian Curriculum and Reporting Agency (ACARA). (2018). Australian Curriculum, Mathematics. https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Fuson, K. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction, 7*(4), 343-403. https://doi.org/10.1207/s1532690xci0704_4

Houdement, C., & Tempier, F. (2019). Understanding place value with numeration units. *ZDM Mathematics Education, 51*, 25–37. https://doi.org/https://doi.org/10.1007/s11858-018-0985-6

Kortenkamp, U. & Ladel, S. (2014). Flexible use and understanding of place value via traditional and digital tools., In *Proceedings of PME 38, Vancouver, Canada, July 15–20, 2014, Vol. 4,* IGPME.

Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT Framework to Evaluate the Design of Two Geometry Apps: an Exploratory Study. *Digital Experiences in Mathematics Education, 5*(1), 59-92.

LISUM. (2015). *Rahmenlehrplan Teil C Mathematik, Jahrgangsstufen 1 – 10*. Landesinstitut für Schule und Medien Berlin-Brandenburg.

Ministerium für Kultus, Jugend and Sport Baden-Württemberg (Ed.) (2016). *Bildungsplan der Grundschule. Bildungsplan 2016. Mathematik.*

Rogers, A. (2012). Steps in Developing a Quality Whole Number Place Value Assessment for Years 3-6: Unmasking the “Experts” *Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia)*. Singapore.

Ross, S. H. (1989). Parts, Wholes, and Place Value: A Developmental View. *The Arithmetic Teacher*, *36*(6,), 47–51. http://www.jstor.org/stable/41194463

Learning the function concept in an augmented reality-rich environment

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Although the function concept is central in mathematics and is included in many curricula worldwide, many students find it difficult to understand. In this study, we consider function as covariation between two variables. This study aims at exploring students' understanding of covariation while learning in an augmented reality (AR) environment. Groups of 16-year-old students carried out the Hooke's Law experiment using AR headsets. Students' interactions were video-recorded, and semiotic lenses were used to analyze their covariational reasoning. Findings show that the AR-rich environment promoted the students' covariational reasoning with mostly elementary levels, but also with some indications of high levels.

Keywords: augmented reality, covariational reasoning, multimodality

INTRODUCTION

Function concept is central in mathematics and is included in many curricula worldwide. However, many students find it difficult to understand and graduate from high school with a lack of knowledge of this concept (Akkus, Hand and Seymour, 2008). Several studies were conducted on teaching the function concept, some using digital technologies that include computers and simulations. These studies found that the use of dynamic technology tools fosters students' understanding of the function concept (e.g., Hoffkamp, 2011). Although several attempts have been made in recent years to integrate AR technology into science and mathematics education (e.g., Yen, Tsai and Wua, 2013), less is known about augmented reality (AR) affordances to foster pre-calculus concepts. In this paper, we aim at shedding light on the role of AR technology in fostering students' understanding of the function concept. For this reason, we designed an innovative AR tool and explored its effectiveness in fostering covariational reasoning as an indication of understanding the function concept.

The reported study is innovative for two reasons: (1) It proposes a new prototype for employing AR in an educational setting using a special headset, presenting a dynamic object in a real environment with virtual representations. In contrast to the design presented in this paper, typical ways of using this technology involve augmenting static objects rather than dynamic ones (for example, a 3D view of a cell when observing a cell picture on a biology book page). (2) It explores covariational reasoning as mathematical representations juxtaposing a dynamic real-world object, showing that it is innovative in mathematics education.

THEORETICAL FRAMEWORK

Covariation reasoning: In this study, we address function as a dynamic process of covariation. Thompson and Carlson (2017) described understanding covariation as holding a sustained image in the mind of two quantities values (magnitudes) that change simultaneously. They discussed understanding function by describing meanings and thinking styles that can be attributed to someone who understands the essence of function and suggest five levels of covariation: (1) pre-coordination of values; (2) gross coordination of values; (3) coordination of values; (4) chunky continuous covariation; and (5) smooth continuous covariation. In the first level, students can predict the change of each variable value separately but cannot create pairs of values. In the second, students perceive a loose link between the overall changes in the two quantities values, such as “this quantity increases as the other decreases.” In the third, students can match values of one variable (x) to values of another one (y), creating a discrete set of pairs (x, y). In the fourth, students may perceive that the changes of two variables occur simultaneously and that they vary in piecewise continuous covariation. In the fifth, students perceive that an increase or decrease in the value of one variable occurs simultaneously with changes in the value of the other variable and see that both variables change smoothly and continuously. In this paper, we observe students’ actions regarding the function concept through the theoretical lens of the Action, Production and Communication (APC) space.

Multimodality and the APC space: “Multimodality” refers to the importance and mutual coexistence of a variety of cognitive, material and perceptual resources in mathematics learning processes, and in general, in the creation of mathematical meanings. Radford, Edwards and Arzarello (2009) argued that “these resources or means include verbal and written symbolic communication, as well as drawing, gestures, manipulation of physical and electronic devices, and various types of physical movements” (ibid, pp. 91-92). Earlier studies show gestures playing an important role when students solve problems and explain mathematical concepts (e.g., Edwards, 2009). Gestures are just part of a whole arsenal of students’ resources available to bridge their experiences with daily life phenomena and formal mathematics (Arzarello & Sabena, 2014). Data analysis in this study was done according to the APC space perspective (Arzarello & Sabena, 2014). This method considers multimodal resources and analyzes learning processes to understand conceptual knowledge. The APC space model consists of three main components - body, physical world and environment that are vital in mathematical activities in the classroom social context. Hence, it is important to examine them in analyses of learning processes. The three crucial APC components (action, production and communication) of students’ learning processes in mathematics highlight their active role in the learning process (Arzarello & Sabena, 2014). According to this perspective, a suitable mathematical learning environment for students must include three specific activities: action and interaction (e.g., with classmates, teacher, tools or students themselves); production (e.g., answering or asking questions, conjecturing); and communication (e.g., delivering a solution to a teacher or classmate, verbally or in writing, using appropriate representations). The APC space considers the students' environment, including the tools they use, as being crucial for learning, thinking and the inquiry process. Our analysis addresses all students' multimodal resources: speech, body gestures, drawings and interactions with the physical model.

Augmented reality: AR is an innovative technology that incorporates a wide variety of techniques for presenting computerized materials (such as text, images and video) about the real world as seen in the normal state (Kaufmann et al., 2005). AR combines several layers of virtual objects that are augmented over physical objects in the real world, creating a unique reality in which virtual objects and the real environment coexist. AR technology has many advantages. It contributes greatly to student learning because it integrates the benefits of physical and virtual learning experiences (Bujak et al., 2013). It can help students learn challenging scientific content thanks to its ability to present information and details visually that are not naturally visible. In addition, it allows students to experience interactive 3D simulations, leading to deeper insights about phenomena that might be difficult to understand; it simplifies objects' visual appearance and helps students think about their symbolic representations; and it enables students to observe virtual objects in a perspective they choose while still being able to see other students (Bujak et al., 2013). Two research questions guided this study: (1) What levels of covariational reasoning arise among students as they learn in an AR-rich environment? (2) How does the use of an AR-rich environment facilitate covariational growth?

METHOD

Research context and participants: This study is based on qualitative research method. Research experiments were conducted with four groups of three 10th- and 11th-graders studying in two high schools in southern Israel. The participants study advance mathematical topics (geometry, algebra, and calculus). They had studied linear functions in 8th grade and quadratic functions in 9th grade. The meetings were held in a scientific laboratory at Ben-Gurion University of the Negev. Each session lasted for 90-120 minutes. Each group carried out the physical Hooke's Law experiment, which examines the relationship between mass and elongation of a spring (Figs. 1, 2). At the beginning of the experiment, students received an explanation about each part of the experiment, as well as about using the technology. Each group worked on task sheets (see link) corresponding to both physical experiments.

Figure 1. Hooke's law experiment

Figure 2. Virtual data as seen through the AR headset

Data collection and analysis: Learning activities were video-recorded and students' interactions, gestures and materials (written notes, files) were collected. Thus, a solid set of data was obtained that was analyzed using the deductive approach (Patton, 2002).

(1) All videos were observed to get a general impression about the process. (2) The analysis refers to the covariational reasoning levels (Thompson and Carlson, 2017). (3) Transcription of the observations was done. Students' statements and interactions, peer-to-peer interactions or interactions with the model, were recorded and documented. (4) Accurate encoding of transcripts was done and statements and instances of our data categories were sought for, including scans of transcripts to identify expressions indicating levels of covariational reasoning. These expressions were categorized into appropriate levels of covariational reasoning.

FINDINGS

Covariation level 2 - the evolution of quantities: This example illustrates how students were engaged in the second level of covariational reasoning. It happened when they coordinated the weight of cubes and the shape of the graph. Tal, Hila and Maya wore the AR headset and observed the spring elongation. After identifying the virtual object that displayed the length of the spring (blue line in Fig.2), they added cubes one by one while looking through the headset to observe the variation in spring length. Twenty minutes after beginning the experiment, Hila noticed for the first time that the graph began to appear on the screen. She went on to say, "It seems to me that if we add more cubes, the function will continue." Later on, the girls changed the spring with a new one and again added cubes. The changes were observed simultaneously through the AR headset. Here, Maya also saw the graph and described its direction through body gestures (a sloped increasing hand movement) while stating "it is from zero to 10" after continuing to add cubes, Hila reported (excitedly) that she saw the graph "from zero – absolutely increasing... really inclines to the side." Afterwards, the researcher asked them to share their insights.

Hila: The more weight we add, the greater the graph function.

Maya and Hila: It inclined to the right. (gesture with their hands to the right (Fig. 3a).

Researcher: What do you mean?

Hila: If the graph started out like this, the more I add, the more it inclines to the right (Fig. 3b).

Maya: Increasing.

Hila: Yes, increasing, pulling more up like this.

Figure 3. (a) Hila and Maya gesture the inclination to the right"; (b) Hila gestures "if the graph started out like this"; (c) Hila gestures the inclination to the right."

Hila explains her insights and says: "as we add weight" and refers to hanging more cubes on the spring, "The graph inclines more to the right." Maya agrees with Hila's insight. Tal, the third student, does not take part in the discussion and continues to observe the model through the AR headset. It is worth noting that Hila uses non-mathematical terms to describe the graph as "inclining to the right." The gestures in Fig. 3b&c suggest that the students express the graph rotation without using the word "slope". Maya immediately adds that the graph is "increasing" and adds a mathematical element here to the description Hila had provided.

In this example, Hila and Maya describe a relationship between two objects: cube weight and graph shape. In the experiment, they hung cubes one by one while observing the graph generated by AR. As a result of their actions, they conclude that "as we add weight, the graph inclines to the right." This type of covariation could refer to the second level: students see general changes in the values of two quantities, but do not coordinate specific pairs of values. As we argue in the Discussion, this phenomenon can be seen as one of the greatest potentials of using AR: the students deal with a real experiment and add cubes to the spring. They see changes in the graphical visualization simultaneously. They initially use the concept of "weight" (quantity) to explain the observed phenomenon. Hence, the quantities the students coordinate (weight and changes over time) emerge while conducting the experiment.

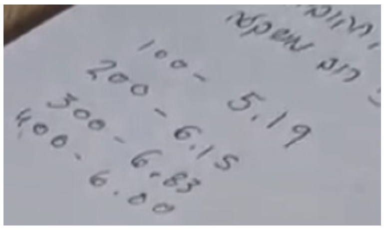

Figure 4. Matching cube weight and spring length

Covariation level 3 - the mathematization process: We show how Uri, Shilat and Shahar engage in the third level of covariation they encountered when coordinating between cube weight and spring length. The students performed the experiment using the AR headset and observed that the addition of each cube really increases spring length: "Every time we add 100 grams, we get 5.19 (cm) for the first 100 grams. For the second 200 grams, 6.15. At 300 grams, 6.83 and then 6.00 at 400 grams". The students performed several actions during the experiment: while Uri and Shilat added cubes and observed the resulting values on the spring through the headset, indicating its length at each moment, Shahar documented the values on the notes page. She prepared a table with the left column representing cube weight and the right column the corresponding spring length value (Fig. 4). Shahar describes the relationship between two covariation variables: cube weight and spring length. The way she read and wrote the data suggests that the covariation type she expresses is of level three (Thompson & Carlson, 2017). In terms of using AR, this example represents an important step in the mathematization process: function is an abstract object. A representation (graph or table of values) is not a mathematical object itself; it is a representation of an abstract object. Hence, the real phenomenon as well as the mathematical representations are not self-explaining, rather they must be connected conceptually by the students. This example gives insights into strategies students use in order to reconstruct a mathematical conceptual meaning behind observed phenomena (real situation and augmented reality experiment).

Covariation level 4 - the multimodal process of meaning-making: We illustrate how students engaged in the fourth covariation level they encountered when coordinating between cube weight and spring length. In the experiment, a few minutes after Xenia concluded that table and graph were changing simultaneously, Ronnie explains her insight about the connection between spring length and number of cubes: Ronnie: “Here you see this is at 0.7, and here at 0.4, it means that the slope between each other is lower, meaning that at the beginning, the line was more... less… seems steep, more like this... (Fig. 5), and then such a break point, and then it will become more (Fig. 6) ...., and then you understand? It will increase at such a velocity.”

Figure 5. Ronnie shows a slight slope with her hand

Figure 6. Ronnie shows a steep slope with her hand

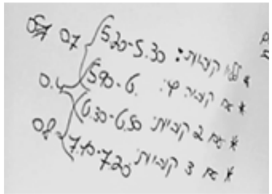

Figure 7. Table prepared by the students

Ronnie explains her insight to the group members. She finds it difficult to communicate it verbally and therefore uses body gestures to convey the message to her classmates. After generating a table connecting spring length and number of cubes (Fig. 7), Ronnie refers to the difference between the first two rows in the table as being 0.7 and between the second and third rows as being 0.4 (“This is 0.7 here and 0.4 here”). She describes that the graph starts out at a more moderate slope “less steep" (Fig. 5), and at some point becomes a section of the graph that is steeper (Fig. 6). As Ronnie relies on the table they had prepared together with the differences, we conclude that she links the two variables: number of cubes and spring length. This type of coordination may be associated with the fourth covariation level (Thompson & Carlson, 2017) because Ronnie describes the graph as two continuous segment lines separated by a point, which she describes as a “break point” at which the change in slope occurs. In terms of using AR technology, this example shows an important step in the process of meaning-making: here, the students use a mathematical strategy (determining differences) in order to reconstruct the conceptual (in this case, functional) relationship. Although, the result leaves questions unanswered (e.g., How to explain the “different differences” 0.4 – 0.7 – 0.8?), it still reveals how students use concepts (like analyzing differences between given quantities) in order to reconstruct mathematical structures.

FINAL REMARKS

The findings show that the participants engaged in the second, third and fourth levels of covariation. We also found that the students mostly engaged in the second and third levels of covariation, and less in the fourth one. This could be attributed to the nature of the Hooke’s Law experiment, namely, hanging cubes one by one and observing the spring elongation after each addition. Students who focused on the table of values mainly engaged in the third level, while students who focused on the length-mass graph mainly engaged in the second or fourth levels of covariation. The second and fourth covariation levels illustrated in examples 1 and 3 differ somehow from the covariational reasoning defined in (Thompson & Carlson, 2017). In their definition of covariational reasoning, they refer to two values of quantities that vary simultaneously. In the two examples presented above, students covary quantity (mass) with object (graph) (e.g., the more mass we add the more the graph is inclined). Inspired by Arzarello (2019), to distinguish between the covariational reasoning defined in (Thompson & Carlson, 2017) we introduce the term "*second-order covariation*," which refers to two objects that covary simultaneously.

The first and third examples help us hypothesize that second-order covariation is complicated just like covariational reasoning. In both the first and third examples, the students resorted to gestures as a semiotic means to thinking with and through them, while in the second example, their thinking was mainly mediated by words and text. We also conjectured that the higher level of second-order covariation is more complex than the lower covariation level. In contrast to the students' gestures in example 1, which were in tune with the students' statements and could be considered as a means of communication to answer the researcher's question, in the third example, gestures exchanged the students' statements, which they were unable to express through their thoughts. Hence, it seems that the students resorted to gestures as a means of thinking (Vygotsky, 1978) The three examples show that the students continue to interact with the virtual objects displayed by the headset even after removing it. Considering that the students removed the headset after a short time of use, continuing the interaction with the virtual objects suggest that the design of the AR, which juxtaposes real-world objects with virtual objects, was found to be effective for transferring external signs such as the mass-length graph or table of values of mass-length to become instruments for the students (Trouche, 2003). Other than identifying the covariation levels, the three examples also show results in terms of the *process* of meaning-making. Example 1 shows that the use of AR technology can support the emergence of quantities that the students deal with (Thompson and Carlson, 2017). When the students deal with a real object and a graphical representation, they ascribe a quantity (weight) to the box. In order to make sense of the graph, they introduce a second quantity (length). Such observations must be connected conceptually, and example 2 shows how students organize such a process of mathematization (focused collection of observed quantities). Finally, example 3 gives insight into the multi-modal process of meaning-making within the small group discussion: the students determined the relevant quantities (compare example 1), they focused, prepared and organized the (for them relevant) information (compare example 2), and they now interpret and give meaning to the phenomena using different (mathematical) strategies and corresponding mathematical gestures, as well as verbal and written signs. These insights are closely linked to the categories of Thompson and Carlson (2017) in such a way that their procedural character provides insight into the emergence, evolution and connection between the different covariation levels.

REFERENCES

Akkus, R., Hand, B. & Seymour, J. (2008). Understanding students' understanding of functions. *Mathematics Teaching*, 207, 10–13.

Arzarello, F., & Sabena, C. (2014). Analytic-structural functions of gestures in mathematical argumentation processes. In L. D. Edwards, F. Ferrara, & D.Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment*(pp.75–103). Charlotte, NC: Information Age Publishing, Inc.

Arzarello F., Sabena C. (2014) Introduction to the Approach of Action, Production, and Communication (APC). In Bikner-Ahsbahs A., Prediger S. (Eds.), *Networking of Theories as a Research Practice in Mathematics Education*. Cham: Springer.

Bujak, K. R., Radu, I., Catrambone, R., Macintyre, B., Zheng, R., & Golubski, G. (2013). A psychological perspective on augmented reality in the mathematics classroom. *Computers & Education*, 68, 536-544.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics*, 70, 127–141.

Hoffkamp, A. (2011). The use of interactive visualizations to foster the understanding of concepts of calculus: Design principles and empirical results. *ZDM–Mathematics Education*, 43, 359–372.

Kaufmann, H., Steinbügl, K., Dünser, A., & Glück, J. (2005). General training of spatial abilities by geometry education in augmented reality. *Annual Review of Cyber Therapy and Telemedicine: A Decade of VR*, 3, 65–76.

Patton, M.Q. (2002). *Qualitative research & evaluation methods*. Sage publications.

Radford, L., Edwards, L., & Arzarello, F. (2009). Beyond words. *Educational Studies in Mathematics*, 70, 91–95.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: NCTM.

Trouche, L. (2003). From artifact to instrument: mathematics teaching mediated by symbolic calculators. *Interacting with Computers*, 15(6), 783-800.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Yen, J. C., Tsai, C. H., & Wu, M. (2013). Augmented reality in the higher education: Students' science concept learning and academic achievement in astronomy. *Procedia-social and behavioral sciences*, 103, 165-173.

A digital worksheet for diagnosing and enhancing students' conceptions in functional thinking

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This paper reports on results related to one out of six digital worksheets designed by me for a qualitative study conducted in an Austrian middle school class. All of these digital worksheets were designed based on typical problems related to functional thinking discussed in the literature, and they aim at diagnosing and enhancing students' conceptions in this field. In the study utilizing these materials, grade 7 students participated in diagnostic tests and interviews aimed at examining their conceptions. Furthermore, they were video recorded while utilizing the digital worksheets in order to investigate their interactions and a possible enhancing influence on their conceptions. In this article, I briefly introduce the research project and focus on results related to one of the digital worksheets.

Keywords: digital task, functional thinking, representations, lower secondary school.

INTRODUCTION AND THEORETICAL BACKGROUND

Functional thinking is an important concept in mathematics education. In this project, it mainly comprises Vollrath's (1989) *relational* and *covariational* aspects. Literature review reveals various problems in this field such as graph-as-picture errors or difficulties related to the aspects of functional thinking, especially in comprehending the dynamic covariational aspect (e.g., Clement, 1989).

Concerning dynamic aspects of functional thinking, the development of technology-based resources offers new opportunities. Dynamic mathematics software supports students' development of functional thinking because it allows to examine multiple, dynamically linked representations and thus to emphasize relational and covariational functional aspects (Lichti & Roth, 2018). However, representations also must be considered critically, as they can constrain students' thinking about the concepts involved and are interpreted by students according to their prior knowledge (Vosniadou & Vamvakoussi, 2006). Therefore, in this project the overarching question arose whether digital mathematics tasks can support students in an early stage of learning functions (lower secondary education) in developing mathematically correct conceptions.

In addition, the term conceptual understanding emerged during this project. I follow the interpretation of Barmby, Harries, Higgings, and Suggate (2007) who defined understanding as making "connections between mental representations of a mathematical object" (p. 42) which result in a network of representations for a concept. In the next section, I present one of the digital tasks designed for this project.

DIGITAL TASK DESIGN: THE BILLIARD WORKSHEET

The presented digital worksheet *Billiard* (Figure 1) is based on an example by Schlöglhofer (2000) and designed for discussing a possible graph-as-picture error. The included applet simulates the following situation: From point *P* a billiard ball is shot along an indicated path. The distance *d* of the ball from the upper boundary of the billiard table is a function of time *t* and is modelled as a piecewise linear function.

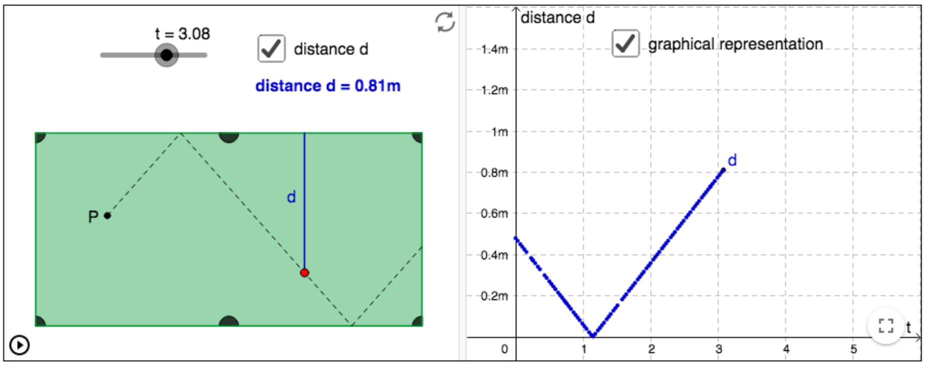


Figure 1: Digital worksheet *Billiard*, https://www.geogebra.org/m/sqqgkkmz

Due to the prior knowledge of the participating students, the applet focuses on the transfer of the functional dependency between iconic situational model (left) and graphics window (right) with the option to display the graphical representation in trace mode. Due to the characteristic feature of this task, the graphical representation mirrors the path of the ball in the iconic situational model, but the boundary between the two windows should not be interpreted as "mirror". Both representations of the functional dependency are positioned side by side to avoid the impression that the *x*-axis corresponds to the length of the billiard table or horizontal distance of the ball. On the right side of the applet, the function of distance over time *d(t)* can be displayed as trace in a Cartesian coordinate system. This pointwise appearance is intended to highlight the relational aspect of the functional dependency as each point visualizes the relation between specific values of time and distance. The dynamic feature of the animation aims at emphasizing the covariational aspect because during the animation (or by moving the slider), students can observe the distance changing as a function of time. In this research project, I additionally used a second version of this digital worksheet using two parallel coordinates – called *dynagraph* – instead of perpendicular axes (see Task 2 in https://www.geogebra.org/m/tnqamu4x).

To design the worksheet, I considered several student tasks. First, students should explore the situation, identify the dependent variable, and explore relationships between distance and time. Afterwards, students were asked to predict the general course of the function graph. Digital materials were designed with the option to include graphical representations; when included, students were encouraged to examine connections between graphical and situational representations. I also included several questions addressing either relational (e.g., to determine function values for given arguments *x* from graphs) or co-variational aspects (e.g., to describe changes of the function value as a function of the independent variable). Moreover, students should additionally examine characteristics of functions such as extrema.

RESEARCH DESIGN OVERVIEW

Like previously mentioned, I designed several digital worksheets addressing student difficulties outlined in literature. These materials were implemented in a qualitative case study integrating features of Grounded Theory (Charmaz, 2006) conducted with 28 students of grade 7 for examining (i) the intuitive students' conceptions concerning the selected tasks, and (ii) students' interactions with the designed digital worksheets as well as possible influences of the digital worksheets on their conceptions. I collected several types of data within five stages: (1) diagnostic test 1, (2) diagnostic interviews, (3) intervention with the digital worksheets, (4) diagnostic test 2, and (5) diagnostic interviews. In the following, I concentrate on information and results concerning the presented digital worksheet "Billiard".

First, all 28 students solved a paper-based version of the billiard task in the first diagnostic test, and I conducted six diagnostic interviews. The results should reveal possible student responses and levels of conceptual understanding and thus examine the first question. For investigating the second question, all participants worked with both versions of the presented digital billiard worksheet (Cartesian coordinates and dynagraph) approximately for one lesson and without teacher guidance because I first wanted to focus on the influence of the digital tasks; for data collection, ten students were videotaped. Afterwards, I conducted a second diagnostic test with a slightly varied version of the presented billiard task and further five diagnostic interviews to enlighten the influence of working with the digital worksheets on students' conceptions. During data analysis I followed qualitative coding procedures according to Grounded Theory – initial, focused, and theoretical coding (Charmaz, 2006).

RESULTS

In this section I first present an analysis of student responses to the first diagnostic test and interviews. This part highlights possible student answers as well as indicates levels of conceptual understanding; in addition, these results could be further developed into a diagnostic tool for formative assessment. Then, I continue to outline exemplary results of students' interactions with the digital worksheet. In a final version, the paper-based task of the diagnostic test was implemented within the digital worksheet (see Task 1 in https://www.geogebra.org/m/tnqamu4x).

Analysis of student responses to a paper-based version of the billiard task

In the first diagnostic test, students worked on the billiard task in a paper-based version of task presented in Figure 1. Students were asked to describe verbally the change of the distance *d* during the movement of the ball, to translate it into a graph representing the distance *d(t)* as a function of time and to explain their considerations. Students' responses were analyzed, condensed and divided into eight different types:

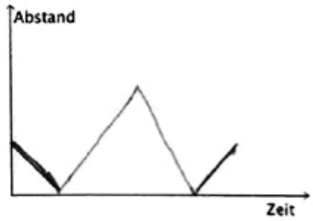
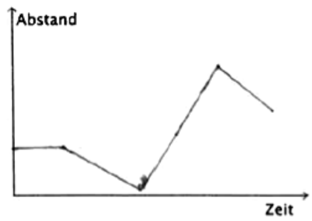
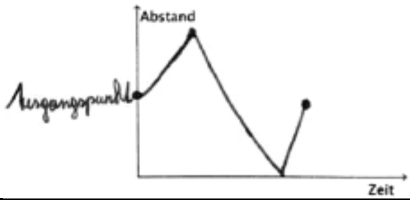
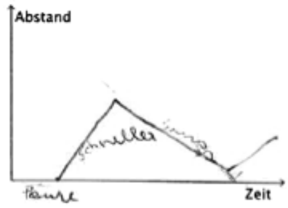
Solution type	Description
1: Correct solution	This type includes a qualitatively correct graphical representation of the distance d as a function of time t including a correct verbal description.
2: Distance to nearest boundary	Distance d was interpreted as respective distance to the nearest boundary. Based on such interpretation, the graphical representation would be correct.
3: Graph too long	The first part of the graphical representations resembles a qualitatively correct solution. Still, two additional linear pieces involving a minimum at the end of the graph are included. Possible reasons for this error could be a misinterpretation of the distance d (see solution type 2) or a confounding influence of the presented iconic situational model. 
4: Confused axes	A further incorrect student response revealed a confusion of x and y-axis resulting in a graph rotated by 90°.
5: Incorrect first part	The changing distance was correctly described; however, it did not start at $t=0$ but included an additional horizontal line from $t=0$ to the point where the correct part started. This solution may imply a local correspondence graph-as-picture error, as the graph resembles the situation where the starting point P is not located on the left border. 
6: Graph-as-picture error	The solution resembles the shape of the path on the billiard table. The image shows a typical graph-as-picture error, with the label "starting point" at the point representing $d(0)$ of the graph. 
7: Measuring path length	This happens when students estimate the length of the billiard ball travelled path instead of considering the distance from the upper boundary. Such an error is possibly influenced by the students' prior knowledge.
8: Merging situational model and speed	Here, two different misinterpretations can be identified. First, the solution resembles the shape of the path on the billiard table and thus indicates a graph-as-picture error. Moreover, the student interpreted the graphical representation as speed-time diagram probably caused by her previous experience in mathematics teaching. 

Table 1: Student solutions task *Billiard*

I analyzed the student responses for possible reasons indicating their conceptual understanding and condensed the solution types within five categories. The solution types could not be related unambiguously to one category as incorrect solutions could be explained by different reasons. The categories and their relations are visualized in

Figure 2. On the left, a violet shaded area visualizes problem solving from a situational perspective, which includes understanding of the presented situation. On the right, the green shaded area represents students' ability to abstract the functional dependency and to transfer it to a graphical representation. The solution categories can be related to these representations and thus additionally demonstrate students' progress in the representational transfer between situational and graphical representation.

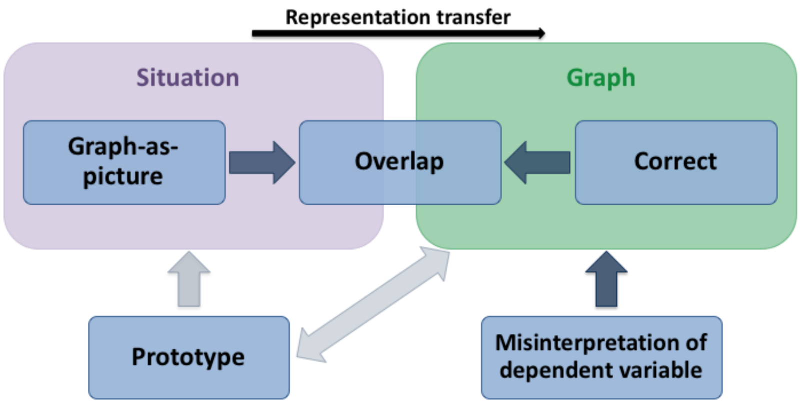


Figure 2: Solution categories developed by the researcher for the billiard task

First, the position of category *Graph-as-picture error* (types 6 and partly 8) indicates that students argued from a situational perspective, and they did not manage to translate the functional dependency to a graphical representation. There appears to exist a *comprehension gap* possibly based on students' difficulties in understanding and interpreting Cartesian coordinates. Second, category *Overlap* (types 3, 5, and possibly 4) contains both graph-as-picture misinterpretations and partly correct translations, illustrated by the two dark arrows pointing at this category. Therefore, students with such type of solutions partly managed to abstract the functional dependency and translate to a graphical representation but were also influenced by situational representations. Third, students who created a qualitatively *correct* graphical representation were able to interpret function graphs by reasoning from an abstract perspective. On the right-hand side, below the green shaded area representing the ability to abstract a functional dependency from a situational model to a graphical representation, the category *Misinterpretation of dependent variable* (types 2 and 3) is visualized. As illustrated by the arrow pointing upwards, based on this misinterpretation of distance students assigned to this category were able to translate correctly to a consistent graphical representation. Finally, the influence of prior knowledge is represented by category *Prototype* (types 7 and 8). Students partly were not able to grasp the presented situation when trying to identify the dependent variable. Data analysis revealed that students' interpretations of the presented situation and identification of the dependent variable seemed to be influenced by previous teaching experiences, which is visualized by an arrow directed upwards. The arrow pointing to the right top symbolizes students who were able to create a consistent graphical representation based on a misinterpretation of the dependent variable. The opposite direction of this arrow exemplifies that additionally function graphs were misinterpreted in a way that reveals the influence of prior knowledge. The pale arrows symbolize less definite and strong relations to the representational registers.

Exemplary results from interacting with the digital billiard worksheet

Depending on students' prior conceptions and their understanding of Cartesian coordinates, the digital worksheet appears to have an adaptational influence on their conceptions, which means that they are not able to alter conceptions to a greater extent but adapt them partly to the direction of a correct conception. In addition, for students whose conceptual understanding can be assigned to the situational perspective (Figure 2), the animation in the situational model of the worksheet can support them in understanding and (dynamically) visualizing the situation as well as for identifying and qualitatively describing the dependent variable. For example, during an interview Mario first confused variables time and distance. However, after watching the billiard task animation, he managed to identify the dependent variable and to describe it qualitatively correctly during the movement of the ball as follows:

67. Mario: So, it [the distance] would have to be smallest here (*first touch of ball on side*) and here largest (*starting point*) . . . Then it increases (*second part of path*) . . . and here (*last part of path*) . . . it decreases again.

Certainly, one cannot determine from this study how lasting such adaptational influence would be particularly for students who were not able to abstract functional dependencies (completely) and to represent them in Cartesian coordinates. For instance, Konstantin managed to solve the billiard task almost correctly in the second diagnostic test after the intervention but could not further explain his answer later in the diagnostic interview conducted nine days later. This agrees with the conceptual change approach that emphasizes that actual restructuring existing conceptions would require more effort and time (Vosniadou & Vamvakoussi, 2006). Data analysis further indicates that if students are not able to abstract and translate the functional dependency to Cartesian coordinates, they were unable to overcome their comprehension gap by working with the digital worksheets without teacher guidance. Probably, they would profit from teachers' assistance to help them reflect and reconsider their perceptions and interpretations.

Participating students tended to interpret visual characteristics of graphs in *trace mode*. In trace mode, the graphical representation does not appear as line but as "sequence" of points. Depending on the speed of the animation, these points do not necessarily appear at the same distance. On the one hand, such behavior could be problematic if students, for example, try to interpret the denseness of points, which is mathematically not relevant but depending on the processor performance. In such case, an option would be to implement a line instead of trace mode for graphical representations. Also, in dynagraph representations a drawn trace pattern appears to make it more difficult for students to detect and read an actual functional value from the y-axis. On the other hand, trace mode can emphasize the relational aspect of functional dependencies. It can be used to mark extrema in dynagraph representations and consequently to compare function values with a maximum value as, for instance, two students did when working with the dynagraph version of the presented task.

During the study a new misinterpretation, which I called *reflection,* evolved. Reflection is a visual feature that students assign to graphical representations when the shape of the graph resembles a reflected image of the situational model. Pia, for example, perceived this visual feature when discussing this digital worksheet (Fig. 1).

102	Interviewer:	How do the graphical representation in the coordinate system and situational model fit together?
103	Pia:	It is upside down, no. It has been reflected.
104	Interviewer:	Why?
105	Pia:	Because it is a graphical representation?

A following discussion did not lead to any further explanations. Her uncertainly expressed last answer shows that Pia only recognized this visual characteristic, but she could not interpret it with respect to the functional dependency. In case of an overgeneralization of this characteristic, it could lead to misinterpretations related to graph-as-picture errors. Consequently, results also reveal a potential difficulty of utilizing the digital worksheet without teacher guidance, especially for students who are not able to interpret mathematically relevant features in a correct way.

In sum, this project resulted in a model representing various levels of students' conceptual understanding of the billiard task and helped to understand the role of the digital worksheet in the learning process. Now the question arises, how to combine these results and implement this digital task into regular teaching for diagnosing and enhancing students' conceptions.

DISCUSSION AND OUTLOOK

In the preceding section, I identified several typical student errors connected to a billiard task grouped within five categories visualized in Figure 2. Although this model could be refined in further studies, it outlines among others the main stages of the representational transfer that students working on the task had to conduct from situational to graphical representation. The analysis of all student responses further suggests a comprehension gap between both representations that can be interpreted as obstacle students have to overcome for managing to translate to and interpret graphical representations. Possible reasons for this gap could be students' difficulties in understanding and interpreting Cartesian coordinates or students' inability to focus on more than one feature or variable. The latter can be examined under the more general term of covariational reasoning (Johnson et al., 2017). In essence, the results reveal different levels of conceptual understanding, which are relevant for implementing the digital task in regular class and could be utilized for diagnostic purposes in a formative assessment tool. Results concerning students' interactions with the digital worksheet show (only) an adaptational influence on conceptions. Especially lower achieving students seem to need a teacher's or peer's support to draw their attention to mathematically relevant features and to reflect their perceptions and interpretations.

Considering design of dynamic materials, using trace mode could either support or hinder learning. Therefore, teachers should consider whether in their case it could provide an opportunity for learning or is better avoided.

These results lead to considerations how to utilize the digital task in regular teaching. New developments in classroom collaboration including dynamic math element question types (e.g., Zöchbauer & Hohenwarter, 2020) will enable to implement the digital task presented in this paper for formative assessment. GeoGebra enables teachers to save student responses, which could be analyzed and assigned to the model presented in Figure 2, thus enabling diagnosis of students' conceptions concerning this task. Based on this categorization, teachers could design specific lessons plans for each group of students including the digital worksheet to enhance their conceptual understanding.

REFERENCES

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2007). How can we assess mathematical understanding? In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 41-48). Seoul: PME.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. London: Sage Publications.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics*, *11*(1–2), 77–87.

Johnson, H. L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM Mathematics Education*, *49*(6), 851–864.

Lichti, M., & Roth, J. (2018). How to foster functional thinking in learning environments using Computer-Based Simulations or Real Materials. *Journal for STEM Education Research*, *1*(1–2), 148–172.

Schlöglhofer, F. (2000). Vom Foto-Graph zum Funktions-Graph. *Mathematik lehren*, *103*, 16–17.

Vollrath, H.-J. (1989). Funktionales Denken. *Journal für Mathematik-Didaktik*, *10*(1), 3–37.

Vosniadou, S., & Vamvakoussi, X. (2006). Examining mathematics learning from a conceptual change point of view. In L. Verschaffel, F. Dochy, M. Boekaerts, & S. Vosniadou (Eds.*), Instructional psychology* (pp. 55–70). Amsterdam: Elsevier.

Zöchbauer, J., & Hohenwarter, M. (2020). *Developoing a collaboration tool to give every students a voice in a classroom discussion.* Paper presented at the ERME Topic Conference ETC 7. Montpellier, France.

Relations between mathematics and programming in school: juxtaposing three different cases

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In this paper we juxtapose and give examples of mathematical aspects of how programming is included in teaching in three different countries; Denmark, Sweden and England. We look at cases of both curriculum standards and resources in order to describe the nature of the relations between programming and mathematics. The methodology consists of a case-based analysis, and can be seen as a first step in developing an understanding of the nature of the relations between programming and mathematics as it is enacted in different educational systems. We discriminate between specific, explicit, implicit and weak relations and use these terms to describe the differences between the cases.

Keywords: programming, computational thinking, mathematics education, digital competencies.

INTRODUCTION

Many countries include programming as a part of the curriculum in compulsory education. This is done in relation to different academic topics and to different degrees. The aim of preparing students for the digital society is a general trend across many educational systems (Bocconi et al., 2016). Mathematics has a special role in relation to this ambition. Computer science shares many aspects of methods and objects with mathematics and the preferred thinking styles and learning objectives to some extent coincide (Misfeldt & Ejsing-Dunn 2015, Wing, 2006), which of course has to do with the fact that computer science as a discipline originates from that of mathematics.

The widespread ambition of teaching programming in compulsory school has increased significantly over the last years and different countries are adopting different routes. Currently, relatively little work has been done in comparing these approaches (Bocconi et al., 2016). In this paper, we make a first attempt at comparing the mathematical aspects of how programming is adopted in compulsory school. We address this by looking at cases of governmental curricular descriptions and teaching materials from England, Sweden and Denmark in order to address the research question: *Based on three different case studies, what types of relations between mathematics and programming in school exist*?

THEORETICAL PERSPECTIVE AND METHODOLOGY

The ambition of using programming as a means to reform mathematics education has been around for the last 40 years, and has led to educational innovations such as programming languages for kids and theoretical frameworks describing the learning of mathematics with programming (Papert, 1980). However, it was not until Jeanette Wing's (2006) much-cited paper was published that the effort of making programming into an integrated part of compulsory education became mainstream (Bocconi et al., 2016). Wing (2006) described computational thinking as decomposition, data representation and pattern recognition, abstractions and algorithms. Educational research and practice has attempted to clarify and activate computational thinking as teachable competencies. This is often done by highlighting how computational thinking relates to mathematical processes such as abstraction, problem solving, modelling and algorithm building (Kafai & Burke, 2013).

From a methodological perspective, we conduct an open juxtaposing (Bereday, 1967) of different cases of including programming into schooling with focus on both the official mathematics curriculum and the specific language used in teaching materials. Our argument builds on case-based reasoning, in the sense that the three combinations of teaching materials and governmental curricular documents that we look at are seen as having the particularity of specific cases (Yin, 2011). In the following three sections, we describe cases of how programming is included in the compulsory school in England, Sweden and Denmark. We focus on the rules and curriculum standards that underpin this movement and on examples of curriculum materials dedicated to support such a change.

CASE 1: PROGRAMMING IN COMPUTING CLASS IN ENGLAND

In England, the former ICT curriculum was replaced by the 'new' subject of computing in the National Curriculum of England in 2014. This new computing curriculum was developed with the support of representatives from the industry, computer scientists, government officials and teachers (Larke, 2019). It aimed at providing school students, from the age of 5 onwards, with the necessary skills, knowledge and thinking to become digital literate and be able to actively participate in a digital world. Common computing activities in secondary classrooms in England are, for example 'Pair Programming', where one pupil programs whilst the other looks on offering advice and they swap at key points or after a certain time; 'Debugging/Programming', where pupils are given some code with errors they need to fix and/or parts they need to amend to add new functionality; and 'Predict and test', where students are given code snippets they need to read (trace) and understand to predict output.

We focus on an activity (Unplugged: Polygon Predictions) taken from the ScratchMaths (SM) project (www.ucl.ac.uk/ioe/research/projects/scratchmaths), which aimed at investigating the learning of computing alongside mathematical concepts. The SM project team has designed an integrated curriculum to teach mathematical ideas by using Scratch, a programming environment, targeting 9-11 year old students (end of primary school in England). Students are presented with three scripts and are asked to "Read each of the scripts. Draw and/or explain in words the picture that it will create (it creates various polygons). Students may think that the scripts will deliver the same outcome due to the same codes being used, even if they are not in the same order. All three scripts lead to the creation of a square with a side of ‘50 steps’, but each square will ‘look’ different. Students would need certain computational thinking skills to identify those differences (Ainley, 2019). The teacher needs to model how to predict the outcome of a script, ideally after allowing students to spend a few minutes on this themselves. The teacher would then run a class discussion to model how to deconstruct each script. Students are expected to ‘access’ their mathematical knowledge of the properties of different polygons, in this case those of a square. As Ainley (2019), a teacher who has used this activity with her students, reported, “in my own experience, ScratchMaths has improved teacher subject knowledge, computational thinking, problem-solving, and my student’s understanding of block coding in Scratch. From a computing point of view, that’s pretty good!” (p.21).

CASE 2: PROGRAMMING IN ALGEBRA IN SWEDISH MATH CLASS

In 2017, the Swedish national curriculum was revised in order to strengthen students’ digital competence (Swedish National Agency of Education, 2018). The main idea was to teach programming in mathematics and apply it in technology. In the mathematics curriculum, a major part of the programming is included in the core content of algebra. An additional item connected to algorithms is included in the core content of problem solving. The revised curriculum is reflected in new editions of Swedish mathematics textbooks, where stepwise instructions and algorithms have been incorporated in close connection to the algebraic content ‘patterns’ (Bråting, Kilhamn & Rolandsson, 2020).

We provide an example of a programming activity for grades 7-9, taken from a Swedish government-provided online material. The aim of the activity is to show how programming can be used to explore new mathematical ideas. The task is to program an algorithm for finding prime numbers. The idea is to use programming as a tool for mathematical problem solving, and at the same time develop a deeper understanding for divisibility and number sense. The activity consists of three steps. In the first step, the students work with the algorithm ‘Sieve of Erastothenes’ with pen and paper. Especially, the students are encouraged to discover general patterns. Some students may even be able to conclude that they only need to investigate the numbers up to $\sqrt{n}$. In the second step, the students together with the teacher are supposed to construct an algorithm with pseudo-code that can answer whether a number is a prime or not. In the third step, the students are to translate the pseudo-code to Python or a similar programming language. In the material, the meaning and usage of algorithms are discussed as a help for the teacher. It is emphasized that the algorithm can solve all problems in a given class, and not only a procedure to solve a single problem. That is to say that the mathematical definition of a problem is considered.

CASE 3: PROGRAMMING IN A NEW SUBJECT IN DANISH SCHOOL

The Danish government is investigating a two-tier strategy to implement programming in compulsory school. This involves the integration of programming into a wide range of topics and the development of a specific topic ‘technology comprehension’ which is being tested in Danish classrooms. The curriculum was piloted in 2017, and is currently (2018-2021) tested in a larger project. The key learning objectives of technology comprehension are: (1) students engage in digital production; (2) students learn to develop, modify and evaluate digital products; and (3) students learn about the role of informatics as a change agent in the society. Technology comprehension is described as an individual topic as well as in relation to arts, design, science, social science, first language and mathematics. The test curriculum standards for introducing technology understanding in mathematics has six focus areas: (1) digital design and design processes, (2) modelling, (3) programming, (4) data algorithms and structures, (5) user studies and redesign, and (6) computer systems. The test curriculum for technology comprehension as an individual topic – which we will focus on here, since it currently seems the most likely decision regarding how to move forward after the test phase – consists of four areas: (1) digital citizenship, (2) digital design and design processes, (3) computational thinking, and (4) technological agency.

An example activity designed to support the work with ‘technology understanding as its own topic’ in grade 8 is the well-known two-person hand-game ‘rock-paper-scissors’. Students are to use data generated from playing this game to get an understanding on how data can help with predicting future outcomes. The associated teaching material is divided into three different stages: coincidences; from coincidences to patterns; and challenges. In the first stage, the students will develop a simple computer script in Python for playing ‘rock-paper-scissors’. The script will have the player choose between 1 and 3, corresponding to the different outcomes a player can make, likewise for the computer, but this time it will be chosen randomly. The students reflect on data based on multiple runs and consider to what extent knowing the probability will affect one’s game. In the second stage, the students will keep on collecting data to see how the probability changes as a consequence of the previous game. The students will also work with bigger sets of data and use all outcomes to measure the probability. In the third stage, the students will use what they learned to create a new game, where they can gather data, use it for predictions and thereby win the game. This will show the students that a game, which has a completely random outcome in the beginning, can be programmed so that the popabilites of certain outcomes changes as a fuction of data about previous played games. The students will present their program to each other and provide feedback.

JUXTAPOSING THE THREE DIFFERENT APPROACHES

The different material involves programming in various programming languages. In none of the countries, the curriculum standards endorse specific programming languages. Yet, while both Danish and English curricula talk about the learning outcomes of programming activities on rather abstract terms (value for the society, engagement in digital production), the Swedish case shows that curriculum standards can be quite specific in relation to what type of programming language (e.g. visual versus textual programming) the students should work with at different levels. If we look at the examples of teaching sequences from the different countries, they all point to specific programming languages/environments that students and teachers should work with. This is no surprise since the specification of course is a help in terms of instructing teachers and students on how to handle specific difficulties and develop specific solutions in these environments. All three cases of teaching activities point to the ability to work with and handle algorithms as an important part of the work with programming. In the following, we consider the cases of curriculum standards and materials that relate to programming and computational thinking and compare how they refer to mathematics. In order to describe the differences between the three cases we are distinguishing between four different ways to see relations between mathematics and programming: (1) *specific relations* to mathematical concepts or processes, when a curriculum standard text or an educational resource states the relation to a specific area of mathematics or a specific mathematical process; (2) *explicit relations* to mathematics, when mathematics, mathematical working processes, and mathematical competencies are referred to explicitly; (3) *implicit relations* to mathematics, where we can interpret the activity or educational intention as embodying mathematical work, but where this relation is not uttered in an overt way; and (4) no or *weak relations* to mathematics.

The English Computing curriculum puts emphasis on students' developing programming skills and computational thinking, understanding and applying "the fundamental principles and concepts of computer science, including abstraction, logic, algorithms and data representation" (DfE, 2013, p.1), whereas the National Mathematics curriculum in England refers to the use of ICT tools when necessary based on the teacher's judgement (DfE, 2014), without any reference to programming, computational thinking or digital competencies. Therefore the computing curriculum argues for an *explicit relation* to mathematics stating examples of relatively *specific relations* (e.g. abstraction, logic, algorithms, data representation), but the mathematics curriculum only refers to the use of ICT tools when necessary, and without any mentioning of computational thinking, which suggests a documentational *weak or no relation* between programming and mathematics. Even in computing lessons, the links to mathematics may not be as strong due to the subject knowledge of the teacher, who may or may not be a computer science specialist, and may or may not have strong mathematical knowledge (Larke, 2019; Mee, 2020). The Scratchmaths example we presented was an intervention especially targeting the potential that Scratch offers for the learning of certain mathematical concepts and was chosen due to its potential to showcase how the relations between programming and mathematics can be *specific and explicit*, even if in the national curriculum documents these relations are *implicit, weak or non-existent*. In the example we presented, the *specific relation* between the characteristics of polygons and the ways in which a square in particular can be constructed in Scratch are suggested. In the class discussions, students are expected to explore and de-construct the scripts by relying both on their mathematical thinking and competencies (knowledge of what a square is, its properties and ways to construct a square), but also digital competencies (knowledge of Scratch language and familiarity with programming in Scratch), hence making the relation between programming and mathematics *explicit*. Still, despite Scratchmaths being a great initiative, the current reality in the English educational system remains as having mostly *implicit*, *weak* or *no relations* between programming and mathematics in practice.

In Sweden, the connection between programming and mathematics as a subject is quite clear. The Swedish mathematics curriculum document has incorporated programming in the core content of algebra, but also within the mathematical problem solving content. However, there is no explicit formulation or explanation regarding *how* programming connects to algebra or algebraic concepts in the curriculum document; in that sense the relation is *explicit* and relatively *specific*. The main focus of the programming content in the mathematics curriculum is on algorithms. In the described teaching activity, there is a *specific* relation between programming and mathematics in the sense that the students are to learn programming as well as developing a deeper understanding of divisibility and number sense. It is also noticeable that the students are encouraged to discover patterns, which could be seen as a *specific* relation between computational thinking and algebraic thinking. According to the instructions to the teacher, the aim with the activity is to use programming as a tool for problem solving, which also is in line with the mathematics curriculum document for grades 7-9. The usage of programming as a tool for solving mathematical problems may be interpreted as an *explicit, but not specific* relation between programming and mathematics in the sense that the students do not learn mathematics primarily. The main focus in this activity, as well as most of the activities in the government provided teaching material, is to work with algorithms, structure and the approach of breaking down a problem in smaller steps. This is interpreted as an *explicit*, *non-specific* relation between programming and mathematics.

In relation to Denmark, it of course makes a difference if we talk about technology comprehension as its own topic or as a part of mathematics. Yet, in both cases, the relations to mathematical concepts and processes are rather un*specific*. In the latter case, there is of course an explicit relation to mathematics. Both because of the *placement* of the curriculum standards in relation to the topic mathematics, but also because the standards highlights aspects that are *meaningful from a mathematical perspective* such as modelling and data algorithms and structures. The educational material is built around a challenge that focuses on the breakdown of a well-known game into a programmable structure. It is strongly related to algorithmic thinking and modelling. The inferences that students make are scaffolded to move from naive decisions based on prejudiced opinions towards inferences based on data and probabilities. This has *explicit*, but not *specific*, relations to statistics. The design of a game also allows the students to enact design thinking and problem solving. From a mathematical point of view, this teaching sequence highlights statistics and probabilities as well as modelling and problem solving. In terms of *specific relations to mathematics*, these appear in the curriculum standards regarding technology comprehension as part of mathematics, but also in the specific curriculum material (especially relations to statistics). When it comes to *explicit* but not *specific relations* to mathematics, the entire version of the curriculum, where mathematics is part of technology comprehension is exemplifying this. *Implicit relations* to mathematics are present in the standards for technology comprehension as an individual topic.

CONCLUSION

In order to answer the question about the differences and similarities in the approaches taken by England, Sweden and Denmark to incorporate programming into school, and in particular how specific and explicit materials and curriculum standards relate programming and computational thinking to mathematics, the materials in the presented cases suggests two observations. The first observation is that the curriculum standards of different countries have different disciplinary affinities to programming. In Sweden, programming is clearly related to mathematics and the standards are both explicitly and specifically related to algebra and programming. Neither England nor Denmark have the same level of specificity in this relation, mainly because none of their curriculum standards has chosen to relate programming to specific mathematical topics. The second observation is that even though the curriculum standards differ in specificity of the relations to mathematics, this difference cannot be seen in the example cases of the teaching materials. Clearly, the data presented is too sparse to allow general claims. Nevertheless, the *explicit* relations to mathematical problem solving are seen in all the activities, and this together with some relatively *specific* relations to mathematical content applied in the materials. In conclusion, we point to the fact that all three countries – England, Sweden and Denmark – to some extent build on mathematics, when introducing programming in their school curriculum. Yet, there is a large difference in specificity as to how mathematics enters curriculum standards.

REFERENCES

Ainley, M. (2019). Hands on with teaching Scratchmaths: Examining ScratchMaths as a vehicle for developing computational thinking – and making lessons effective. *helloworld cc*, Issue 10, October, pp.19-21.

Bereday, G. Z. (1967). Reflections on comparative methodology in education, 1964-1966. Comparative Education, 3(3), 169–287.

Bocconi, S., Chioccariello, A., Dettori, G., Ferrari, A., & Engelhardt, K. (2016). Developing computational thinking in compulsory education – Implications for policy and practice; EUR 28295 EN; doi:10.2791/792158

Brennan, K. & Resnick, M., (2012). New frameworks for studying and assessing the development of computational thinking. In *Proceedings of the 2012 annual meeting of the American Educational Research Association*, Vancouver, Canada.

Bråting, K., Kilhamn, C., & Rolandsson, L. (2020). Integrating programming in Swedish school mathematics: description of a research project. Presented at MADIF12, the twelfth research seminar of the Swedish Society for Research in Mathematics Education, Växjö, Jan 14-15. In press.

Department for Education. (2013). National curriculum in England: Computing programmes of study. Retrieved from https://www.gov.uk

Department for Education. (2014). The national curriculum in England: Key stages 3 and 4 framework document. Retrieved from https://www.gov.uk

Kafai, Y. B., & Burke, Q. (2013). Computer programming goes back to school. *Phi Delta Kappan*, 95(1), 61-65.

Larke, L. R. (2019). Agentic neglect: Teachers as gatekeepers of England's national computing curriculum. British Journal for Educational Technology, 50(3), 1137-1150.

Mee, A. (2020). Computing in the school curriculum: a survey of 100 teachers. DOI: 10.13140/RG.2.2.19883.59689 Retrieved from https://www.researchgate.net/publication/339536481_Computing_in_the_school_curriculum_a_survey_of_100_teachers

Misfeldt, M., & Ejsing-Duun, S. (2015). Learning Mathematics through Programming: An Instrumental Approach to Potentials and Pitfalls. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2524–2530). Prague, Czech Republic: Charles University in Prague.

Niss, M. & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*, 9–28.

Papert, S. (1980). *Mindstorms: Computers, children, and powerful ideas*. NY: Basic Books.

Swedish National Agency for Education. (2018). Läroplan för grundskolan, förskoleklassen och fritidshemmet. Stockholm: Fritzes.

Wing, J. M. (2006). Computational thinking. *Communications of the ACM, 49*(3), 33–35.

Yin, R. K. (2011). *Applications of case study research*. Sage.

An investigation on the use of GeoGebra in university level calculus

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Geogebra has shown great potential in school mathematics and some areas of university level math. However, it seems unclear how integration into a more abstract calculus should be done. The purpose of this paper is to describe the approach taken and reports some promising results.

Keywords: Calculus, Geogebra, logic, formal mathematics

INTRODUCTION

The shift to formal mathematics is a major obstacle for first year university students and it is currently of broad interest in Germany (e.g. Hoppenbrock et al. 2016) and internationally (e.g. Gonzales-Martin et al. 2017). In Germany students enter university after having gained a high school diploma. The curriculum of these schools includes some basic calculus (derivatives and integral) but on a very informal level where proofs play almost no role at all. Thus, when starting at the university they experience a substantial gap that results in high failure rates in examinations after the first term (typical failure rates 70-80%).

The learning of calculus has been investigated by many researchers. A recent overview is given by Bressoud et al. (2016). Insight has been gained into many problems that students face when learning university level calculus, e.g. problems with logic (e.g. Selden& Selden (1995), Shipman (2016)) and proofs (e.g. Stavrou (2014)). A wider overview is also given in (Winslow 2018).

The use of technology is discussed in a variety of papers as well. Tall (2003) has argued that technology allows for an embodied approach to teaching calculus by making notions dynamic and visible. Similarly, Moreno-Armella (2014) argued that the traditional teaching approach is not able to bridge the tension between intuition and formalism. He suggests some dynamic activities that illustrate limiting processes and involve differentials as small changes.

A lot of research has investigated the use of dynamic math software such as Geogebra (Hohenwarter 2019) for the learning of mathematics in general and also of calculus. However, the majority of research concentrates on the high school level. Beyond high school college calculus is investigated to some extent but there are only a few investigations about using Geogebra at the university level of analysis. In Tall et al. (2008) an overview is given that aims mainly at the high school level but presents also ideas beyond that. One paper that focusses on university level analysis is Attorps et al. (2016). They find positive effect in teaching Taylor approximations using a variation-theory based approach. d'Azevedo Breda and dos Santos dos Santos (2015) investigated complex numbers. Nobre et al. (2016) have positively evaluated the use of Geogebra in a calculus course for computer science students. However, the topics touched are more of the college style calculus. Much the same can be said about Machromah et al. (2018).

The contribution of this paper is new as it addresses rigorous university analysis.

THE STUDY DESIGN

The course

The course "Analysis I" was taught by the author in the summer term 2019 (duration 14 weeks). 180 students were enrolled into the course with 141 taking the examination at the end. Students' age and sex was not recorded for reasons of privacy but age was approximately 20 and sex distribution almost equal.

The main learning objective of this course is to introduce students to the rigour of mathematics. This course is taken mainly by students aiming at a bachelor in mathematics, but also students from physics and trainee teachers for high schools. The content includes logic, axiomatic theory of natural, rational, real and complex numbers, sequences, series and convergence, limits, continuity, differentiability, sequences of functions, Taylor series, and integral. The approach is rigorous, i.e. all statements are proven and exercise for students included a lot of proof tasks. The course consists of 4 h lecture per week, 2 h exercises in a huge group, and homework exercises which are graded and discussed in small groups (2 h / week).

The setting implies that many concepts have to be re-learned by the students, e.g. in high school the sine and cosine functions are defined geometrically while in this course they are defined by the exponential series. Traditionally, computers are practically absent from such courses. However, for the redesigned course reported here, computers were used to some extent (also Mathematica). This paper concentrates on the use of Geogebra. About half of the students reported that they knew Geogebra from high school. The use of Geogebra was twofold:

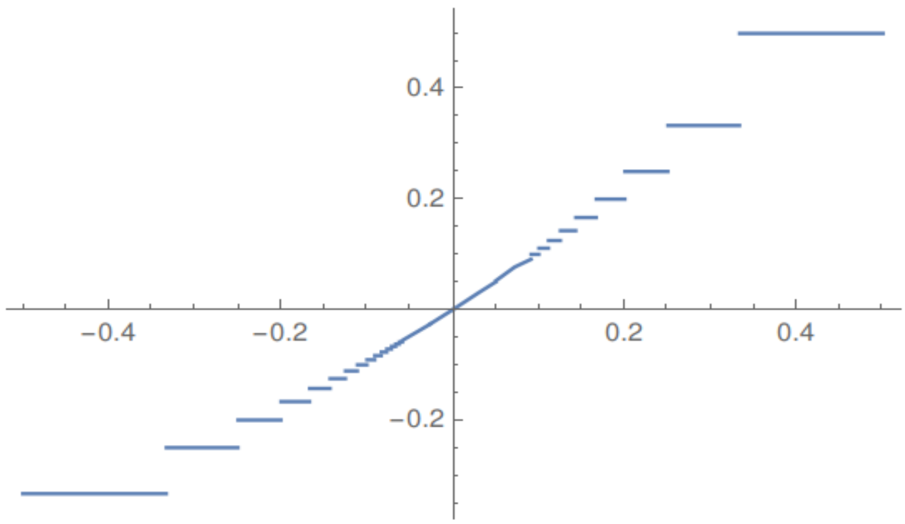
- Demonstrations in the lectures. Many concepts, e.g. addition and multiplication of complex numbers, epsilon-strip-concept of convergence, convergence of function sequences (in general and particular for Taylor series), epsilon-delta-definition of continuity, local linearity of differentiable functions etc were visualized.
- Non-mandatory home work. Every week a set of homework assignments were given and some of them were mandatory and graded, however, for legal reasons, the computer assignments were voluntary.

Task design

The following example illustrates the use of Geogebra in homework assignments:

Exercise: Investigate where the function $f\colon\left(\frac{-1}{2},\frac{1}{2}\right)\to\mathbb{R}, f(0):=0; f(x\neq 0):=\frac{1}{\left\lceil\frac{1}{x}\right\rceil}$ has a derivative.

In doing this it is very useful to plot the function as this gives the idea that it may be differentiable in the origin with $f'(0) = 1$, which is a bit contra intuitive. (I learned this nice example from Peter Quast, Augsburg).



The didactical principle behind this task design is a kind of variation theory (Maton & Booth 1997). In mathematics education this theory has been mainly applied in elementary school mathematics. A very typical example is the use in a teaching experiment on logarithms (O'Neil & Doerr 2015). In my own conceptualization the theory says that learning materials should be arranged to allow the individual genesis of a concept by contrasting examples and counter-examples, experience relations to hold of a variety of examples, identify single aspects, exclude counter examples and fuse several aspects to the general concept. Applied to the concept of differentiability this leads to the following learning trajectory: Students learned the concept definition $f'(x) \coloneqq \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ already in high school. This definition emphasizes the aspect of rate of change which is applied to determine the slope of tangents. Thus, in my course the derivative was introduced in the following varied manner: f is differentiable in x_0 if there is a function $q: U \to \mathbb{R}, x_o \in U$ defined on some open neighbourhood of x_0 and continuous in x_0 such that $f(x) - f(x_0) = q(x) \cdot (x - x_0)$, i.e. $\Delta y = q(x) \cdot \Delta x$. This definition emphasizes local linearity and students were demonstrated in the lecture that graphs of differentiable functions appear straight when zoomed in at a sufficient scaling factor. Variation theory then suggested to explore a bunch of functions to sharpen the concept. The example given above in the example is the most challenging in this series.

Assessment

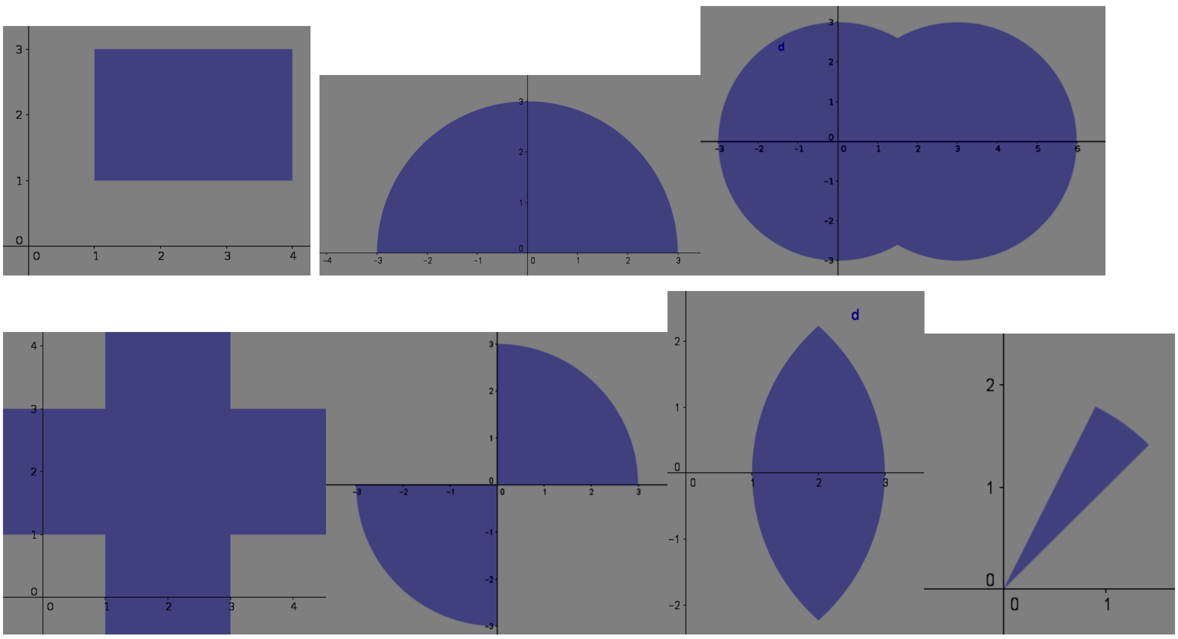
The general research question would be if this kind of using Geogebra helps students to master the course. In this generality, of course, the question cannot be answered empirically, and more precise questions will be posed later on.

In general, empirical intervention studies at university level are not easy to carry out. Ideally, one would randomly split courses into groups with different treatment and measure results. However, splitting a course requires teaching resources that are rarely available and spitting also raises the ethical issue if some students are offered better conditions than others. In this situation the problem that computer exercises could not be made mandatory turned out to offer a new possibility for research: Students themselves decided if they did the computer exercises or not. Hence, this provided two groups without ethical problems. However, one should not assume these two groups to be equivalent. It seems likely that students doing the exercises might be more interested, more motivated and thus stronger overall. The methodological trick to solve this problem was to give two different kinds of tasks: One that could potentially profit from the computer exercises because the mathematical content was related and another that was not expected to benefit from doing the computer exercises. The categorization of the tasks into these groups was done by my own expertise; the tasks used for assessment are detailed below. They were chosen to reflect some of the many teaching goals of this course, especially they should assess the understanding of the logical argumentation about sets, sequences and functions.

FIRST STUDY

The mathematical topics dealt with in the beginning were logic and sets. During the first week the following (non-mandatory) computer exercise was given:

> **Task (voluntary)**: Logic with Geogebra
>
> a) Geogebra can plot the set of solutions of certain (not too complex) inequalities in the two variables x, y. Try this out using the following inequalities:
>
> 1) $x + 1 > y - x/2$ 2) $x \cdot y < 4$ 3) $x^2 + y^2 < 9$ 4) $2x^2 + y^2 > 5$
>
> b) One may also plot logical combinations of inequalities. Try out: $x > 1 \wedge x < 4,\ x > 1 \vee x < -1,\ x > 0 \wedge \neg y > 0,\ x > 0 \rightarrow y > 0$
>
> c) Find ways to describe these sets:



The rationale behind this task should be obvious: Students should have the opportunity to work in visually appealing setting with logical operators that gives direct feedback. The importance of feedback is widely acknowledged (e.g. Hattie & Timperley 2007), so this should be effective.

During the second week the students had to do homework exercise that had to be done on paper and were graded. Two of these mandatory exercises are given below:

Exercise 1

a) Prove: $M = N \Leftrightarrow M \subset N \wedge N \subset M$.

b) Prove both *de Morgan* laws for sets.

c) Illustrate the symmetric set difference $M \Delta N \coloneqq (M \cup N) \setminus (M \cap N)$ and prove: $M \setminus N = M \Delta (M \cap N)$.

Exercise 2

Find pairs of equal sets and prove equality resp. inequality:

$M_1 = \{(x,y) | \neg(x > 2 \wedge x < 3)\}, M_2 = \{(x,y) | x \cdot y > 0\}$

$M_3 = \{(x,y) | x > 2 \wedge y > 0 \vee y < 0\}$

$M_4 = \{(x,y) | x > 0 \wedge y > 0 \vee x < 0 \wedge y < 0\}$

$M_5 = \{(x,y) | x \leq 2 \vee x \geq 3\}, M_6 = \{(x,y) | \neg((x \leq 2 \vee y \leq 0) \wedge y \geq 0)\}$

My expert classification was that Exercise 2 might benefit from doing the Geogebra task, while little effect of Geogebra use on exercise 1 was to be expected. Thus, the hypothesis was that students who decided to do the Geogebra tasks would perform substantially better on exercise 3 but not better or only slightly better on the other tasks.

To assess which students took the voluntary Geogebra task students were asked explicitly to indicate if they did do the Geogebra task and then they were asked to rank the intensity on a Likert scale from 0 (not done) to 5 (intensely).

Unfortunately, several of the master students that ranked the students' papers forgot to write down these engagement variables and due to privacy issues, it was not possible to get this information. Hence, the usable data set consists of a rather small sample of n=23 students, 11 of them indicated that they had worked on the Geogebra task (group G), 12 indicated that they didn't (group N). Statistics (all done in R, www.r-project.org) is thus limited but here are the results:

E1, E2 denote the score students achieved at exercises 1 and 2 respectively. These variables can be considered to be normally distributed as the Shapiro test gives p-values for $E1$ of 0.33 for the whole group and of 0.48, respectively 0.45 for the N and G groups. However, $E2$ cannot be considered to be distributed normally. Thus, the Wilcox test is applied to discover group differences between the N and G groups.

Exercise	Wilcox-Test	Cohen d: G-N
$E1$	0.27	-0.371
$E2$	0.014 *	0.842

Conclusion: The students who worked on the Geogebra task scored significantly better on the third tasked, as expected. The fact that they performed worse (although not significantly) on exercise 1 came as a surprise and there is no good explanation yet. It is likely that this is just due to the small number of students, but it may also be that good and theoretically-minded students did not do the computer exercises.

Another way to explore the findings statistically is to use a linear regression model that includes the information (provided by the students) on the intensity of their technology use T. Although this is not normally distributed, a linear model was devised: $E2\sim T+E1$ and it turned out, that T is significant, the whole explained variance is $R^2=0.44$. Given the fact that many other issues influence performance on such tasks this should be regarded as being rather high.

SECOND STUDY

The second study was conducted almost at the end of the course, in week 12. The methodology was the same as in the first study. While the first study focussed on a very small intervention the second study was more designed to account for the whole learning effect during the term.

A total of $n=97$ students' exercise responses could be used in the statistics. First, there were two Likert-scale items to judge agreement with a statement from 0 to 10:

a) "I used Geogebra regularly for this course." Mean: 3.4, Std. dev.: 3.0

b) "Geogebra is a useful tool for learning in this course." Mean: 6.7, Std.: 2.6

Those students who marked 5 or more on the first question were considered to be the Geogebra user group (G, 37 students), the others the non-users (N: 60 students)

The marked mandatory exercises that were used in this study were the following:

Exercise 1 Prove for which $k \in \{1,2,3\}$ the functions $f_k: \mathbb{R} \to \mathbb{R}, f_k(x) := \begin{cases} \sin(x) + x^k \cdot \sin\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases}$ are differentiable and if the derivatives are continuous.

Exercise 2: Prove: If $f: \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $\exists c > 0 \exists d > 0: \forall x \geq d: f'(x) > c$, then $\lim_{x\to\infty} f(x) = \infty$. Give an example that shows that the conclusion is not valid if one only demands: $\forall x \geq d: f'(x) > 0$.

The choice of these tasks was mainly driven to match the topics of the lecture in that week but some theoretical considerations came into play: Given the application of Geogebra to explore the concept of derivative (explained above) I assumed that

Geogebra-affine students can use the tool to foster their intuition about what is going on here. Thus, it was expected that this task benefits from using Geogebra. However, it seems not obvious that the transfer from the graphical setting to a written proof that as required here can be made. The second task does not invite for plotting as no concrete function is given. Moreover, it deals with quantifiers that are not touched on in any Geogebra activity. As above, students' solutions were marked and graded by points by master students. For all these variables, the hypothesis of normal distribution was checked using the Shapiro test and had to be rejected.

Our general hypothesis is that students who used Geogebra regularly performed better than others. More specifically: Use of Geogebra should boost results of Exercise 1 because students who used Geogebra regularly could be expected to investigate this functions' graphs and used zooming in to investigate the limit empirically. For exercise 2 I didn't expect a benefit of using Geogebra besides the baseline effect caused by the fact that Geogebra use was likely to correlate with motivation and engagement. Wilcox tests were performed to test this hypothesis.

Exercise	Wilcox-Test	Cohen d: G-N
E1	0.00 **	0.52
E2	0.63	0.05

These results nicely confirmed the hypothesis.

DISCUSSION AND CONCLSION

Geogebra is a tool that can be used both in high school and at the university level and thus offering the advantage that students experience some continuity in the tool as the experience the rather radical change of mathematical culture from school to university. The study adds evidence to the proposition that Geogebra can be used to boost students' performance on certain tasks of rigorous analysis. Besides the usefulness to visualize graphs in this study the plotting of solutions to logical combinations of inequalities proved to be a useful teaching tools that should be studied in more detail.

REFERENCES

Attorps, I., Björk, K., Radic, M. (2016) Generating the patterns of variation with GeoGebra: the case of polynomial approximations. Int. J. Math. Educ. Sci. Technol. 47, No. 1, 45-57.

d'Azevedo Breda, A. M., dos Santos dos Santos, J. (2015). Complex functions with GeoGebra. In: Amado, N´elia (ed.) et al., Proceedings of ICTMT 12. Faro: University of Algarve (ISBN 978-989-8472-68-7). 277-284.

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., Törner, G. (2016). Teaching and Learning of Calculus. Springer Open.

Demarois,P., Tall, D. (1996). Facets and Layers of the Function Concept, Proceedings of PME 20, Valencia, 2, 297–304.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. Educational Studies in Mathematics, 61, 103-131.

González-Martín, A. S. et al. (2017). Introduction to the papers of TWG14: University mathematics education. Proceedings of CERME 10, 1953-1960.

Hattie, J., Timperley, H. (2007). The Power of Feedback. Review of Educational Research, 77(1), 81–112.

Hohenwarter, M. (2018). Geogebra. https://www.geogebra.org/

Hoppenbrock, A., Biehler, R., Hochmuth, R., Rück, HG. (Eds). (2016). Lehren und Lernen in der Studieneingangsphase. Springer, Wiesbaden.

Machromah, I. U. , Purnomo, M. E. R., Sari, C. K. (2018). Learning calculus with Geogebra at college. IOP Conf. Series: Journal of Physics: Conf. Series 1180.

Marton, F., Booth, S. (1997). Learning and Awareness. Mahwah, NJ: Lawrence Erlbaum.

Moreno-Armella, L. (2014). An essential tension in mathematics education. ZDM Mathematics Education, 46, 621–633.

Nobre, C. N., Meireles, M. R. G., Vieria Junior, N., Resende, M. N., da Costa, L. E., Rocha, R. C. (2016). The Use of Geogebra Software as a Calculus Teaching and Learning Tool. Informatics in Education, 2016, Vol. 15, No. 2, 253–267

O'Neil, A. H., Doerr H. M. (2015). Using variation theory to design tasks to support students' understanding of logarithms. Proceedings of CERME 9.

Selden, J., Selden, A. (1995). Unpacking the logic of mathematical statements. Ed. Studies in Math. Vol. 29, 2, 123-151.

Stavrou, S. (2014). Common errors and misconceptions in mathematical proving by education undergraduates. Issues Undergrad. Math. Prep. Sch. Teach.,1.

Tall, D. (2003). Using technology to support an embodied approach to learning concepts in mathematics. In L. Carvalho & L. Guimara˜es (Eds.), Historia e tecnologia no ensino da matematica (Vol. 1, pp. 1–28).

Tall, D., Smith, D., Piez, C. (2008). Technology and Calculus. In: M. K. Heid & G. W. Blume: Research on Technology and the Teaching and Learning of Mathematics, pp. 207-258.

Winsløw C. (2018) Analysis Teaching and Learning. In: Lerman S. (eds) Encyclopedia of Mathematics Education. Springer, Cham.

Eliciting students’ thinking about change: filling a vase in a computer application

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Computer applications and digital tasks can be designed in ways that support mathematical learning. In this paper we discuss the results of a teacher experiment involving a computer application that was designed to elicit students' covariational thinking when learning about graphs. The application provides adaptive feedback based on students' graphs, that are created using free-hand drawing. We focus on students' learning with two animations of the application. Analysis of students' graphs and written explanations showed that most students improved their initial graph or they kept the same graph as it was originally correct after working with the application. These results and its implications will be discussed.

Keywords: dynamic graphs, covariational reasoning, task design, digital application.

INTRODUCTION

The design of digital resources and tasks that promote students’ covariational reasoning has shown to be a challenge in the field of mathematical education. Several researchers have designed and investigated tasks involving dynamic computer environments to elicit covariational thinking. However, in some cases students working on the tasks don’t engage in the intended reasoning (Johnson, McClintock, & Hornbein, 2017). One of the aspects that affect students’ engagement in covariational reasoning is students’ conceptions of quantities and change (Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, 2017). There is few research that focuses on this aspect and how computer applications can be designed to afford it in ways that promote covariational reasoning (Johnson et al, 2017).

In our research, we developed an interactive digital application, Interactive Virtual Math (IVM), that elicits students to generate and adapt their graphs. The innovative aspect of the application is that it generates adaptive feedback based on students’ graphs, they created using free-hand drawing on a tablet screen or with a mouse on a computer (Palha & Koopman, 2017). The aim of this paper is to present part of the results of a teaching experiment with the application that was conducted at secondary school involving 20 students. We focus on the learning occurring with two features of the IVM-application that were designed to elicit students’ covariational reasoning when learning about graphs. The results provide interesting insights into students’ thinking that can be useful for researchers, task designers and curriculum developers.

THEORETICAL PERSPECTIVES AND TASK DESIGN

Covariational reasoning framework

Thompson and Carlson (2017) developed a framework explicating levels of students' covariational reasoning. The levels are presented in Table 1. We framed our research and analysis of students' reasoning in this framework. We illustrate this with the well-known problem filling the bottle (Carlson, Jacobs, Coe, Larsen & Hsu, 2002): a spherical bottle (or vase) is filled with water at a constant rate, imagine how the height of the water in the vase varies as function of the volume of water. At the highest levels the researchers placed *smooth continuous covariation* and *chunky continuous covariation*, which entails that students conceive quantities varying smoothly through intervals simultaneously. The difference is that in a smooth continuous covariation the covariation is continuous within any interval. While at chunky continuous covariation students envision change in chunks. For instance, in the case of the vase-task a student at chunky continuous level imagine the water level rising for each increment of water added, including all values of volume and height between successive values, but without envisioning height and volume passing through those values. At lower levels of the framework, Thompson and Carlson (2017) place *coordination of values* (e.g. focus on the water's height in the vase and the number of cups of water added to the vase with no thought given to intermediate values of volume or height), *gross covariation* and *pre-coordination* (e.g. notice that after some amount of water is poured into the vase, the water level on the vase rises).

Level	Description
Smooth continuous covariation	The student envisions changes in one quantity's or variable's value as occurring simultaneously with changes in another variable's value, and envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The student envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and envision both variables varying with chunky continuous variation.
Coordination of values	The student coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The student forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The student does not envision that individual values of quantities go together.
Precoordination	The student envisions two variables' values varying, but asynchronously.
No coordination	The student has no image of variables varying together.

Table 1: Levels of covariational reasoning

According to this framework within the context of filling the vase, a student "conceiving of how the values of height could vary as a vase continually fills (smooth continuous variation) with liquid would be more advanced than just conceiving of the height as increasing, without attending to values for which it might be increasing (gross variation)"(Thompson & Moore, 2017; Johnson et al, 2017).

Covariational reasoning with the support of technology

Computer applications can support students to develop functional thinking by making multiple representations available and allow for actions on those representations (Kaput, 1992 in Ferrara, Pratt, & Robutti, 2006). Research on covariational thinking in the field of functions show that computer applications can enable students to draw, move and modify graphical representations from covariational relations. Also, with the evolution of technology, software design has ‘increased fluidity between representations” (p. 252, Ferrara et al., 2006), which have been found to be beneficial in exploiting the connectivity between representational systems.

Design of the digital environment

One theory often used in task design is Marton (2015) theory of invariance. According to this theory the task designers develop task sequences that incorporate certain patterns of variation. The idea is to foster students’ discernment of critical aspects. In the design of the IVM-application and sequence of tasks we defined a critical aspect for students to discern: smooth continuous covariation between height and volume. The IVM application contains a sequence of tasks about dynamic phenomena and requires students to draw a graph representing the relation of the two variables that covary. The IVM-application uses the well-known problem filling the bottle (Carlson, Jacobs, Coe, Larsen & Hsu, 2002).

Sequence of tasks: The tasks we developed regard the same dynamic event. The first task is about a spherical vase, based on the bottle filling task. The other four tasks involve different forms of vases. Together the five tasks address main types of rise: decreasing followed by increasing rise (task 1), decreasing rise (task 2); constant rise (task 3), increasing followed by decreasing rise (task 4) and increasing rise (task 5). To solve the vase task, the students need to consider how the dependent variable (height) changes (rise) while imagining changes in the independent variable (volume). The coordination of such changes involves covariational reasoning.

The application incorporates several components that assist students in solving the tasks and support their reasoning. Two of these components are the animations: *fill the vase* and *move the dots*.

Animation fill the vase: At the start, the application presents the learner with the animation shown in Fig. 1 (left). Every time the student pushes the arrow button, a constant amount of water will flow into the vase, with a distinct color. Because of the different colors, the students can easily notice that the same amount of water corresponds with different increments in height. Because the student can fill the vase with more (up to six) layers if desired, he or she can observe the variation in height and try to visualize it. The student can vary the size of the cup.

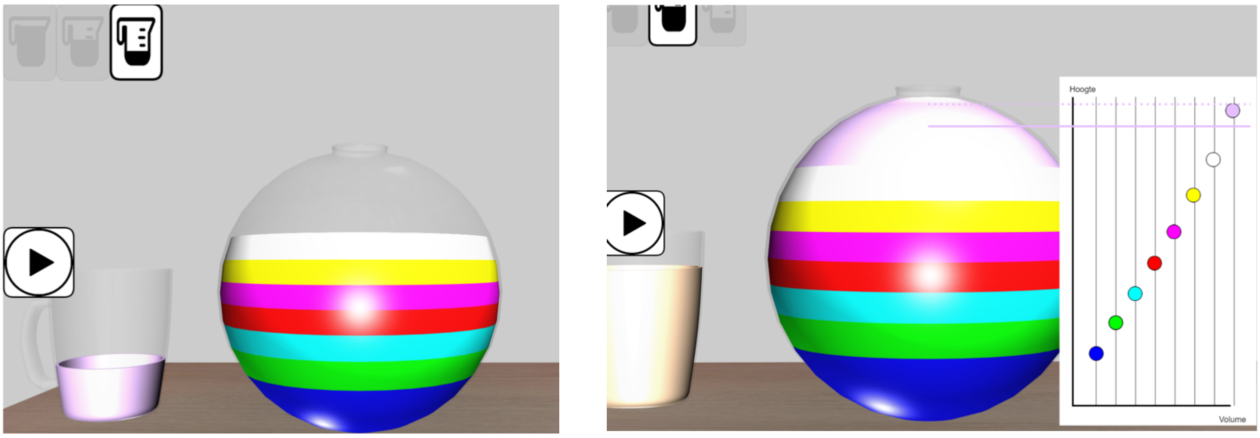

Figure 1. Animation fill the vase (left) and animation move the dots (right)

Animation move the dots: When students fail to represent the covariational relationship correctly the application displays the animation move the dots, see Fig. 1, (right). This animation relates the dynamical event (represented in animation fill the vase) to the more abstract representation (cartesian graph). The student sees an empty cartesian graph in which he or she can move given dots to the estimated height every time the same amount of water is added. The student moves the first dot to the position he/she thinks that the height of the water will reach and then press an arrow button on the water-reservoirs left to the vase. The water fills the vase to a certain height. The student compares the estimated height to the height actually reached and uses the comparison to move the dot to a more precise location. This animation allows students to interact with the elements of a cartesian graph (dots and axes), and it stimulates students to consider the variation of the quantities in relation to each other and at a more abstract level (within a graph representation).

METHOD

Students' covariational reasoning was investigated through qualitative analysis of students' graphs and written answers to the spherical vase task and using the covariational reasoning framework of Thompson and Carlson (2017). The spherical vase task allows for reasoning at all levels and has been used by Thompson and Carlson and several other researchers to investigate covariational reasoning. First, the task is presented to the students as a paper and pencil task at the start of the lesson and again later in the IVM-application (task 1), after they have worked with the animations *filling the vase* and *moving the dots*.

Context and participants

We used data from a teaching experiment conducted in one regular secondary school in The Netherlands. The participants were twenty 10th grade students (14-15 years old) at general education. In the Netherlands, students usually are introduced to graphs in dynamic events during the first year of lower secondary education (12 -13 years old). They are posed tasks similar to the one of the filling vase and requested to draw and interpret the related graph. Later on they learn to draw graphs pointwise and using tables. Typically, when students are asked to draw a graph, they are used to construct the table first and then to draw the graph pointwise. Although mathematical reasoning is valued in the Dutch curriculum there is no explicit attention for covariational reasoning.

Data collection and analysis

The IVM-application was used during one lesson (about 20 minutes). At the beginning of the lesson the teacher asks the students to individually solve the spherical vase task and explain their reasoning. After filling in the task the teacher collected students answers and tablets were distributed. The teacher provided instructions about how to access the computer application. Each student worked individually with the IVM-application. Both authors were present and observed the lesson using a semi structured observation form.

The collected data consists of researchers' records of the lesson observation, students' written answers to the pre-knowledge task and to the tasks in the digital application.

Firstly, students' answers to the pre-knowledge task were analysed using the covariational reasoning framework presented in Table 1. Students' conceptions of the covariational relation were studied from their graphs and written explanations. The graphs were categorized according to its shape. In some cases it was not clear whether parts of the graphs were representing an increasing or decreasing rise. In that case the graph was coded as 'unclear'. Students' explanations were categorized according to the way they refer (or not) to change and different directions of change, as students' perception of change is a key aspect in the covariational framework that distinguish higher and lower levels of reasoning. Based on this information the students were assigned to one of the covariation framework-levels.

Secondly, we focused on students' concepts of change while working with the two animations. In both animations, students are asked to describe in words how the height varies while the water is pouring into the vase. Analysis of these explanations provide insight in the influence of the animation-features on students' discernment of change and covariation.

RESULTS

Students pre-knowledge

Very few students generated an acceptable graph (n=4) in the pre-assessment task. That is a graph with a decreasing rising curve followed by an increasing rising curve. Instead, most students produced graphs representing variation in one direction: decreasing rising curve (n=4), increasing rising curve (n=5), strictly increasing curve (n=6). Analysis of students' reasoning with the covariational framework showed that most students reasoned at lower levels, as shown in the second column of Table 2. The most predominant level was the pre-coordination level. An example of a student reasoning at this level is: "as the amount of water increases, the water becomes more and more". And an example of a student reasoning at gross variation level is: "as the height of the water increases, the amount of water increases". Only one student reasoned at chunky continuous covariation: "because it is smaller at the bottom of the vase, it first goes up quickly, then it widens, so it goes less quickly, then it becomes smaller again, and it goes up faster". No student reasoned at smooth continuous level.

Reasoning levels	Pre-knowledge (N=20)
smooth continuous covariation	0
chunky continuous covariation	1
coordination of values	2
gross coordination	2
pre-coordination	11
unclear	4

Table 2: Students' levels of reasoning at pre-knowledge

Students' graphs with the IVM-application

Twelve students drew an acceptable graph of the spherical vase with the IVM-application; eight of them had previously drawn incorrect graphs in the pre-knowledge assignment and four already produced correct graphs in the same assignment. Specifically, the eight students who improved their graph had previously drawn a linear graph (n=5), an increasing concave down graph (n=2) and a concave up curve (n=1). The graphs of the other students (n=8) were not correct but most of them showed attempts to improve their initial graph. Only one student drew the same incorrect graph again and one student did not provide an answer.

Students' transfer: variation of the vase shape

Analysis of the graphs generated in tasks 2, 3, 4 and 5 showed that the majority of the students who had generated an acceptable graph of the spherical vase-filling also drew acceptable graphs with regard to 3 or more of these tasks. This result suggests that students could transfer their ability to draw a graph by the spherical vase to other vase shapes.

Students perceptions of change while interacting with the two animations

The average time of students' working with the IVM-application was 15.5 minutes. All students get the animation *fill the vase* which is aimed at supporting students in visualizing the variation of the variables height and volume individually and in relation to each other. In the application students were requested to describe how the height varied while the water was pouring into the vase. Eighteen students provided an explanation. Students who draw an incorrect graph in the spherical vase task were led to a second animation *move the dots*. This happened for 15 students. However, students written explanations in the application were too unclear or incomplete and therefore they did not provide much information about students' thinking. For this reason, we focused our analysis in students' explanations provided by the first animation *move the dots*.

Most students' explanations (*n*=14) showed that students' discerned change (*n*=6) or different directions of change (*n*=8) while interacting with the animation *fill the vase*. The other students' explanations were unclear (*n*=3), didn't show discernment of change (*n*=1) or did not provide explanation (*n*=2).

An example of a students' explanation that entailed different directions in change was the following: "first the water went slowly to the middle and then it went faster". The student draw a smooth curve representing a concave up graph followed by a concave down graph, which is not correct. But, the fact that the student perceived different directions in change when playing the animation can mean that the animation elicited students' thinking at the levels of chunkey or smooth continuous reasoning.

An example of a students' explanation that referred to change (but not different directions in change) was: "it continues to fill in the same way and so the height increases more and more". This student draw a concave up graph. In this case students' perceived change in one direction (increases more and more) when playing the animation and we considered that the animation did not elicited students' thinking at higher levels, but elicited thinking about change. We don't know in what extend the student played with the animation and how many times he pushed the button and observed the water falling. These are questions that we could not answer with our data but they could provide more insight in the way the animation can elicit (or not) covariational thinking.

There were also situations in which the students' explanation referred to change but the student generated an acceptable graph with different directions in change. An example of an explanation of this type was: "the height changes per part because the vase becomes wider in some parts". In this cases was not clear if the animation elicited (or not) students thinking at higher levels, but elicited thinking about change.

CONCLUSION

Although there is several research on the learning of functions with technology, there are only a few studies that provide concrete directions on how computer applications can be designed to afford students thinking in ways that promote covariational reasoning (Johnson et al, 2017). A better understanding of how specific task features elicit and impact students' thinking can provide useful directions for designers and task-users. The results of this study suggest that the IVM-application (Interactive Virtual Math) can elicit students' covariational reasoning. Most students improved their initial graph or they kept the same graph as it was originally correct after working with the application. These students could transfer their ability to draw a graph by the spherical vase (task 1) to other vase shapes (tasks 2-5). Moreover, the students discerned change or different directions of change while interacting with the animation *fill the vase*, which is a characteristic of the two higher levels of the covariational reasoning framework of Thompson and Carlson (2017). Key design features of this animation were the request for students to explain how one variable varied in relation to the other and the different colored-layers that were thought to help to visualize the variation of both variables individually and in relation to each other. These results extend the work of several researchers on the relation between students conceptions of change and covariational reasoning (e.g. Johnson et al., 2017).

REFERENCES

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 352-378.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, *33*(3), 31-37.

Ferrara, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus: Ideas discussed at PME over the last 30 years. In *Handbook of research on the psychology of mathematics education* (pp. 237-273). Brill Sense.

Johnson, H. L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM*, *49*(6), 851-864

Palha, S. & Koopman, S. (2017). Interactive Virtual Math: a tool to support self-construction graphs by dynamical relations. In *Proceedings of CERME10.* Dublin, Ireland.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Student use of mathematics resources in Challenge-Based Learning versus traditional courses

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In this study we used qualitative methods to investigate which kinds of resources students used, and how they used them, in two programmes: (1) Challenge-Based-Learning projects (CBL), and (2) Calculus and Linear Algebra courses (Pepin & Kock 2019). The main data collection strategy consisted of focus group interviews with student groups working on the CBL projects. Results show that students working on CBL projects used resources outside the realm of curriculum resources offered to them in traditional courses, including internet resources and digital modelling and signal processing tools. The supervisor appeared important to provide feedback, which for some students led to an iterative Actual Student Study Path. This may have implications for the development of a CBL curriculum.

Keywords: Actual Student Study Paths, curriculum resources, higher education mathematics, challenge-based learning.

INTRODUCTION

In traditional university mathematics courses, students orchestrate different resources and follow various *Actual Student Study Paths* (ASSPs) depending, amongst others, on the course organization and their preferred ways of studying (Pepin & Kock 2019). We have shown this to be the case for first year engineering students studying Calculus (CA) and Linear Algebra (LA) (ibid). Digital resources (e.g. the university's Digital Learning Environment - DLE) constitute part of the resources used by students in these courses (Pepin & Kock, 2019). However, universities of technology tend to move towards more challenge-based projects (Malmqvist, Rådberg, & Lundqvist 2015), in which groups of students work, interdisciplinary, on authentic engineering tasks.

Typically, studies on the use of resources include the curriculum resources made available or recommended as part of mathematics courses. For example, in a review study Biza, Giraldo, Hochmuth, Khakbaz, and Rasmussen (2016) have described the opportunities afforded by introductory university mathematics textbooks, and their actual use by students in traditional lecture/tutor group courses. However, in these courses there are also *social resources* (e.g. lecturers, tutors, peers) that students tap into, and *digital* and other resources mobilized by students themselves. In challenge-based courses students have to develop their own learning/study strategy, including their learning trajectories (i.e. what to do to learn). Subsequently they have to build the resources into the challenge-based project development according to their needs. Hence, we expect that student needs regarding the selection and use of resources in a challenge-based learning environment differ from those in traditional lecture/tutor group courses.

We ask the following research question: What kinds of resources were selected by students, and how did students' ASSPs unfold in CBL projects, as compared to traditional courses?

THEORETICAL FRAMES

In this study we use the theoretical frames of (a) Re-sources, (b) Actual Student Study Paths, and (c) Challenge-Based Learning for our purpose of comparing ASSPs in traditional mathematics courses versus studying mathematics in innovative approaches, such as CBL.

Several studies lean on the notion of *resource* to study what kinds of resources and materials students have access to, use, and orchestrate for their study of mathematics and engineering (e.g. Anastasakis, et al., 2017). To clarify the concept of curriculum resources, Pepin and Kock (2018) referred to mathematics curriculum resources as "all the resources that are developed and used by teachers and pupils in their interaction with mathematics in/for teaching and learning, inside and outside the classroom." Curriculum resources would thus include text resources (e.g. textbooks, teacher curricular guidelines, worksheets); other material resources, such as manipulatives and calculators; and web based/digital resources (e.g. the DLE, videolectures). Digital resources are generally distinguished from digital educational technologies (e.g. digital geometry software) (Pepin & Gueudet, 2018). General resources are the non-curricular material resources mobilized by students, such as general websites (e.g. google Scholar, Wikipedia, YouTube). Cognitive resources are the mathematical frameworks and concepts students work with. In terms of social resources, we refer to formal or casual human interactions, such as conversations with friends, peers or tutors.

Using the lens of resources, we (a) investigated students' own perceptions of how they manage their learning/studying, and (b) coined a term that linked to the patterns we observed in these perceptions. We drew on Simon and Tzur's (2004) Hypothetical Learning Trajectory approach, and other works in this field. However, when investigating students' study trajectories, we considered: (1) the alignment and orchestration of resources, not of tasks or activities; (2) the students' perspective, that is, how they actually orchestrated the resources for their own learning (and not how it was done by teachers/lecturers) and how they gave meaning to these self-created/orchestrated paths. In an earlier study we called these *Actual Student Study Paths* (ASSP- see Pepin & Kock 2019) and outlined selected study paths for the first-year courses of CS and LA.

Malmqvist, Rådberg, and Lundqvist (2015) have defined challenge-based learning (CBL) in the following way:

> Challenge-based learning takes places through the identification, analysis and design of a solution to a sociotechnical problem. The learning experience is typically multidisciplinary, involves different stakeholder perspectives, and aims to find a

collaboratively developed solution, which is environmentally, socially and economically sustainable. (p.1)

Often mathematics is the content area where students struggle to see its value, until they arrive at an engineering problem/challenge that typically cannot be solved without mathematics. Therefore, one way to stimulate students to appreciate the value of mathematics is to frame the mathematics learning around engaging, authentic problems related to a ‘real’ challenge (Rasmussen & Kwon 2007). These learning environments aim not only to foster student learning, but also to promote collaboration between university, business and the public sector and to create societal and technical prototype solutions to difficult, strategic challenges. Students play a key role in these environments, not only in problem-solving, but also in driving a collaborative, multi-perspective dialogue on defining the problem to be solved. In addition to a physical arena, the CBL labs typically provide a set of methods for addressing a societal challenge, from problem identification to solution concept.

THE STUDY

In the Innovation Space (IS) of a Dutch engineering university, groups of 4-5 bachelor students worked together in multi-disciplinary teams (e.g. industrial design, mechanical engineering, innovation sciences) on a challenge, set by a stakeholder from outside the university. The projects were unstructured but the students had to fulfill Bachelor End Project (BEP) requirements of their respective disciplines. This setup provided students with opportunities to investigate and analyze an authentic situation and to develop a (prototype) solution in the form of an artefact. Different projects included various degrees of mathematics in the form of mathematical modelling and signal processing with the help of digital tools. No particular resources were stipulated/provided, except that each group had a supervisor (from one of the university departments) and an outside stakeholder, who supported the projects. Earlier, students had followed lecture/tutor group based mathematics courses (e.g. CS, LA), where they were provided with particular resources.

In our previous study (Pepin & Kock 2019) we had used a case study approach to investigate 24 students’ orchestration of resources (coming from nine different engineering departments) in two first year mathematics courses in the same engineering university. In the present study we explored 6 students’ use of resources for their challenge-based Bachelor End Projects.

Our data collection strategy had a similar approach as in the previous study:

- Individual and focus group interviews with four student groups working on IS projects. In two of these projects mathematical knowledge was important for the students to understand, model, or solve the problem. In this study we focus on these two groups, the *Parkinson project* and the *Garden of Resonance* project. The interviews were conducted in English, which was a second language for most students.

- Students were asked to draw Schematic Representation of their Resource System (SRRS - see Pepin & Kock, 2019), to illustrate the particular resources each student used, and how. During the interviews, students were asked to explain their resource use based on their SRRSs. The SRRSs served as a methodological tool, to help the researchers understand the use of resources.
- Selected observations in the IS environment included observations of mid-term and end-of-project presentations; observations of student working places in the IS labs; observations of supervisor meetings where the projects were discussed (including assessment).
- Examination of documents/curriculum materials provided by the university and lecturers for the students: e.g. syllabi, resources provided by the different engineering departments (mainly to understand the contexts in which students were working in the IS as compared to their 'home departments'). Moreover, we examined two final student reports.
- Informal interviews/discussions with course leaders were conducted, in order to understand the context of and the way of working in the IS.

In terms of analysis, the interviews were first transcribed and interview quotes with a reference to mathematics and mathematics resources were selected and coded. Second, student drawings were compared with their explanations and the selected quotations (within case comparison): how they explained their identification of (for them) suitable resources and the orchestration of these resources; this resulted in selected resources and self-reported study paths. Third, these self-reported study paths were compared across the cases. This resulted in particular types of study paths for the interviewed IS BEP students who referred to mathematics resources. Fourth, the findings from the previous study were compared with the results from this study, taking-into-account the context and course organization.

RESULTS

In the Parkinson project the students focused on the so-called freeze of gait issue. During a freeze of gait episode, patients unexpectedly feel as if their feet are stuck to the ground while they are trying to take a step. Patients have an increased risk of falling as a result of a freezing episode. Present research is directed at understanding the occurrence of these episodes. In the project students were working on the development of a prototype sensor to detect a freeze of gait situation.

The students realised that some of their previous courses and learning experiences were useful as cognitive resources:

> Yea, calculus physics, modelling, and … Like, you have calculus mathematics and modelling, those are very important basic skills for an engineer I think. … I think for example when you want to calculate the stride length, you have to integrate. So, then you need to have had a basic of calculus. (Int Foc C)

Students mentioned the importance of mathematics as an important tool in the project. Students C and D specifically mentioned CS, and the techniques of double integration from an acceleration to a displacement. Moreover, the students were dealing with issues of measurement errors, the filtering of signals, and analysis in the frequency domain. For this, they mastered and employed digital tools, in particular Matlab/Simulink. Student D explained the resources he used to obtain the knowledge needed for the project, as shown in his SRRS (Figure 1):

> Because the double integration was really a hard thing (…) I contacted a master student who helped me further with the Matlab and Simulink part. I also asked my own supervisor or tutor. I also got in contact with another faculty staff member, …He helped me with the sensor part, and filtering signals. …And [the supervisor] helped me a little bit, but he didn't have the time to figure it out himself. So, I was all by myself and looking it up on the internet, and YouTube videos and Matlab documents and such. (Int Foc D)

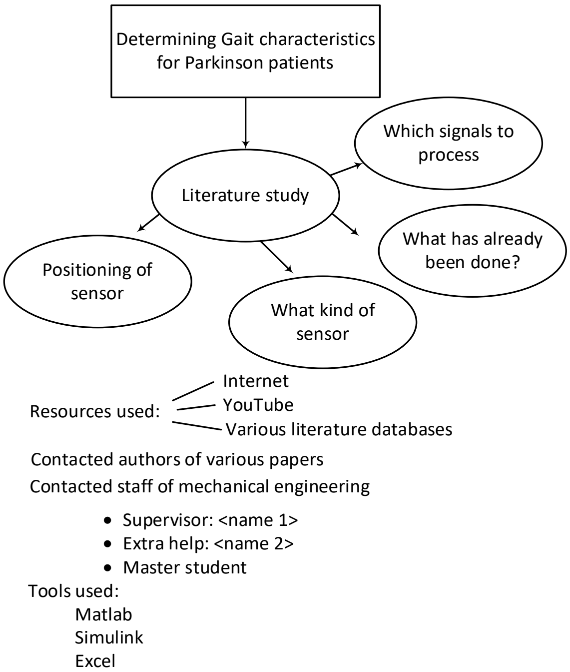

Figure 1. Student D SRRS. (Transcribed from original drawing for readability.)

From the SRRS and interviews, we identified particular self-reported study paths. for example, student D (see Figure 1) put a literature study at the centre of his work. He interacted with several social resources when he got stuck, including the authors of scientific papers, and knowledgeable peers. Student C built knowledge from university courses and knowledge he had developed following his own interests, using internet, literature, and social resources. He appeared confident that he could develop and add "another level of expertise" when needed.

In the *Garden of Resonance project* the students, under the guidance of an artist, assisted in creating a work of art consisting of huge sound scales, that made it tangible how all matter is constantly vibrating (see www.gardenofresonance.com). The students attempted to model and design sound scales producing a sound spectrum with frequencies that had a calming effect on the listener.

The students of this project appreciated the support of social resources: (a) their peers; (b) professors / supervisors; and (c) the artist originating the project:

> I helped [H] in working out a little bit of the calculations for his model. And I was not able to do that, I guess, when I didn't do any mathematics before. So, otherwise you really have to find out how the concepts work. (Int Foc V)

> Yea, and [what] she [the acoustics professor] kind of does is, she lets me go run in the wild with all the knowledge. And then tells me "It's good that you found all that, but you need to especially need to take care of that." And then I go back again, get up and do it again. (Int Foc H)

Digital mathematics tools, such as Matlab and its signal processing toolbox (e.g. Fast Fourier Transform - FFT), were important for the students to model the problem and analyse their data (see Figure 2).

Students referred to previous courses as resources for the project. However, they had wanted to see the relevance of the knowledge in those courses:

> Maybe the knowledge about that process, that takes time. (…) If I would have had my classes as I would have had now. And if I would have had my tests in a different way, in which I needed to use the knowledge that I had obtained into modelling something, or making something. But then making it by myself and not making it in a whole team that all get really specialized in one piece so that you still don't have oversight of what you are doing. That would have created a sense of, kind of a sense of how knowledge becomes. (Int Foc H)

The students explained the different ways of using (and learning) the mathematics in traditional courses and for the project:

> Like the calculus and the physics ones, those are essential. Because they actually teach you like the tools to handle some problems. But where I think like the project work, … like there would be a moment where you sit down with somebody, and actually sketch out the project. Actually, sketch out what steps you need to take, and what information you might want to research on that. (…) What the format is now [of the traditional courses], is that you get a lot of information that you might use sometimes. But because you get like information like "Here you got it." And then you don't do anything with it. "Yea, ok. I'm now just trying to pass the test, and then I'll forget it I guess". (Int Foc H)

In terms of different student study paths, we observed that student H (Figure 2) described an iterative path of developing knowledge using different material and social resources, subsequently being corrected by feedback from his professor/supervisor, after which the process started again. Students N and V drew SRRSs with categories of resources they had used, but without a specific sequence. Their categories contained curriculum materials from the university, but also social resources from within and outside the university and a range of general resources, often web-based. Digital software tools such as Matlab were important resources to shape the mathematical practice of the students and to help develop a solution to the challenge, based on the mathematical concepts involved.

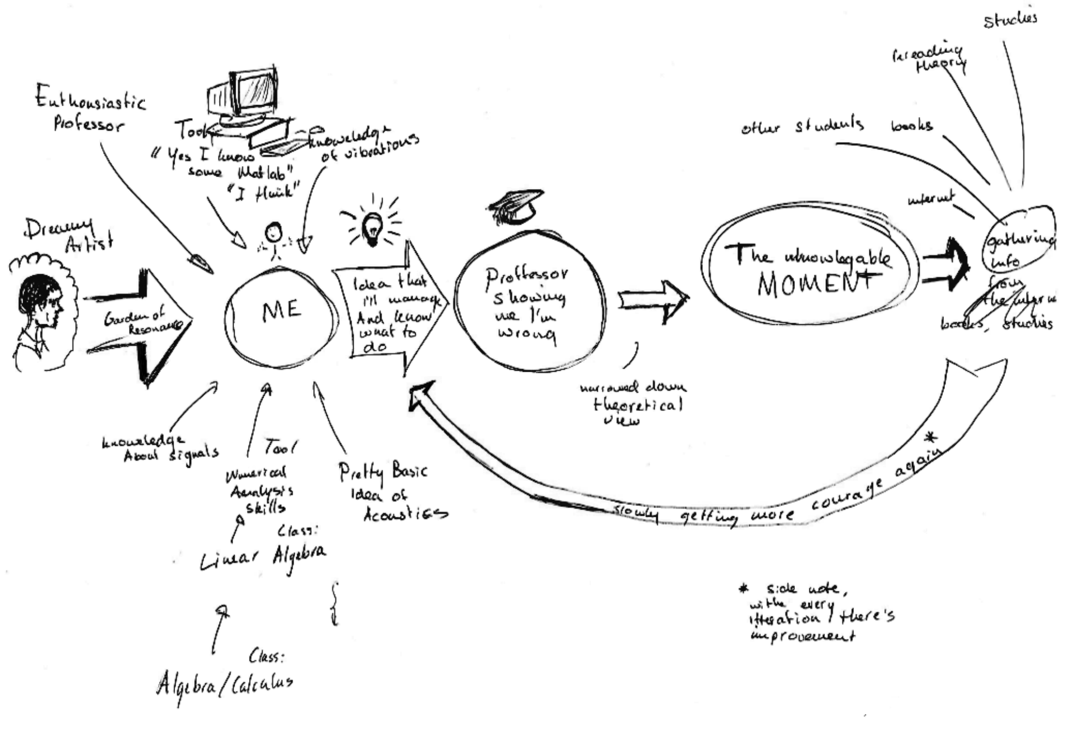

Figure 2. Student H SRRS

Compared to the first year CS and LA courses (Pepin & Kock 2019), the CBL projects had no pre-determined structure or study path. In the CBL projects the multi-disciplinary student group had to find their own ways of defining and solving the problem. Whilst the suggested resources for the CS and the LA course/s were pre-selected, and partly well-defined and aligned, there were no pre-defined resources for the CBL projects, apart from the supervisors and the peer support.

CONCLUSIONS

The findings show that in their CBL projects the students used resources other than the curriculum resources offered to them in traditional courses. These included cognitive resources, digitally accessed scientific papers, digital software tools, peers, and experts in the field. The ASSPs were either (a) iterative/cyclical and feedback based, or (b) focused on the common project goal, or (c) on the supervisor providing advice, with a combination of approaches being quite common. This was in contrast to the more sequential ASSPs, described in our earlier study, which students had used to individually master mathematical content for examination purposes. Indeed, some students mentioned the importance of the basic mathematics courses as cognitive resources. In these, the ASSP appears to have its benefits in terms of students developing confidence with particular mathematical concepts. However, the CBL ASSPs show that students need to develop confidence when dealing with uncertainties in a multidisciplinary group. For example, students experienced uncertainty when they looked for conceptual mathematics resources and digital modelling resources, to address the CBL problem. They acquired confidence with the help of the social resources at their disposal, in particular with their supervisor. An implication is that students have to be supported with CBL trained supervisors, in addition to suitable curricular, technological, and social resources. This will help them develop as self-determined, open-minded, and mathematically-knowledgeable engineers, willing and able to solve engineering challenges.

REFERENCES

Anastasakis, M., Robinson, C. L., & Lerman, S. (2017). Links between students' goals and their choice of educational resources in undergraduate mathematics. *Teaching Mathematics and Its Applications, 36*, 67-80.

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on teaching and learning mathematics at the tertiary level: State-of-the-art and looking ahead*. doi:0.1007/978-3-319-41814-8_1

Malmqvist, J., Rådberg, K. K., & Lundqvist, U. (2015). Comparative analysis of challenge-based learning experiences. In CDIO (Ed.), *Proceedings of the 11th International CDIO Conference*. Chengdu, Sichuan, P.R. China: Chengdu University of Information Technology.

Pepin, B., & Gueudet, G. (2018). Curriculum Resources and Textbooks in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 1-5). Cham: Springer International Publishing.

Pepin, B. & Kock, Z.J. (2019). Towards a better understanding of engineering students' use and orchestration of resources: Actual Student Study Paths. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education.* Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Rasmussen, C., & Kwon, O. (2007). An inquiry-oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior, 26*, 189-194.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking & Learning, 6*(2), 91-104.

Citizen empowerment in mathematics curriculum: design of exemplary digital learning environments

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Mathematics behind social contexts – whether in political or economic concepts – has a low presence in school curriculums despite its importance for a civic participation. This paper is a plea that mathematics education must contribute to citizen empowerment in a subject-specific way. Therefore, it provides an analysis of how digital learning environments can facilitate an access to the mathematics of civic topics. The design of concrete examples illustrates how students can gain elementary insights in civically relevant mathematics through dynamic exploration and autonomous simulation. These digital learning environments give ideas how to promote empowerment through mathematical skills and to enlarge the curriculum.

Keywords: citizen empowerment, digital learning environment design, curriculum accessibility, dynamic exploration, modelling in civic contexts.

MOTIVATION

"It [mathematical literacy] assists individuals to recognise the role that mathematics plays in the world and make the well-funded judgments and decisions needed by constructive, engaged and reflective citizens" (OECD 2017, p. 67). This partial characterisation of mathematical literacy as defined in the PISA framework specifies the pursuit of students' empowerment as one principal objective for a mathematics curriculum. The learners as future fully-fledged citizens should be able to identify the mathematical perspective on a civic issue and to integrate this perspective into the formation of their own opinion. Civic affairs are often complex, interdisciplinary and controversial which used to limit an exploration within mathematics education. Nevertheless, the mathematical side of a public topic should not be ignored in advance solely because it is mathematical.

As a consequence, this paper is a proposition for the constructive design of digital learning environments in which citizen empowerment can be addressed as a topic of mathematics education. The starting point is the thesis that using the advantages of digital technologies facilitates new approaches to discuss the mathematical perspective on civic issues. Firstly, it is important to know the significance and the challenges of citizen empowerment as an objective of a mathematics curriculum. Secondly, the question is to determine features of digital learning material that simplify an examination of relevant civic issues with students. These considerations result in concrete digital learning environments. By analysing exemplary civic topics by means of mathematics, students discover how a mathematical perspective enriches the individual opinion formation and the public debate.

CITIZEN EMPOWERMENT FOR ALL?

To begin with, a common understanding of the extent and content of citizen empowerment is needed, as well as its significance in mathematics education. Principal links between mathematics education and education for democracy consist in the social functions of mathematics and the fact that the learners become autonomous members of society – besides pedagogical methods to promote self-determined cooperative learning (Skovsmose 1998). As the first aspects are content-orientated, they can easily be integrated in a set of learning targets that are specific to mathematics educations. Apple (1992) differentiates between a functional mathematical literacy that prepares mainly for working life and participating in the consumer society and a critical mathematical literacy "to support open and honest questioning of our society's means and ends" (pp. 428f.). To give a concrete example, empowerment through mathematics education is more than making your tax return but understanding the mathematical dynamics of tax rates for different incomes. The social impact of mathematics applies to the mechanism behind the phenomena and therefore often remains invisible:

> Applied mathematics are only interesting and really indispensable for general knowledge, when real life examples illustrate the functioning of mathematical modelling (Winter 1995, p. 38, transl. by the authors).

Winter (1990) names two categories of citizen empowerment through mathematics education: Civic Calculating and Political Arithmetic. While the first comprises a basic understanding and autonomous handling of the everyday mathematics as percentage calculation, rule of three, statistics etc., the second aims for a more critical context-orientated analysis including a deeper mathematical understanding. Winter places the topics of Political Arithmetic at the earliest in the curriculum of an upper secondary education. The hidden mathematics of civic issues is indeed often part of complex modelling relying for example on discrete mathematics or functions with several variables, content beyond the school curricula. Regarding education for all and equal civil rights, it would be a problem if mathematical insights within the scope of citizen empowerment were not accessible for everyone.

Citizen empowerment is not limited to communicative competencies and reflection but also includes specific mathematical knowledge and skills. As to the example of taxes, an understanding of a progressive tax bases upon the differentiation between absolute and relative values. Otherwise, even with one fixed tax rate (e.g. the biblical tithing) a rich person would pay more in absolute numbers than a less wealthy citizen. This argumentation recurs to fundamental mathematics; furthermore, Vohns (2017) analyses the topic of taxes with the perspective of a civic education: the effects of an income tax threshold lead to an asymptotic curve of the average tax rate. In addition, the widely discussed marginal tax rate is a concrete practice of differential calculus. Both are mathematical topics of an upper secondary education, but essential for the understanding and critical judgement of a politically mature citizen who regularly pays taxes and whose vote for a political party can depend on its fiscal policy.

The mathematical side of civic phenomena reveals the "formatting power of mathematics" (Skovsmose 1998, p. 197). Mathematics is used to define and to arrange socially relevant concepts. Whereas a definition of poverty generates one fixed value, tax rates is an instance of establishing functional dependencies. They have in common that mathematics has been used to shape reality (*normative modelling* [1]). Winter's Political Arithmetic not only analyses the current circumstances, but also compares them with possible alternatives. In fact, the formulating and reflecting of different mathematical approaches to the same civic problems are the core of a critical mathematical citizen empowerment.

One principal idea of citizen empowerment through mathematics education is disclosing and questioning the prescriptive function of mathematics in civic contexts. Since the connections between mathematical assumptions and social consequences are manifold and demanding, a way of popularisation of the central dynamics is needed to illustrate how the use of mathematics in civic contexts forms social reality.

NEW ACCESS VIA DIGITAL TECHNOLOGIES

This paragraph presents constructive settings and didactical findings that give guidance for the design of digital learning material towards citizen empowerment. The difficulty while studying topics of citizen empowerment is the high intrinsic cognitive load defined by the simultaneous consideration of multiple factors. Therefore, the task design to this content should reduce the extraneous cognitive load incorporated in the task itself to facilitate the intelligibility in general (Sweller 1994).

One concept for task design is Wittmann's (2001) substantial learning environment: It ties important aims and contents of teaching mathematics to fundamental matter beyond this educational setting while being flexible and offering a multitude of learners' activities. This didactical concept seems suitable for the essential challenges of the above-mentioned mission, the more so as there are further criteria for digital learning environments: They promote explorative learning, focus on relations and on interdependencies, their handling is rather self-explanatory and their inherent tasks can be adapted (Brüning et al. 2008). In such a learning environment, the two main advantages of digital mathematics tools are the systematic variation and the dynamic visualisation (Heintz et al. 2017). On the one hand these explorative activities appeal to the students' curiosity and ludic drive, on the other hand they can reduce efforts in calculation and mathematical notation hindering the accessibility.

Weigand (2017) presents a three-dimensional competence model for the use of digital technologies in mathematics classes with the axes *Activity*, *Representation* and *Understanding* which categorises classroom activities and can be transferred to the context of this paper: with embracing citizen empowerment, learners should engage in open, interactive tasks like discovering and explaining instead of directed exercises as calculation. The dynamic representation of the mathematical aspects of the civic issue is the core of the following exemplary learning environments whereupon multiple dynamic representations are preferable. The objective is to reach a relational understanding in the sense of a qualitative evaluation how a mathematical decision affects the context in question [2].

Designing learning material for citizen empowerment is contextualised in the area of mathematical application and modelling. Geiger (2017) links this part of the mathematics curriculum with the challenge of integrating digital technologies into the task design: digital technologies support principal functions of mathematical applications. They increase the *Accessibility* to complex problems whereby a higher level of *Authenticity* of the example is reached. By giving direct feedback, they stimulate a *Development* of the students' mathematical and contextual knowledge.

One effective way to initiate civic learning and developing a reflective attitude is confronting students with paradoxes. Bokhove (2017) attests the success of intentionally provoking cognitive crises: in this way, students have to react to something unexpected which turns their way of thinking upside down. For achieving citizen empowerment, it is helpful to expose intriguing counter-intuitive phenomena.

However, empirical psychological results show another alternative: once learners have been in the position to produce own fake news in an online game, they are more aware and critical when consuming misinformation afterwards (Roozenbeek & van der Linden 2019). In analogy, learners could be more observant of the formatting use of mathematics if they had their own experience in manipulating mathematical models in civic contexts.

A digital learning environment that targets citizen empowerment must profit from new ways of visualisation and simulation facilitating the access to topics with multiple interdependencies. Interactive exploration, interest awakening paradoxes and experiencing self-efficacy are keys making civic topics accessible for a relational understanding without overstraining students with higher mathematics.

EXEMPLARY CONTEXTS AS DIGITAL LEARNING ENVIRONMENTS

As a first example, apportionment methods in proportional electoral systems are a classic civic issue with relevant mathematical content. While students can easily derive the largest remainder method (also known as Hamilton method) from the rule of three, its most famous disadvantage, the Alabama Paradox [3], has traditionally been taught by tables of historical examples or by constructed minimal numbers. The dynamic digital exploration via the developed material represents a new approach despite the wide range of literature about apportionments and their geometrical visualisations (e.g. Gauglhofer 1988, Bradberry 1992). In fact, apportionment methods are functions with multiple variables (numbers of seats, number of parties and their votes); they are based on algorithms and not defined by an equation. Varying the parameter *S*, the number of seats, leads to an analysis of the Hamilton method: Are there any regularities? Which party gets which seats? In this environment (Fig. 1) students can discover the Alabama paradox: Why do two parties get the 272th seat? Why does one party lose a seat? Once the students can explain the visualisation, they can interpret this illogical phenomenon in its contextual meaning. For a further understanding, a spreadsheet next to the visualisation presents the typical table with quota and remainders for the chosen *S*, so that a combined representation leads to the resolving of the paradox. Instead of focussing on calculation, students elaborate their skills on authentic data (as the numbers of the German federal election 2017 [4]). Unlike abstract single cases, the dynamics make the paradox more accessible while giving a bigger picture of the Hamilton method.

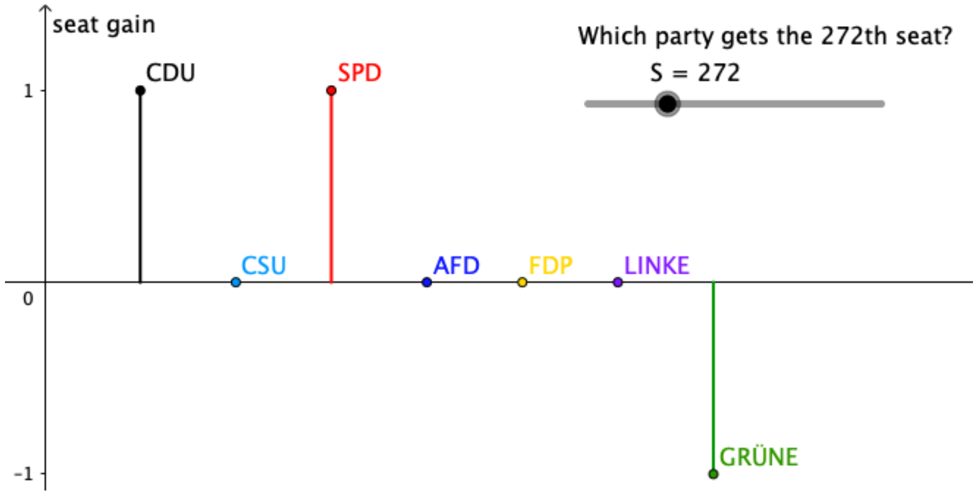

Figure 1: Dynamic Exploration of the Alabama Paradox

Taxes are a good second example for socially relevant mathematics (examples for didactical analyses are Daume 2016, pp. 42-50, or Henn 2017). As stated above, citizen empowerment in this area consists in revealing fundamental mechanisms. Figure 2 shows a digital learning environment in which students can determine an income tax threshold on a simple linear or quadratic tax rate: While manipulating the threshold, they see in the left graphic the resulting tax rate (the dashed line is the tax rate without threshold) and in the right one the average and marginal tax rate. In this dynamical simulation students discover that the bigger a threshold the gentler is the slope of the average tax rate. The average tax rate approaches the marginal tax rate more slowly. Graphically and without calculus, the latter can be understood as the tax rate of the next euro earned. Based on this learning environment fundamental concepts as fiscal drag become obvious; in analogy, other crucial points of a progressive tax system as the transitions between different rates can be analysed. In this manner, the students can model important characteristics of a subjectively fair tax system. The dynamics not only support a visualisation in order to perceive a certain phenomenon but display a central lesson for citizen empowerment: mathematics behind civic topics are not static but manipulatable. Formulas in such context are not laws of nature but conventions. By being able to identify mathematical assumptions, civic engagement can lead to another outcome.

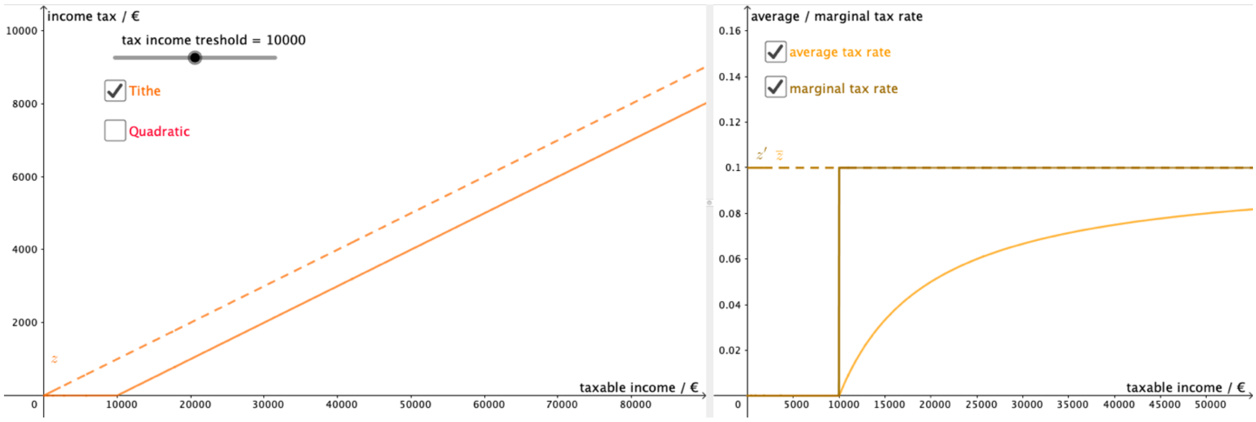

Figure 2: Dynamic Simulation of an Income Tax Threshold

Lastly, business valuation (e.g. Hitchner 2017) gives an economic context for a third example. Market capitalisation regularly gives cause for headlines [5]. A digital analysis easily shows the unstable character of this value as it is based on the share price. Reflecting the news values on the knowledge that the business value – measured as market capitalisation – can change every day promotes a critical attitude towards such information. In a next step, students can correlate the market capitalisation of different companies or examine long-term effects by referring to mean share prices. For an own judgement it is not only important to verify parameters within one method, but to know that there are different methods as well. Putting business valuation on the agenda of a mathematics curriculum is the occasion to discuss the principle of an income approach as an alternative. This financial context of discounting allows a real-life introduction of the geometric series. The main advantage in this digital learning environment is that students are relieved of excessive and/or complicate calculation by a (multiple) dynamic representation of the results. Within a short space of time, students can check variants of one model and contrast them with other approaches. By doing so, they can focus on mathematical interrelationships and their contextual meaning as well as on a reflection about the role of mathematics in economic models in general.

EXPERIENCES AND CONCLUSIONS

In workshops it could be shown that the understanding of the learning environments doesn't differ between lower and upper secondary students supporting the initial thesis of an enlarged accessibility through the digital opportunities. Besides, the developed material includes units enriching the regular curriculum like apportionments as dynamical deviations from the intercept theorem.

The use of mathematics is not limited to science, technology or engineering but is indispensable in many social, economic and political applications. In addition to cognitive competencies a mathematics curriculum should aim for citizen empowerment: politically mature students reassess mathematical decisions in civic affairs and formulate alternatives. Even if the mathematics in such topics often belongs to upper secondary level, digital technologies can make principles accessible, such as assumptions and characteristics of the modelling process. Mathematical-contextual interdependencies can be visualised dynamically and are open for exploration and manipulation. Digital learning environments promote examining paradoxes, retracing simulations and comparing different approaches rather than calculating single situations. They are particularly apt for a visualisation of interrelationships and alternatives in socially relevant contexts. This dynamic approach enlarges the perception of mathematics: it is not only a tool for description but also shapes reality.

NOTES

1. For an elementary characterisation of normative modelling see Freudenthal 1978.

2. The original meaning of relational understanding is specific to the contexts of functions (Weigand & Bichler 2010).

3. The Alabama paradox is the phenomenon when a party (or a state in the American Congress) loses a seat although the number of seats has been increased (Balinski & Young 2001).

4. Effectively, the Webster method is used for the German federal election, discovering the Alabama paradox only serves here as a means to introduce the actual method.

5. E.g. "Alphabet Becomes Fourth U.S. Company to Reach $1 Trillion Market Value", The Wall Street Journal, 16 Jan. 2020, https://www.wsj.com/articles/alphabet-becomes-fourth-u-s-company-to-ever-reach-1-trillion-market-value-11579208802 (last accessed 9 Feb. 2020).

REFERENCES

Apple, M.W. (1992). Do the standards go far enough? Power, policy, and practice in mathematics education. *J. Res. Math. Educ. 23 (5)*, 412-431.

Balinski, M.L. & Young, H.P. (2001). *Fair Representation. Meeting the Ideal of One Man, One Vote* (2nd ed.). Washington: Brookings Institution Press.

Bokhove, C. (2017). Supporting Variation in Task Design Through the Use of Technology. In: A. Leung & A. Baccaglini-Frank (Eds.). *Digital Technologies in Designing Mathematics Education Tasks. Potentials and Pitfalls* (pp. 239-257). Springer.

Bradberry, B.A. (1992). A Geometric View of Some Apportionment Paradoxes. *Math. Mag. 65 (1)*, 3-17.

Brüning, A. et al. (2008). Lernen mit einer digitalen Lernumgebung. *MU 54 (6)*, 19-26.

Daume, P. (2016). *Finanz- und Wirtschaftsmathematik im Unterricht. Band 1 Zinsen, Steuern und Aktien*. Wiesbaden: Springer Spektrum.

Freudenthal, H. (1978). *Weeding and Sowing. Preface to a Science of Mathematical Education*. Dordrecht & Boston: D. Reidel.

Gauglhofer, M. (1988). *Analyse der Sitzverteilungsverfahren bei Proportionalwahlen*. Grüsch: Rüegger.

Geiger, V. (2017). Designing for Mathematical Applications and Modelling Tasks in Technology Rich Environments. In: A. Leung & A. Baccaglini-Frank (Eds.). *Digital Technologies in Designing Mathematics Education Tasks. Potentials and Pitfalls* (pp. 285-301). Springer.

Heintz, G. et al. (2017). *Werkzeugkompetenzen. Kompetent mit digitalen Werkzeugen Mathematik betreiben*. Menden: medienstatt.

Henn, H.-W. (2017). Die Mathematik der Einkommenssteuer. In: H. Humenberger & M. Bracke (Eds.): *Neue Materialien für einen realitätsbezogenen Mathematikunterricht 3* (pp. 95-105). Wiesbaden: Springer Spektrum.

Hitchner, J.R. (2017). *Financial Valuation. Application and Models* (4^{th} ed.). Hoboken: Wiley.

OECD (2017). *PISA 2015 Assessment and Analytical Framework: Science, Reading, Mathematic, Financial Literacy and Collaborative Problem Solving* (rev. ed.). Paris: OECD Publishing. http://dx.doi.org/10.1787/9789264281820-en.

Roozenbeek, J. & van der Linden, S. (2019). Fake news game confers psychological resistance against online misinformation. *Palgrave Communications 5*. https://doi.org/10.1057/s41599-019-0279-9.

Skovsmose, O. (1998). Linking Mathematics Education and Democracy: Citizenship, Mathematical Archaeology, Mathemacy and Deliberative Interaction. *ZDM 30 (6)*, 195-203.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learn. Instr. 4 (4)*, 295-312.

Vohns, A. (2017). Einkommensteuertarife mit Schulmathematik erkunden – Ein Zugang aus bildungstheoretischer Perspektive. *MU 63 (3)*, 3-13.

Weigand, H.-G. (2017). Competencies and Digital Technologies– Reflections on a complex Relationship. In: G. Aldon & J. Trgalová (Eds.). *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (pp. 40-47). https://hal.archives-ouvertes.fr/hal-01632970.

Weigand, H.-G. & Bichler, E. (2010). Towards a competence model for the use of symbolic calculators. *ZDM 42 (7)*, 697-713.

Winter, H. (1990). Bürger und Mathematik. *ZDM 22 (4)*, 131-147.

Winter, H. (1995). Mathematikunterricht und Allgemeinbildung. *M. GDM 61*, 37-46.

Wittmann, E.Ch. (2001). Developing mathematics education in a systemic process. *Educ. Stud. Math. 48 (1)*, 1-20.

The logic of inquiry when using augmented reality

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The paper focuses on a case study in which three Israelian students are engaged in an Augmented Reality activity. By wearing special glasses, the involved students see mathematical semiotic representations juxtaposed to the real incline plane experiment (Galileo law). The virtual AR signs consist in a table and a graph made with the data caught from the Galileo experiment. Students are supposed to make sense of these virtual data guided by some questions contained in a worksheet task. Our hypothesis is that in order to connect and make sense of the semiotic representations observed within AR environment, students develop an inquiry approach to mathematics.

Keywords: Augmented Reality, Logic of Inquiry, Galileo law, semiotic representations.

INTRODUCTION

Technological development has allowed new ways of teaching and learning mathematics experimentally. Previous research has shown the possibility to simulate physical experiments, to catch data and to make mathematical model of them. Augmented Reality (AR) technology empowers these affordances, by embedding virtual mathematical signs inside the learning environment where real physical experiments are performed. This is the first result reached by the group of the augmented reality projects[1], in which through special glasses it is possible to see both the real word phenomenon and, simultaneously, its mathematical model (Swidan et al. 2019). In fact, the AG technology, developed by the Ben-Gurion University research team, is able to trace moving object and to augment them with virtual mathematics representation.

In this paper I will focus on a case study to examine how the AR experiments of real world phenomena boost students' inquiry based-learning (for more information about the development of the AR technology, see Schacht & Swidan (2019) and Swidan (2019)). Inside this learning environment, students are not passive receiver of knowledge but active inquirer who have to find ways to interpret the real experiment through the mathematical signs provided by the augmented environment. Inquiring processes are essential to mathematical reasoning. As highlighted by J. Dewey (1938, p. 3-4) "[…] all logical forms (with their characteristic properties) arise within the operation of inquiry and are concerned with control of inquiry.

In order to analyse the way in which augmented environments may assist students in inquiring mathematical knowledge, I will adopt the *logic of inquiry* approach, which has already been used to analyses students cognitive process within dynamic geometry environments (Arzarello & Soldano, 2019).

Logic of Inquiry

The logic of Inquiry is a new type of logic, coherent with the classical one, developed in the 1970 by the Finnish logician J. Hintikka (1998, 1999). According to Hintikka, the study of logic should not concern only the study of logical rules which allow the mathematician to avoid to make logical mistakes but should also concern the study of good ways of reasoning. *Interrogative games* describe the model for developing logical reasoning: they are two players games between an *inquirer* and an *oracle*. The oracle represents the source where new pieces of information comes from, it can be "a databased stored in the memory of a computer, a witness in a court of law, or one's tacit knowledge partly based on one's memory" (Hintikka, 1998). The inquirer asks the oracle strategic questions and uses the answer to develop his reasoning. Imagine that the inquirer starts from a theory T and would like to reach a conclusion C (see Figure 1). The inquirer asks a first question to the oracle and, with the received answer, he is able to add a piece of information T_1 to the theory T. The inquirer repeats this questioning process until he obtains all the pieces of knowledge that allow him to deduce the conclusion C.

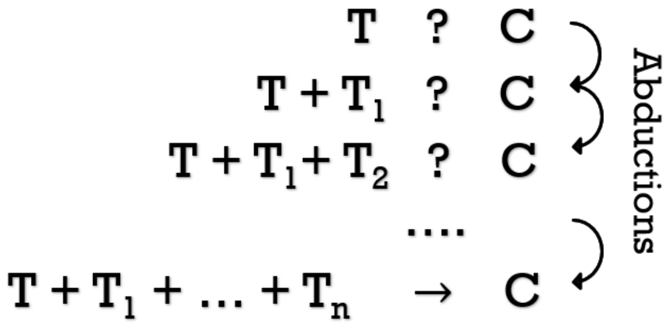

Figure 1: Model of interrogative games

The way of reasoning described by interrogative games allow the introduction of new hypothesis/theories in the discourses: it is an abductive way of reasoning.

Abductive reasoning is a natural way of thinking, which characterizes everyday life. Paying attention to our way of reasoning, it is possible to notice that we are continuously formulating abductions. For example, imagine the situation in which you are in the car and there is a long queue in front of you. You cannot see the end of the queue because a curve limits your vision. Naturally, you start making hypothesis for explaining the queue: maybe there is a traffic light after the curve or maybe a car crash has blocked the circulation.

The notion of abduction has been introduced by Peirce (1960, p.372), who describes abductions as follows:

> […] abductions look at facts and looks for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is:
>
> - a fact A is observed;

- if C was true, then A would certainly be true;

- so, it is reasonable to assume C is true.

Using the car queue example, A is "there is a car queue curve" and C is "there is a traffic light after the curve". I know that if there is a traffic light after the curve then, probably, there is a car queue curve just after. This reasoning can be wrong, in fact also if there is a car crash after the curve, then there is a car queue just after. Abductive way of reasoning is subjected to a certain degree of probability to be wrong. Hintikka (199) observes that the probability to be right increases if the selection of the theory that allows to explain the facts is made in a strategic way, namely using the same strategic principles which are activated while playing strategic games. In order to make a good move during a chess game match, a player should know definitory rules, namely how chessmen may be moved on the board, what count as checking and checkmating, but he should also know strategic principles, namely how to make the moves, which of the numerous admissible moves in a given situation it is advisable to make. In the example of the car queue, you will select the most probable C evaluating the time you stay in queue, the sound of ambulance, etc.

Previous researches have highlighted that dynamic geometry environment can develop a Logic of Inquiry approach to mathematics (Soldano & Arzarello (2018), Soldano & Sabena (2019)), our hypothesis is that, the same result may be observed inside AR environment.

To test our hypothesis, AR activities based on physical experiments, have been designed. The case study described in this paper is based on the Galileo law that describes the movement of a cube which slides down on an incline plane. The AR glasses display AR data juxtaposed to the real experiment. The data are represented in the graphical and numerical register (Duval, 2006). The table shows numbers which represent the distance the cube moves in real time times. The graph displays a distance-time function as a set of discrete points made with the same data contained in the table. The simultaneous display of the graph and the table of numers give to students the opportunity of making conversions from a semiotic representation to the other (Duval, 2006).

METHODOLOGY

The case study involved three voluntary students of grade 11 who have studied quadratic function at school but are new to Galileo's law. The experiment was performed inside the laboratory of Ben Gurion University in Israel, supervised by Osama Swidan. The researcher plays the role of observer and intervenes only to help the students to manage technology or to trigger the discussion. Collected data consist in video and audio recording of two-hours experiments.

The technological devise used in the experiments are AR headsets which uses cellular phones app. The app, trace dynamic objects and represent the dynamicity of the object by mathematical representations.

The three students had at their disposal two pairs of AR glasses, hence while two students are looking in the glasses, the third one takes care of the experiment. The experiment on which students are required to investigate is what we called the “AR Galileo experiment”, because reproduce with AR technology the experiment made by Galileo to verify the hypothesis that a falling object would gain equal amounts of velocity in equal amounts of time, namely the distance made by a falling object is proportional to the square of time. In order to test it, Galileo decelerated the motion of the falling object by using a ball rolling down an inclined plane. In doing that, he assumed that a ball rolling down a ramp would speed up in the same way as a falling ball would since free falling is essentially equivalent to a completely vertical ramp.

The AR experiment consists in a cube which slides down a slanted surface as seen in Figure 2.

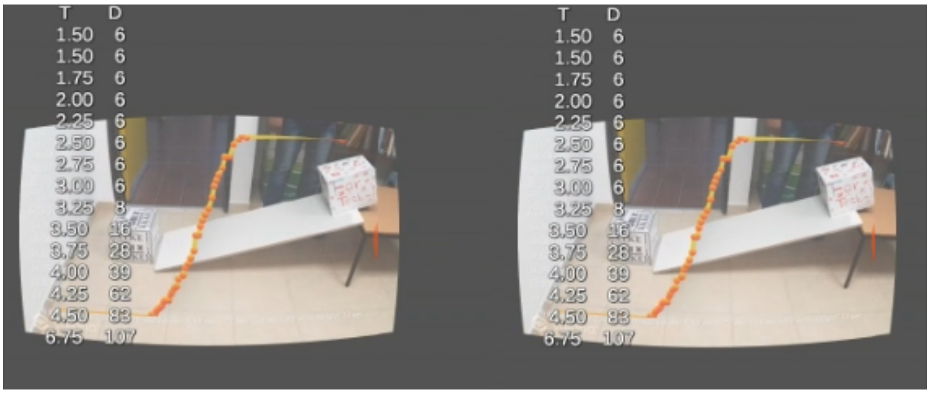

Figure 2: Screenshot of the AR experiment as seen through AR glasses

Through the glasses, students see the cube movement, numbers which represent the real time distance run by the cube and their graphical representation. Students should make sense of the observed virtual mathematical signs and connecting them with the cube experiments. In this attempt they are guided by the following questions:

Hypothesis:	Without using the AR glasses, what kind of relationship do you expect to observe? Discuss your conjecture with your classmates and try to reach a consensus.
Experiment:	Conduct an experiment with your AR-device to check your conjecture.
Reflection:	- What do you observe? - Could you confirm your conjecture?

Table 1: Worksheet task given to the students

In order to design the tasks (see Table 1), we referred to the three epistemological phases suggest by Barzel et al., (2013). The *exploration phase* consists of rich and open exploration tasks (Freudenthal, 1973), which allow students “to actively and collaboratively re-invent ideas, concepts, procedures and relations” (Barzel et al., 2013,

p. 286). In our task, this phase is triggered by the “Hypothesis” and the “Experiment”. The *organization* of knowledge phase which is meant “to establish a shared understanding of the core concepts, theorems and procedures” (Barzel et al., 2013, p. 286). This phase is mainly triggered by the first question of the “Reflection”. The *practice* phase in order to render students’ “knowledge and skills more stable and flexible by repeated practice and transfer” (Barzel et al., 2013, p. 287). This phase is mainly triggered by the second question of the “Reflection”.

ANALYSIS

All the conjectures made by the students before the AR experiments are formulated within the physical field. They conjectured relationships between the slippery of the incline plane and the velocity of the cube and between the inclination of the plane and the slippery of the cube.

After the first experiment, in which one student releases the cube on the inclined plane while the other two are looking through the AR glasses, students start describing the virtual mathematical signs that they observed with their AR glasses, namely the AR graph and numbers. Here it is reported the transcript:

1	Student 1:	It makes a straight line.
2	Student 3:	Me too! Straight line upward then rightward.
3	Student 1:	It has a lot of numbers in it, 22, 21, 21...

The observed mathematical signs are not linked to the cube experiment, the students use the personal pronoun (it) (line1 and 3) without explaining its reference.

The following excerpt refers to the moment in which the students are making the first *controlled e*xperiment. It is a type of experiment suggested by the researcher in which, in turn, one student moves the cube upward and downward on the inclined plane, while the other two students are observing in the AR glasses.

4	Student 3:	Now move it fast up. Now much less dense.
5	Student 1:	Correct!
6	Student 3:	So when she moves it (cube) up it (graph) goes down and when she moves it (cube) down it (graph) goes up.

During this controlled experiment the students start to observe relationships between the cube movement and the AR graph. In particular they observe a relationship between the velocity of the cube and the density of the points (cube fast $\rightarrow$ point less dense, see line 4) and also between the position of the cube and the shape of the graph (cube upward $\rightarrow$ graph downward, cube downward $\rightarrow$ graph upward, see line 6). All their observations are correct since when the cube goes faster, in the same interval of time it runs more distance and so the points captured by the device are more distant and appear less dense. The controlled experiment plays a central role from the inquiry point of view, enabling students to create connections between the cube movement and the observed AR graph.

After that, students make other experiments without controlling the cube movement. After the fourth experiment, the students start taking in consideration the other AR representation, namely the table, as the following excerpt shows:

7	Student 1:	Do you see that on the side it is written ‘centimetre’?
8	Student 2/1:	Yes!
9	Student 1:	It seems that it starts from 68,7. Thereafter it decreases, arrived to 58,3.
10	Student 3:	Maybe it is the distance it does?

Tring to making sense of the observed numbers, Students 3 makes an abduction. Her hypothesis is right: these numbers shown on the AR device do represent the distance made by the cube. However, she does not make explicit the starting point from which the distance is computed and that the other column of the table shows data of the time. Differently from what students did with the graph, in this case they are not able to formulate relationships between the numbers shown in the table and the cube experiment, namely they do not say that when the cube is moving up, numbers represented the distance decrease and when the cube is moving down, numbers increase.

Then students decide to make a second controlled experiment. This time, students start looking in glasses when the cube is still at the end of the plane.

11	Student 3:	Listen to me! When there is not move, there is not ascent or descent. Only when it made its way, it (graph) makes the changes. When it arrives here, in a situation that it is not move it was a straight line

Thank to this controlled experiment, Student 3 observes the relationship between the absence of movement and the type of graph produced by the AR devise, namely an “horizontal” straight line.

CONCLUSION

In the analysed excerpts from the video we observed the activation of the Logic of Inquiry approach. First of all, when the students start observing the AR mathematical signs after the first experiment. Then after the two controlled experiments when the students start linking the AR signs with the cube experiment. Finally, when students formulate an abduction in order to make sense of the numbers in the table. Noticing mathematical signs, observing relationships between them and the cube movements and formulating abductions are first steps of every inquiry. The AR environment plays the role of oracle to which students made implicit questions in order to make sense of the observed signs and to discover relationships between them. This role is emphasized during controlled experiments suggested by the researcher.

The students involved in the case study deeply explore the relationship between the cube movement and the AR graph, whereas they have some difficulties in finding the relationship between the cube movement and the information containing in the AR table. Figure 3 summarizes with a continuous two-side arrow the mathematization that students successfully made though their inquiry (the cube movement is mathematized in the AR graph) and with a broken arrow the mathematization they are not able to made or they made only partially (cube movement is not mathematized in the AR table).

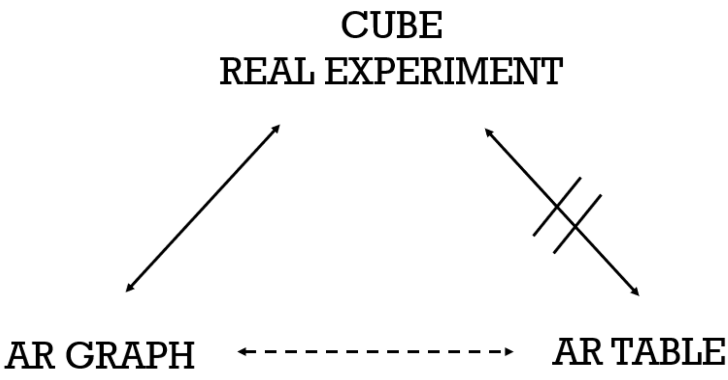

Figure 3: Students mathematization of the cube movement

We suppose that if students were able to mathematize the cube movement through the data contained in the AR table, they will be able to convert the AR table in the AR graph (dotted arrow in Figure 3). The reason of the partial unsuccessful result is partially due to some technological bags which are no more present in the last version of developed technology. Through the results of this case study we understood the importance of making controlled experiments and we decide to add them in the design of the new version of worksheet task.

NOTES

1. The team of the project is made by: Arzarello F. (University of Torino), Abu-Asbe, O. developer of the AG technology, (Ben-Gurion University of the Negev), Fried M. (Ben-Gurion University of the Negev), Jaber O. (Ben-Gurion University of the Negev), El-Sana, J., the AR advisor (Ben-Gurion University of the Negev), Sabena C. (University of Torino), Schacht F. (University of Duisburg-Essen), Soldano C. (university of Torino), Swidan O., the research project coordinator, (Ben-Gurion University of the Negev).

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REFERENCES

Arzarello, F., Soldano, C. (2019). Approaching Proof in the Classroom Through the Logic of Inquiry. In: Kaiser G., Presmeg N. (Eds). Compendium for Early Career Researchers in Mathematics Education. ICME-13 Monographs. Springer, Cham

Barzel, B., Leuders, T., Prediger, S., & Hußmann, S. (2013). Designing tasks for engaging students in active knowledge organization. In A. Watson et al. (Eds.), ICMI Study 22 on Task Design – Proceedings of Study Conference (pp. 285–294). Oxford: ICMI.edia.

Dewey, J. (1938). Logic: The theory of inquiry, (Vol. 12, The Later Works of John Dewey, 1925–1953). Carbondale: Southern Illinois University Press.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational studies in mathematics, 61(1-2), 103-131

Freudenthal, H. (1973). Mathematics as an educational task. Dordrecht, The Netherlands: Reidel.

Hintikka, J. (1998). The principles of mathematics revisited. Cambridge University Press.

Hintikka, J. (1999). Inquiry as inquiry: a logic of scientific discovery. Springer Science & Business M

Schacht, F., Swidan O. (2019). Exploring pre-calculus with augmented reality. A design-based-research approach. In: U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), Proceedings of CERME 11-Eleventh Congress of the European Society for Research in Mathematics Education. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Soldano, C., Arzarello, F., (2018). Approaching secondary school geometry through the logic of inquiry within technological environments. In: Calder N. Larkin K. & Sinclair N. (Eds), Using Mobile Technology in the Teaching and Learning of Mathematics, Mathematics Education in the Digital Era (vol 12., pp.247-261), Springer, Cham.

Soldano, C., Sabena, C. (2019). Semiotic potential of inquiring-game activities. Proceedings of PME 43- International Group for the Psychology of Mathematics Education, Pretoria (Sudafrica), 7-12 luglio 2019.

Swidan, O., Schacht, F., Sabena, C., Fried, M., El-Sana, J., & Arzarello, F. (2019). Engaging Students in Covariational Reasoning within an Augmented Reality Environment. In Augmented Reality in Educational Settings (pp. 147-167). Brill Sense.

Design criteria of proof problems for mathematically gifted students

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Mathematically gifted students have needs that deserve to be considered by the mathematics education research. One of these is learning proof and the incidence of dynamic geometry software in such a process. In this document we present some considerations for the design of 3-dimensional geometry problems that contribute to achieve this goal. We rely on hypothetical learning trajectories and a characterization of the types of proofs to show how dynamic geometry software can favour the learning of proof while solving problems based on 3-dimensional geometric concepts and properties.

Keywords: mathematically gifted students, proof learning, dynamic geometry software, design of problems, 3-dimensional geometry.

INTRODUCTION

Nowadays it's common to find students with different mathematical abilities in any classroom, including some mathematically gifted students (MGS) (Benedicto, Acosta, Gutiérrez, Hoyos, & Jaime, 2015). MGS have a mathematical ability higher than average students with the same age, grade or learning experiences. Despite this, many teachers do not recognize that MGS require special attention, they believe that MGS learn easily on their own (Jaime, Gutiérrez, & Benedicto, 2018). Research has shown that mathematical talent, like any other skill, must be fostered through experiences, appropriate teaching, and challenges. This leads to investigate MGS's mathematical thinking processes and the way these students process and assimilate new mathematical ideas (Dimitriadis, 2010).

Research on the understanding of mathematical proof and the development of proving skills is a lively field of mathematics education. Research has shown that, in a proving process, there is an epistemological discontinuity between the phases of identification of a conjecture and elaboration of its proof. However, the ways in which dynamic geometry software (DGS) may influence aspects of proving, such as exploration, conjecture, and explanation, make this resource a strong mediator between those phases (Sinclair & Robutti, 2013). Considerable research efforts have been made on designing DGS environments based on plane geometry to teach proof and deductive reasoning (Sinclair & Robutti, 2013; Marrades & Gutiérrez, 2000). The recent availability of 3-dimensional (3D) DGS offers a new context for teaching and learning proof, with differential characteristics, that need to be explored. Research on 3D DGS environments is just starting; particularly, the design and analysis of 3D DGS environments based on space geometry and focusing on the learning of proof require specific attention, since research on this topic is scarce.

The design of a teaching sequence based on problem solving requires anticipation of possible students' behaviour and outcomes. The hypothetical learning trajectories (Simon & Tzur, 2004) are an efficient tool for the achievement of this goal. This is particularly true with MGS.

The objective of this document is to present a 3D DGS environment based on a sequence of space geometry proof problems aimed to promote the learning of proof by MGS. Such an experimental environment can provide useful information about MG students' processes of reasoning and can serve as an inspiring example to prepare sequences of space geometry problems based on 3D DGS to teach proof. We present an ongoing research, since, at the time of writing this document, we are completing the design of the experimental setting, to start the experiments later. We first present the theoretical framework supporting the environment, consisting of a classification of the types of proofs produced by students and the construct of hypothetical learning trajectories, as the organizer of the sequence of problems. Then, we present and analyse some of the designed activities.

THEORETICAL FRAMEWORK

Hypothetical learning trajectories and digital technologies

Hypothetical learning trajectories (HLT) are a construct for the design of mathematical instruction and conceptual learning (Clements & Sarama, 2004; Simon, 2014). A HLT involves three components: i) a goal about students' learning, ii) a set of mathematical problems that are expected to lead to the stated goal, and iii) a hypothetical learning process, i.e., an expectation on the way students' thoughts and understanding shall evolve when they engage with the designed problems.

Simon (2014), echoing other authors, mentioned that this framework allows to describe students' thinking and learning in specific mathematical domains. It contemplates projection of routes, through mathematical problems, that promote mental processes and higher level of mathematical thinking. Adopting a HLT requires a clear learning goal and awareness that common aspects are recognized in ways of learning by students. It is also necessary to recognize that HLTs are permeated by opportunities emerging throughout the designed instruction, since their hypothetical character gives rise to the possibility that teachers modify aspects of the intervention when they consider it necessary (Simon & Tzur, 2004). In a HLT, learning goals are seen as a guide and the point to reach, hence they provide elements for the selection of problems and contribute to build the hypothetical learning process (Simon & Tzur, 2004). According to these authors, this reveals a relationship between the last two components of the model, since problems are selected considering a hypothesis about the learning process, while the learning process is conceived through the problems selected.

Sacristan et al. (2010) studied nuclear ideas of HLT in the light of digital technologies and their impact on learning. For them, student's learning varies and takes different forms according to the situations proposed to them and the tools involved, DGS in our case. This recognition leads to deep conceptual levels not commonly reached in the school context, because interaction between students and digital technologies promotes transitions from particular to general, concrete to abstract, intuition to formalization, among others. Despite this, mathematics education research on HLT with digital technologies, in particular with 3D DGS, is in a primary state of development.

Characterizing the types of proof

Some researchers have attempted to recognize students' conceptions of mathematical proof and what is convincing for them, although they focused on particular aspects and left others aside. Marrades & Gutiérrez (2000), based on Balacheff (1988) and Harel and Sowder (1998), propose an analytical framework allowing a broad understanding of students' actions and productions when they solve proof problems. For Marrades and Gutiérrez, the term *proof* encompasses reasons given to convince someone about the truth of a mathematical fact. This model allows analysing all activity performed by students when they generate a conjecture and establish a way to prove it. The model contemplates two categories, empirical and deductive proof.

Empirical proofs take examples as the main element of conviction. Observing regularity in different cases leads students to establish a conjecture and prove it based on those examples. Examples can be used to prove a conjecture in different ways: in a perceptual or intuitive way, by choosing examples without any specific planning (*naïve empiricism*). A carefully chosen special case can be used to verify a property and consider it true in general terms (*crucial experiment*). A specific example can be selected as a representative of the family it belongs to and used to identify abstract properties after its observation and handling (*generic example*). In *deductive proofs*, a decontextualization of the arguments involved takes place. Generic aspects of the problem, mental operations, and logical deductions are used to organize proofs, so conjectures are deductively validated. Examples may be used as a help to organize arguments, but specific characteristics of the examples are not part of the proof. Deductive proofs can be organized and supported by specific examples (*thought experiment*) or based on abstract mental operations, without specific examples, and pertinent mathematical definitions and properties (*formal proof*). Marrades and Gutiérrez (2000) consider that these categories allow evaluating the improvement or changes of students' proof skills across a learning process.

CONSTRUCTION OF A HYPOTHETICAL LEARNING TRAJECTORY

The goal of this HLT is to favour the learning of proof by Spanish MGS studying lower secondary education (11-14 years old). In addition to the ordinary schooling, these students have participated in programs of attention to giftedness (AVAST) and mathematical talent (ESTALMAT). As students are not in the same grade or school, the teaching experiments are organized as individual clinical interviews.

GeoGebra is the DGS environment to solve problems. We have created 22 geometric problems, asking to make a construction and prove its correctness, based on objects and properties related to spheres, lines, planes, parallelism, equidistance, perpendicular bisectors, mediator planes, tangency, etc. Students, through dragging of points and mobilizing perceptual clues, should discover mathematical ideas that will be used later in other problems. To benefit from the integration of 2D and 3D representations in GeoGebra, some problems ask to explore a property of 2D figures and then work on the corresponding property of 3D figures. For instance, when studying the equidistance between points in space, it is possible to start working in 2D with perpendicular bisectors and circumferences and then ask students to extend their findings to bisector planes and spheres in 3D. This organization of problems should allow students reaching a deeper level of understanding by integrating and articulating different objects of 2D and 3D geometry.

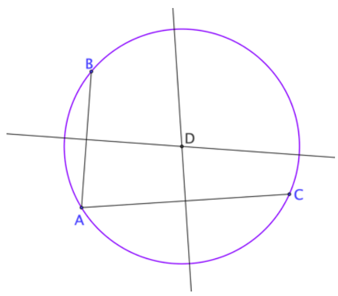
Regarding learning to prove, construction problems offer an opportunity in which students must use the tools provided by the software, as well as the geometric relationships learned from previous problems, in order to construct objects with some properties associated to equidistance and prove that the constructions are correct. In this sense, transit through various equidistance relationships, knowledge and gradual mastery of various software tools, as well as geometric relationships, configure a scenario in which students have availability of more theoretical and instrumental elements to solve new problems. Hence, we expect that the teaching sequence will induce an increment in MGS's deductive abilities. Teacher's role becomes relevant, since he has the possibility of talking to the students and asking them questions based on their productions, to induce them to express ideas in a higher level of proving.

As can be seen, this instrumental and conceptual integration combines objects of 2D and 3D geometry, as well as a non-basic knowledge of GeoGebra tools, in situations that require making particular geometric objects and proving that the construction works. Such an orchestration is not usual in conventional curricular configurations at this school level, so we consider that these problems are challenging for MGS. We present two problems to illustrate two moments of the HLT designed.

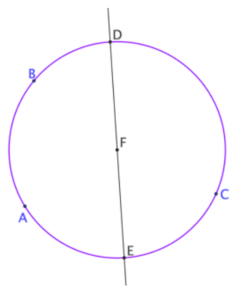
Two problems in the sequence

> Problem 6: Open GeoGebra and activate the 2D view. Construct three points, A, B, and C. Use the *Circumference by three points* tool to construct the circumference that contains the three points.
>
> Determine the centre of the circumference shown on the screen. How do you guarantee that the construction is valid? Do you think that this point is unique? How could you justify your answer?
>
> Open the 3D view and close the 2D view. Construct a point M which is at the same distance from A, B and C, but is not located in the same plane as these points. How could you guarantee that M is at the same distance from the other three points? Do you think that M is unique? If so, why do you think so? If not, what property do points M have?

Problem 6 has two parts. First, the centre of the circumference determined by three points has to be found and the construction has to be justified. Before solving this problem, students will have solved another problem and learned that perpendicular bisector is the locus of points equidistant from two fixed points, so now students will be able to find the intersection of the perpendicular bisectors of two pairs of given points and use that property to prove that this point is the centre (Fig. 1a): the intersection of the perpendicular bisectors is the centre of the circumference because each perpendicular bisector contains the points that are equidistant from the two points that determine it, and their intersection is a point simultaneously equidistant from the three points. Another way to determine the centre is by using the perpendicular bisector of only one pair of the given points. This line determines a diameter of the circumference, so its midpoint is the centre (Fig. 1b).



a)



b)

Figure 1: Finding a point equidistant from three given points in 2D

Those two solutions illustrate deductive proofs. However, MGS may also produce empirical types of proof. For example, students may create a point and place it on the centre of the circumference with the help of the distances from this to points A, B and C (*naive empiricism*). Next, the problem asks about the uniqueness of this point. We expect an affirmative answer to this question. One way to support it is to construct any point and determine the distances from it to points A, B and C, to show that the only point equidistant from them is the centre (*experiment crucial*). If the centre of the circumference is the intersection of two perpendicular bisectors, it is possible to guarantee its uniqueness because the intersection of the two lines is unique (*thought experiment*).

The second part of the problem is based on the construction made in 2D, now seen as part of the 3D space. Students are asked to create a point M at the same distance from points A, B, and C, but not located in the plane of these three points. We believe that students may produce quite diverse approaches, but having in common the combined use of the dragging of point M and the distances from it to points A, B, and C (Fig. 2). When point M is located fitting the condition, dragging it vertically will allow students to find other solutions. A proof of this result may be based on showing that distances are equal when the point is dragged vertically (*naive empiricism*) or involve geometric objects such as the sphere with centre M that contains any of the other points and note that this kind of dragging does not affect that points A, B and C always belong to the sphere, so the equidistance is conserved (*generic example*). We do not expect a deductive justification because it needs an element that is not known by students, namely perpendicularity between lines and planes, which is the learning goal of the last question in this problem.

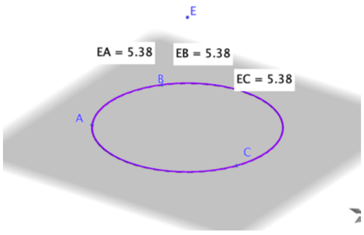


Figure 2: Finding a point equidistant from three given points in 3D

> Problem 11: Open GeoGebra and activate the 3D view. Construct three non-collinear points, A, B, and C on the base plane (grey). Construct a plane equidistant from the three points. Is this plane unique? What property does this plane satisfy, in addition to being equidistant from the three points? Explain why the property is true.

a) b) c)

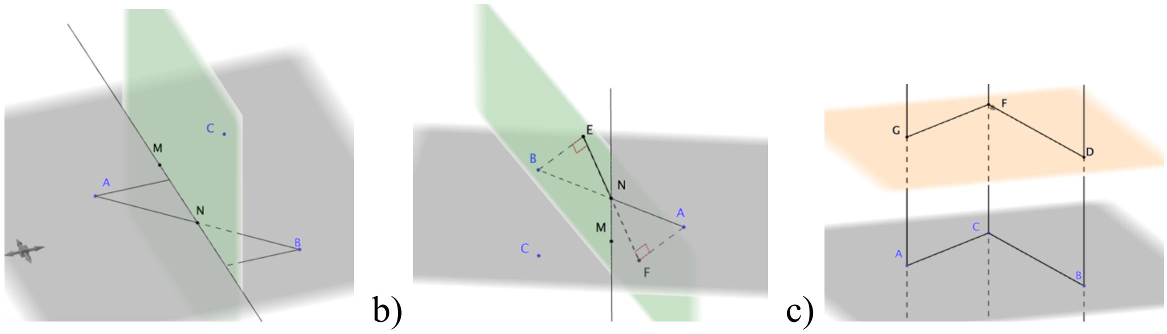


Figure 3: Finding an equidistant plane

In a previous problem, students will have constructed in 2D a line equidistant from three non-collinear points and they will have learned that the line is determined by two of the midpoints between A, B, and C. Problem 11 has two solutions. A solution can be formulated with the help of the mentioned property, that is, by constructing the line MN, where M and N are midpoints of A - B, and A - C respectively. Then, a perpendicular plane to the plane containing A, B, and C is constructed, which also contains the line MN (Fig. 3a). In this case, the proof of such a result will require constructing perpendicular lines to the new plane through A, B and C, as well as segments AB and AC. As these segments determine congruent triangles, compliance with the requested property in the problem can be proved. Students will be expected to recognize that any plane containing the line MN satisfies the stated property (Fig. 3b). In this case, the proof is similar to the previous one.

Another solution is any plane parallel to the plane determined by A, B and C. In this case, the proof can be based on constructing the perpendicular lines to the planes through A, B, and C (Fig. 3c). Since these lines are parallel, and their points of intersection with the two planes (A, B, and C; G, D, and F) determine pairs of parallel segments, three parallelograms are formed, so their opposite sides are congruent, in particular, AG, BD, and CF are congruent, thereby reaching the desired result. These solutions are deductive proof where some examples are used to help to find the properties necessary to build the proof (*thought experiment*). However, it is also possible to develop different types of empirical proofs based on specific examples and numerical values of distances.

CONCLUSIONS AND CONSIDERATIONS

We have created a HLT which has served as an organizer for designing a sequence of problems of spatial geometry in a 3D DGS environment, whose objective is to support learning of proof by lower secondary school MGS. On the other side, the Marrades and Gutiérrez (2000) framework allows us to characterize proofs produced by the students and recognize their progress from empirical to deductive proofs along the sequence. We consider that this is an advance in the research on the use of HLT to design teaching sequences of space geometry problems in 3D DGS environments to promote MGS's learning of mathematical proof.

Nature of learning is related to the means through which it occurs (Sacristan et al., 2010). In our case, GeoGebra provides the possibility of exploring simultaneously related concepts and properties in 2D and 3D configurations (e.g., perpendicular bisector and mediator plane), as well as allowing MGS explore geometric contexts usually not studied at school level. This allows students to integrate different geometrical objects and get experience to move from empirical and perceptual proofs to deductive and formal ones (Sinclair & Robutti, 2013).

The HLT we have created offers the possibility of integrating concepts and properties from school geometry and others that are not studied in secondary school. This integration, framed in situations that demand the construction of specific geometric objects, as well as the proof of the validity of such constructions, produces contexts that are challenging and interesting for MGS, due to their non-routine nature. Furthermore, GeoGebra allows MGS to explore the proposed situations, thus favouring their creativity and problem-solving strategies. These are aspects that, in line with Dimitriadis (2010) and Jaime and Gutiérrez (2017), make this proposal a valid option to promote the talent of MGS, with 3D DGS being a relevant part of it.

REFERENCES

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder & Stoughton.

Benedicto, C., Acosta, C., Gutiérrez, A., Hoyos, E., & Jaime, A. (2015). Improvement of gifted abilities in a 3d computer environment. In N. Amado & S. Carreira (Eds.), *12th International Conference on Technology in Mathematics Teaching* (pp. 24–28). Faro, Portugal: University of Algarve.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Dimitriadis, C. (2010). *Developing mathematical giftedness within primary schools: A study of strategies for educating children who are gifted in mathematics* (Doctoral dissertation). Brunel University School of Sport and Education, Uxbridge, UK.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (Vol. 7, pp. 234-283). Providence, RI: American Mathematical Society.

Jaime, A., & Gutiérrez, A. (2017). Investigación sobre estudiantes con alta capacidad matemática. In J. M. Muñoz-Escolano, A. Arnal-Bailera, P. Beltrán-Pellicer, M. L. Callejo, & J. Carrillo (Eds.), *Investigación en Educación Matemática XXI* (pp. 71–89). Zaragoza, Spain: SEIEM.

Jaime, A., Gutiérrez, A., & Benedicto, C. (2018). Problemas con extensiones. Propuesta para estudiantes con alta capacidad matemática. *Uno, 79*, 7–14.

Marrades, R., & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*(1–3), 87–125.

Sacristán, A. I., Calder, N., Rojano, T., Santos-Trigo, M., Friedlander, A., Meissner, H., … Perrusquía, E. (2010). The influence and shaping of digital technologies on the learning - and learning trajectories - of mathematical concepts. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics education and technology. Rethinking the terrain* (pp. 179–226). Boston, MA: Springer.

Simon, M. (2014). Hypothetical learning trajectories in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 272–275). Dordrecht, The Netherlands: Springer.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104.

Sinclair, N., & Robutti, O. (2013) Technology and the Role of Proof: The Case of Dynamic Geometry. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & K. F. S. Leung (Eds.), *Third International Handbook of Mathematics Education* (pp. 571-596). New York: Springer.

Equation Lab: fixing the balance for teaching linear equations using Virtual Reality

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This paper presents the theoretical foundation, design and evaluation of a teaching experiment using a virtual reinterpretation of the balance model. We present the design of the virtual reality application Equation Lab. The design of the application intends to address issues of equation solving with negative numbers concerning the balance model, identified in the literature. The application was used in a teaching experiment in a Danish lower secondary school. We report findings indicating that a particular student exhibited interesting behaviour, both during and after the teaching experience, indicating an affordance for students in acquiring new equations solving strategies as well as learning about negative numbers.

Keywords: Virtual reality, equation solving strategies, negative numbers, balance model.

INTRODUCTION

Vlassis (2002) asks if we should reject the balance model as a tool for teaching equations. Traditionally, the balance model has been a common tool for teaching and discussing the concept of linear equations for algebra beginners in lower secondary school (Otten et al., 2019; Pirie & Martin, 1997; Rhine et al., 2018). However, the balance model also comes with some limitations as a tool for teaching linear equations. In fact, several studies show that the balance model has severe shortcomings in representing and letting the learner work with negative numbers (Pirie & Martin, 1997; Vlassis, 2002). The present study describes a virtual reinterpretation of the balance model as a tool for teaching linear equations using Virtual Reality (VR). Specifically, for the teaching of linear equations involving negative numbers. VR allows teaching to surpass the constraints of real-world physics (e.g., allowing for the creation of objects with gravitationally repulsive behaviour, conceptually representing negativity). This reinterpretation of the balance model is done with the intention of still maintaining identified affordances of the balance model as a tool for teaching linear equations (Otten et al., 2019).

THEORETICAL BACKGROUND

In this section, we outline the theoretical considerations leading to the first design iteration of Equation Lab. Initially, we will cover the different aspects of difficulties that children in lower secondary school experience while learning the concept of equations and equation solving. Jankvist and Niss (2015) present two categories of difficulties related to understanding the concept of equations:

> “The first kind of difficulty ... is to do with goal-oriented transformation of equations (and, more fundamentally, algebraic expressions) into equivalent ones by way of permissible operations.” (...) ”The second kind of difficulty, which appears to be of a more fundamental nature, is to do with what an equation actually is, and with what is meant by a solution to it.” (Jankvist & Niss, 2015, p. 276)

Traditionally, the balance model has often been used for introducing and teaching linear equations (Otten et al., 2019; Pirie & Martin, 1997). According to Otten et al. (2019) the balance model is used in teaching equations due to three rationales related to equality concept, physical experiences, and learning through models and representations. The balance does an excellent job in representing the equal sign. In fact, using the balance model is seen to enhance the understanding of the concept of equality in general (Otten et al., 2019). When the balance is levelled, the two sides represent equal value and are thereby interchangeable; making it good for demonstrating the idea of the scales facilitates the use of the rule of elimination of like terms. Regarding the second rationale Otten et al. (2019) state that learning about equality through the physical experience of using the balance model is beneficial, giving learners a greater understanding of the concept of linear equation. Research studies underline the importance of movement and gestures working with the balance to develop mental models of mathematical ideas (Otten et al., 2019). Also, offering students experiences with manipulation of balance, equality can be recognized, defined, created, and maintained. Suh and Moyer (2007) emphasize that using manipulatable concrete objects can help provide meaning, through linking procedural knowledge and conceptual knowledge of algebraic equations. However, caution when using such manipulatives for teaching formal equation solving is necessary, because not all students automatically connect their actions on manipulatives with their manipulations on abstract symbols (Suh & Moyer, 2007). The real-time feedback some models provide, allows students to verify the results of their manipulations and their reasoning processes in order to construct knowledge. Learning through the use of models and representations is beneficial, because the learner can use the representation of the model to give sense of the abstract algebraic object (Otten et al., 2019).

When students engage in solving equations, different strategies come into play. Linsell (2009) suggests that instead of trying to determine how difficult a given equation is, it is more useful to look at the strategies that the students can apply. Furthermore, these strategies do not only describe alternative approaches to solving linear equations, but also represent the stages of conceptual development (Linsell, 2009). This indicates that teaching equation solving strategies and the rationale behind these strategies is beneficial to understanding the concept of linear equations. Teachers occasionally try to help students to learn the working backward strategy, by introducing phrases like "change side – change sign" (Rhine et al., 2018). However, this strategy can lead to students making errors or getting an improper conception of the equal sign. Herscovics and Linchevski (1994) found that students tend to detach the minus sign preceding a number. This way the students do not handle numbers correctly when doing transformations (working backwards or change side-change sign) of an equation leading to them inversing subtraction with subtraction.

Vlassis (2002) emphasises that the introduction of negatives (both negative terms and negative coefficients) is what detaches linear equations from concrete models such as the balance. She categorizes the linear equations that are detached from models, as abstract versions of both arithmetic and non-arithmetic equations. Furthermore, Vlassis (2002) identifies equations that include negatives are perceived as being particularly difficult. Pirie and Martin (1997) mention that subtracting a negative number in order to cancel it out is a common error when learning with the balance model. A further specific difficulty is that a solution needs to be perceived as a weight and not just a number (Pirie & Martin, 1997).

VR can provide educators with teaching tools that overcome physical constraints identified in research on the traditional use of the balance model. VR allows designers to create compelling and engaging learning experiences with a modified balance model. This paper seeks to investigate how such an experience can influence a student's understanding of the concept of linear equations, while identifying affordances and limitations of the design. Research show that gamified VR has been linked to high student engagement and motivation (Checa & Bustillo, 2019). When being exposed to the multisensory stimuli of VR, a user may experience a sensation of presence. Different factors of the design contribute to this sensation (e.g., virtual avatar, interactivity). The sensation of presence can induce the two psychological effects of *place illusion* (the illusion of "being there" in the virtual environment) and *plausibility illusion* (the illusion that virtual events are really happening) (Slater, 2009). These effects may contribute to students' perceiving the experience with the modified balance model as compelling and plausible.

Analysis of the presented literature has led us to establish two research questions related to teaching and learning equation solving in VR:

1. *With emphasis on manipulation of negative numbers and equation solving strategies, how can an immersive experience with a reinterpretation of the balance model support students' understanding of the concept of linear equations?*
2. *Based on a user study, what influential factors can be identified and how can we overcome or enhance effects related to transferability from the virtual experience to pen-and-paper?*

REQUIREMENTS

This section will specify the design requirements of the virtual environment, based on analysis of the theoretical background. The requirements of the design will relate to the characteristics of representation and interaction of the immersive experience.

The system should depict the balance model in a recognizable way, while indicating equality dynamically through real-time simulation, as this is described as a clear advantage over a static representation (Otten et al., 2019). It will be necessary to modify the programmed behaviour of the simulation to alter the physical laws allowing for negative numbers in the context of balance. Moreover, each term of the equation should be depicted as an independent virtual object. This means that each term should be represented by encasing the algebraic notation inside a virtual object. Thereby, the terms of $7x$ and 5 cannot be distinguished from each other, apart from the algebraic notation itself. Due to the focus on negativity, the terms of equation in the teaching sequence has been limited to only encompass integers and integers coefficients. Decimals, fractions and parentheses should be investigated in future research with new objectives.

To ensure that any equation can also be solved, we see it necessary to set up the following interaction and transformation capabilities. First, it should be possible to move a virtual object from one side of the scales to the other, not to mention remove an object from the balance entirely. In addition, it must also be possible to transform a virtual object (i.e., the term) to achieve a solution in the traditional way. Therefore, we consider it necessary to be able to add, subtract, invert and divide. We want to make it possible to remove equal parts from each side of the weight bowl. At the same time, we want it to be possible to undo transformations. For example, dividing of $7x$ by 7, should create 7 virtual objects with the value x. Because of this, it will be possible to undo the division by reassembling the 7 virtual objects. We limit the set of possible divisors to natural non-trivial factors of the object or coefficient of the object. Moreover, it is not possible to multiply virtual objects, since it is not a necessity to solve equations with integers and integer coefficients.

DESIGN

In this section, we explain the design of the application called Equation Lab. The task of the user is to solve linear equations using the modified balance model in VR.

The objective of the task involves the user applying transformations (specified in design requirements) that results in the unknown being isolated (as a term) on one side and a number (as a term) on the other. While the user transforms the equation at hand, the user can check the status of equality by reading the deflection of the balance (see Figure 1). For readability, an additional indication of equality can be read in the colour of the beam between the pans (i.e., green for equality).

The virtual environment surrounds the user with a desk containing elements characterized as mechanics and dynamics. The mechanics of Equation Lab can be divided into two types related to interaction and transformation. The user can either move objects (see Figure 1) from one side to another on the balance, remove objects from the balance or transform objects via User Interface (UI) (see Figure 1). The user can move objects by reaching for it with a virtual hand and pressing a button to grasp. While the button is pressed, the user can freely move the object around and place in a new position by letting go of the button. The UI appears when an object is placed on the workspace. The UI lets the user split, invert or divide by a number when a single object is placed. The user chooses a setting with the slider and executes the command pulling the down rope corresponding to the desired mechanic. When multiple objects are placed on the workspace, the user can add the objects, if they are compatible.

Figure 1: Virtual environment of Equation Lab

The mechanics vary depending on the value of the number or the coefficient represented by the object in the workspace area. The balance reacts to the objects in real-time based on the sum of values in interactive objects on either side of the balance. Moreover, the balance instrument incorporates the modified physics simulation that allows objects with ultimate negative value to have repulsive gravitational force. In other words, an object with a negative value placed on the pan will lift it up.

As described in the requirements section, the overall purpose of designing and developing a virtual setting featuring the balance model is for the user to be able to solve any linear equation within the limitations we set. When implementing the mechanics deemed necessary to solve any equation, one must naturally explore what side effects, good or bad, are relevant. Traditionally the balance model only affords formal transformations of equations within the limitations of the mathematics that are representable. This means that a user can only remove or add positive weights. By formal transformations, we refer to Kieran (2007) and Linsell (2009), which connects these to performing the same legal action to both sides of an equation. Not all legal transformations are accessible using the traditional balance. For example, a subtraction resulting in a negative amount cannot be performed. Manipulations with additive inverses are not a part of the set of actions on the traditional balance. Therefore, the traditional balance model does not afford the strategy of change side-change sign and do only support a limited amount of formal transformations. The consequences of being able invert elements in Equation Lab together with the ability to move terms between the two sides create possibilities regarding equations solving strategies. The emergence of different possible strategies for solving equations with the balance model are not specified directly within the design requirements but are an immediate consequence hereof. We shall touch more upon this discussion in a later section.

USER STUDY

In the following section, we outline the methodology and procedure of the user study, in form of a teaching sequence, performed using Equation Lab. Initially we explain how data was gathered before, during and after the teaching sequence. Afterwards, we discuss and conclude on how the data we collected led to answering the research questions presented in the introduction.

Participants were ten students (six male and four female) from the same Danish 7th grade and their male teacher. The teacher of the class chose the ten participants based on gender, performance level in math and game experience. Gender and performance level were the most important to us. The students engaged in the teaching sequence with Equation Lab, individually for approximately 30 minutes, conducted by the authors. Before the actual teaching sequence with Equation Lab, the participants answered a pre-test survey consisting of 14 questions including demographic questions and solving tasks with different types of linear equations.

After the teaching sequence with Equation Lab, the teacher of the class agreed to do a two-week period teaching linear equations. This two-week period was not taught using Equation Lab, but in a way, the teacher would normally teach linear equations. After these two weeks, we did an interview with the teacher to hear about the possible differences, learning with Equation Lab had made.

FINDINGS FROM TEACHING SEQUENCE WITH EQUATION LAB

Two of the equations the students solved in the environment were $-6x=24$ (Vlassis, 2002) and $7x-3=13x+15$ (Bodin, 1993). These equations fall under the abstract categories (Vlassis, 2002). These equations were chosen because they are great examples of an arithmetic and a non-arithmetic equation with negative numbers.

In this section, we showcase and discuss the behaviour of a particular student who at first glance responded very well to the teaching with Equations Lab. In an interview, two weeks after the teaching sequence, the teacher of the class pointed out that Albert (the male student in question) showed interesting behaviour when solving equations in class. The teacher of the class explained that Albert had taken the experience of Equation Lab to heart, when solving equations. In class Albert had used the invert mechanic when solving equations similar to the ones from the teaching experience ($-6x=24$ and $7x-3=13x+15$). Data from the pre-test suggests that Albert was not able to solve $3x-4=5x-12$, $12-x=15$ or $7=3-x$ before the teaching experience with Equation Lab. Additionally, Albert solved lesser abstract equations such as $3x=15$, $14-x=8$, $x+5=21$, and $3x-4=23$, but only using the ‘guess and check’ or a counting strategy.

We did an interview with Albert, after interviewing his teacher, to try to understand how he adopted and used to the workflow from Equation Lab on pen and paper. In this interview, we let him solve additional equations with negative numbers using pen and paper. This interview was not part of the pre-planned data collection. Video from this interview shows that Albert was not able to guess the solution to the equation $-5x= 25$, however, he was able to solve it using the invert mechanic. Inverting each side before having an additional guess helped him out. He was now able to see that in order to have 5 times an unknown number to result in a negative number the unknown had to be -5. Additionally, Albert was able to apply the version of change side-change sign from Equation Lab to $4x-10=x+2$. However, here he was not successful. He ended up with $4x-x=2-10$ after attempting to collect corresponding terms on either side. When we asked him, why he had chosen to subtract 10 instead of adding and explaining that 10 should be added because it was in fact -10 that he should have inverted. His response was that the minus sign preceding 10 was not close enough to the number 10 for him to consider our proposal.

CONCLUSION

In this paper, we investigated the concept of teaching linear equations in VR. Through analysis of the literature in math didactics and VR, we established requirements related to the task in VR, implemented a novel prototype fulfilling the requirements, and conducted a user study involving a pre-test survey, observational data from teaching sequence and post-test surveys.

This study is highly preliminary, and due to the experimental nature of the concept, the findings may lead to new directions for future user studies in this branch of research in teaching mathematics using VR (e.g., negativity in equations, transferability between different situations, enhancing engagement through VR, the potential of virtual pedagogical agents).

Conclusively, we have identified certain positive prospects of teaching equation solving strategies Equation Lab. With the balance modification, allowing the presence of negative numbers, teachers can use the balance model to teach strategy-oriented equation solving. The transformations make sense and feel intuitive to students, since teachers and students can observe, reflect and discuss the consequences of pressing down on one side and pulling up on the other side. Based on the case of Albert, we believe that learning about negative numbers and how they influence linear equations can help provide understanding of what an equation is, by utilizing the affordances of the balance model in this new setting. Students may get a better understanding of negative numbers as a solution and of the goal-oriented transformations of linear equations. However, several design considerations are eligible for further research to understand the influence of representation and interaction on transferability from Equation Lab to traditional settings (e.g., classroom). The detachment of the minus sign (Herscovics & Linchevski, 1994) was indeed present in the post interview with Albert.

REFERENCES

Bodin, A. (1993). What does to assess mean? The case of assessing mathematical knowledge. In *Investigations into assessment in mathematics education* (pp. 113–141). Springer.

Checa, D., & Bustillo, A. (2019). A review of immersive virtual reality serious games to enhance learning and training. *Multimedia Tools and Applications*, 1–27.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*(1), 59–78.

Jankvist, U. T., & Niss, M. (2015). A framework for designing a research-based "maths counsellor" teacher programme. *Educational Studies in Mathematics*, *90*(3), 259–284.

Kieran, C. (2007). Learning and teaching of algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 707–762). NCTM-Information Age Publishing.

Linsell, C. (2009). A Hierarchy of Strategies for Solving Linear Equations. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 331–338). MERGA.

Otten, M., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (2019). The balance model for teaching linear equations: a systematic literature review. *International Journal of STEM Education*, *6*(1), 30.

Pirie, S. E. B., & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, *34*(2), 159–181.

Rhine, S., Harrington, R., & Starr, C. (2018). *How Students Think When Doing Algebra*. Information Age Publishing, Incorporated.

Slater, M. (2009). Place illusion and plausibility can lead to realistic behaviour in immersive virtual environments. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *364*(1535), 3549–3557.

Suh, J., & Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching*, *26*, 155.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(3), 341– 359.

Designing multiple manipulatives to explore cube cross-section

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The paper describes one of five cube cross-section lessons designed and carried out in our mixed methods research project. The research is focused on how the interplay of physical and digital manipulatives can be integrated into solid geometry to develop students' spatial visualisation. Altogether, a paper workbook, 3D prints and dynamic GeoGebra applets form a five-set toolkit each corresponding to one of the five designed lessons. In this paper, Lesson 3 will be described in detail, which, like the others, has been particularly influenced by the instrumental genesis approach.

Keywords: visualisation, external representations, instrumental genesis, 3D prints, GeoGebra software

INTRODUCTION

Drawings, constructions, pictures or visual models are classic examples of how three-dimensional objects can be represented. These representations can be shown in the paper-and-pencil, physical and digital environment, and in the last two can be dynamically manipulated. The use of digital resources has grown steadily in the past years due to the promotion of technologies at various levels of education. Lieban (2019) pointed out that digital materials often seemed to be developed more to replace physical resources than to supplement them. Focusing on solid geometry, Camou (2012) proposed the positive impact of designing and implementing a multi-representational approach to exploring three-dimensional objects (in Sinclair et al., 2016). In this context, the mixed methods research was conceived with the aim of supporting the development of students' spatial visualisation by integrating the multiple manipulatives into solid geometry lessons. We anticipate that this development could lead to the correct perception of paper-and-pencil representations of 3D objects, and, therefore, to the correct cross-sectional drawings of cubes in worksheets. A paper workbook, 3D prints and dynamic GeoGebra applets form the toolkit designed and implemented in this project. Altogether, the material consists of five follow-up sets each corresponding to one cube cross-section lesson. The aim of this paper is to present and describe Lesson 3 in detail, which, along with the others, has been particularly influenced by the instrumental genesis approach defined in the following section. The objects of the lesson are as follows: (a) to support the development of spatial visualisation, (b) to appropriately manipulate and implement manipulatives in cross-sectioning a cube, (c) to apply solid geometry knowledge and personal experiences with manipulatives in cross-sectioning a cube.

THEORETICAL BACKGROUND

Gutiérrez (2006) emphasizes the importance of close relationships for visualisation and spatial geometry. The concept of a visualisation model was introduced in (Gutiérrez, 1996) to define visualisation or spatial visualisation as a set of four visual elements: mental images, external representations, visual abilities and visual processes. In this paper, the external representations are discussed and considered as any type of graphical (static or dynamic, physical or virtual) or verbal representations of terms, objects or their properties. A detailed revision of the rest visual elements is introduced in (Vágová, 2019; Vágová et al., 2020).

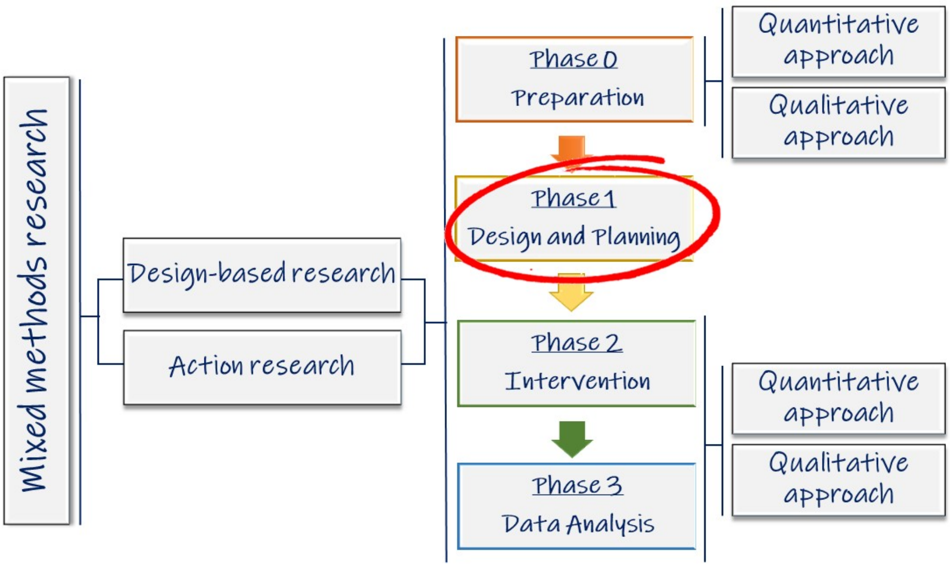

Figure 1: Research design overview

In our mixed methods research, that applies both design-based (see Bakker, 2019) and action research (see Cohen et. al, 2018), two types of external representations were designed in Phase 1 (see Figure 1): 3D physical representations by 3D prints and 3D virtual representations by GeoGebra software. The inspiration comes from two sources: Phase 0, in which the solid geometry lessons were observed in three different grammar schools (Vágová, 2019), and Lieban (2019) who presented and discussed some perspectives of the physical and digital modelling in mathematics education. From our point of view, these representations complement each other and offer a more holistic perspective on the spatial arrangement of solid shapes.

The availability of technology in the mathematics classroom challenges the way to orchestrate the educational process (Drijvers et al., 2009). The process of manipulating 3D prints and GeoGebra applets has played the main role in our project intervention. With this intention, the five-set toolkit has been particularly influenced by the instrumental genesis approach. The essence of this theory lies in the process by which a particular artefact becomes an instrument for a user. An artefact is any device working as a tool, and an instrument refers to a mental construction of the artefact developing and applying by the user while using this artefact (Drijvers et al., 2009; Rabardel, 2002). In our project, the artefacts are represented by physical (3D prints) and digital (GeoGebra applets) resources that become instruments for the students when manipulating them. The idea of such multiple resources is to support the development of students' spatial visualisation (graphic external representations, mental images, visual abilities and processes), which could lead to a better understanding of solid representations in plane and cube cross-sections. A more detail description of this idea is given in (Vágová et al., 2020). Following Lieban's (2019) study, there is a terminological non-convergence using physical and digital resources in education. While some researches (e.g. Hershkovitz, 2016) consider physical and virtual resources to be manipulative, others do not include the digital ones (e.g. Faggiano et al., 2018). In our research, both physical and digital resources are considered to be manipulatives as stated by Lieban (2019). For physical manipulatives, the term 'manipulative' refers to the artefacts that can be manipulated. For the digital ones, it refers to the interaction of the user with the resource and not the manipulation of the physical device itself. In order to get the objects, the following research question is addressed: How should the combined use of physical and digital manipulatives be integrated into cube cross-section lessons to develop students' spatial visualisation? Subsequently, a detailed description of Lesson 3 will be presented and discussed.

DESIGNED CUBE CROSS-SECTION MATERIAL: LESSON 3

Material *Cube Cross-Section. Connecting the digital and the physical world* includes a paper workbook, 3D prints and dynamic GeoGebra applets, where the last two are considered to be artefacts in the sense of the instrumental genesis approach. When students use the printed workbook (3D prints/GeoGebra applets), we say they work in the paper-and-pencil (physical/digital) environment (later PaPE, PhyE, DigE) and it is a resource of 2D paper-and-pencil representations (3D physical representations/3D digital representations).

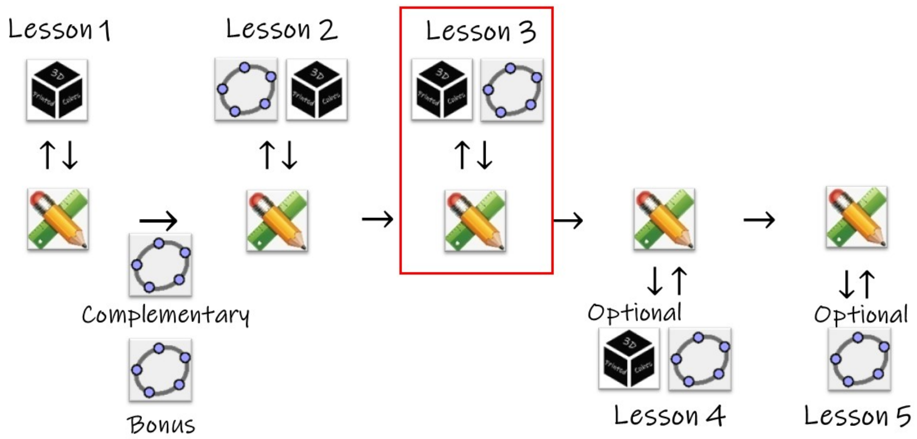

Figure 2: Combination and sequence of the assigned resources in lessons

The designed toolkit is divided into five sets, each corresponding to one of the five cube cross-section lessons (see Figure 2). Each set is specific due to the different combination and sequence of the assigned resources. Every environment (resources) has its own symbol and the movement from one to the other is specifically indicated in the worksheet. The cycle of solution steps in Lesson 3, as well as the transition among the working environments, can be described as follows (see Figure 3):

- **STEP 1** - represents the *transition* from the PaPE to the PhyE. Based on 2D paper-and-pencil representation shown in the workbook, students are looking for the corresponding 3D prints (see also Figure 4 and Figure 5).
- **STEP 2** - represents the *interconnection* of the PhyE and the DigE. By 3D prints observation and manipulation, students cross-section a cube in the applet.
- **STEP 3** - represents the *transition* from the interconnection of the PhyE and DigE to the PaPE. By acquired experiences from the 3D prints manipulation and digital geometric construction, students cross-section a cube on paper in W3 worksheet. The construction steps are discussed and compared with the 3D prints and digital construction.

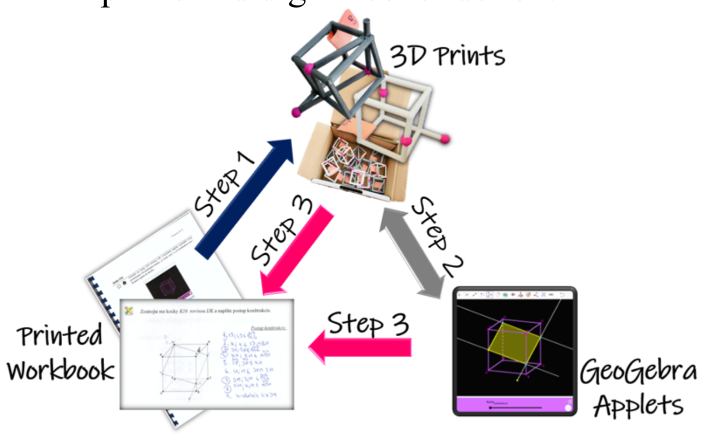

Figure 3: Cycle of solution steps in Lesson 3

To summarise, in Lesson 3, students initially open a task in the printed workbook (Step 1) and then manipulate 3D prints in interaction with the GeoGebra applet (Step 2). Afterwards, they cross-section a cube in the worksheet and write down the construction steps (Step 3). The idea of the interplay of physical and digital manipulatives is to support a better perception of paper-and-pencil (2D) representations of solids and right cross-sectional drawings of a cube by students. The following section describes the individual resources designed for Lesson 3 in detail.

WORKSHEET W3 AND ITS PHYSICAL AND DIGITAL MANIPULATIVES

Paper-and-pencil, physical and digital environments are those contexts in which students solve tasks *Cube C13 – Cube C18* in Lesson 3 (see Figure 2 and Figure 3). The most important and key resource is the printed workbook because of its guide/manual character referring to the transition among the working environments. It means the PaPE is the starting point from which students move to the other environment(s) and always return to it. To underline, a cycle of solution steps of each worksheet begins and ends in the PaPE (e.g. see Figure 3). This relationship is also illustrated in Figure 2, where the symbol of student work, which shown in the PaPE, is centred in a single line. To repeat, the aim of such multiple manipulatives is to support student work in the PaPE. Subsequently, each designed resource is characterised in detail.

Worksheet W3

As mentioned above, the workbook is the key resource and the physical and digital manipulatives are complementary. To make it easier for students to orient themselves on the worksheets, the same layout was followed in the design process. In the beginning, there is a list of working environments in which students will solve the individual tasks. Each environment is assigned a symbol, as well as pictures of 3D prints and GeoGebra applet are shown. Importantly, there is a "gateway" to the digital world (see Figure 4). Using a web page link or scanning a QR code, students move from the PaPE to the DigE.

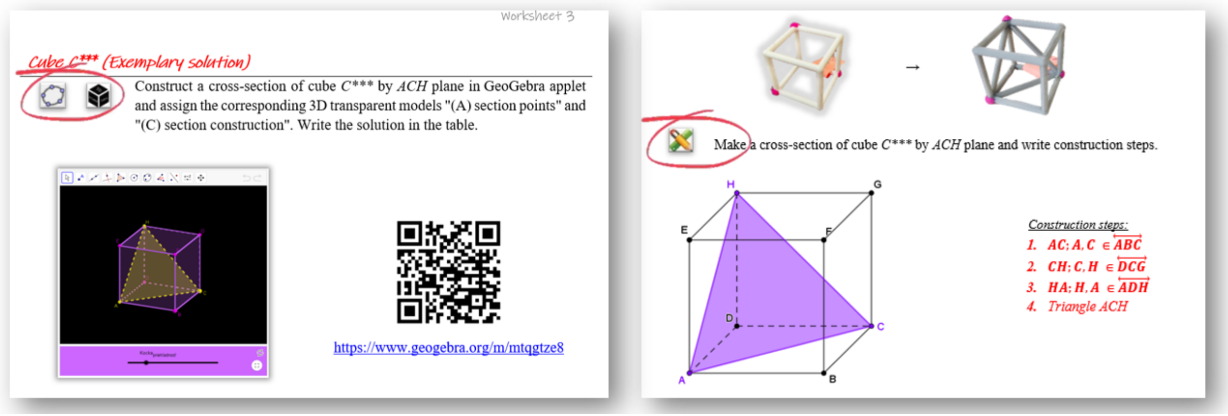

Figure 4: Illustration of worksheet W3

Afterwards, an exemplary well-solved task and six unresolved tasks follow the introductory part of the worksheet. The exemplary solution (see Figure 4) demonstrates when and how to move and work in that concrete environment to support cross-sectioning a cube on paper. As shown in Figure 4, every task is divided into two parts. In the first one, students have to find the corresponding 3D prints and accomplish a cube cross-section by using the GeoGebra applet. In the second one, student cross-section a cube and write down the construction steps.

3D prints

In our research, five categories of 3D prints were proposed, the entire classification of which is given in (Vágová et al., 2020). The W3 worksheet includes the following:

- **Category A** – white transparent cubes demonstrating the cube cross-section points. The section points are painted pink and every print is named. In this category, 3D prints are labelled A10-A16.
- **Category C** – grey transparent cubes demonstrating the cube cross-section construction. The section points are painted pink and every print is named. In this category, 3D prints are labelled C10-C16.

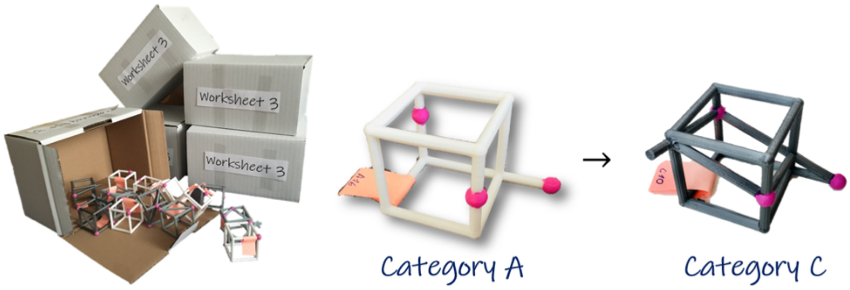

Figure 5: Illustration of 3D prints supplementing W3 in Lesson 3

The 3D prints were labelled strategically. The letter A and C represent the 3D printing category. The first number (1 or 2) represents the transparent or opaque character of the 3D print. The second number (1 – 6) was randomly added to the 3D print regardless of the task number. Moreover, in each set, the cube cross-section points are painted differently. Only the fifth set, which is the last one, is an exception, since there are no 3D prints. As it can be inferred from Figure 5, the 3D prints represent the cube cross-section process. The idea is to allow students to "familiarise themselves" with the different stages of the geometric construction. In this way, students can take, play, observe and discover various geometric problems. Mathematics thus becomes alive, real and physical, because it is on the table and can be captured and manipulated from different perspectives.

Operative GeoGebra Applets

In our project, demonstrative (passive) and operative (active) GeoGebra applets were proposed. The **demonstrative applets** are those which do not require any skills with GeoGebra software. Students manipulate the objects only by moving the sliders. The demonstrative applets are integrated into Lesson 1, Lesson 2 and Lesson 4. A more detail description of these applets is given in (Vágová, et al., 2020). On the other hand, the **operative applets** require basic skills with GeoGebra software. Students construct basic objects such as point, intersection, line, parallel line, polygon and intersection of two surfaces. The operative applets are integrated into **Lesson 3**, which is described in this paper (see Figure 5), and also into Lesson 5.

The W3 worksheet, along with the others, is available to students in the GeoGebra online book published on the GeoGebra platform. By using the web page link or scanning the QR code (see Figure 4), students move from the PaPE to the DigE. In addition, the content of the online book is the same as the content of the paper workbook. The online book is supplemented with the PDF format of the paper workbook as well as 3D prints photos. In Lesson 3, there are six operative applets each corresponding to one of the six tasks *Cube C13 – C18.* All applets have the same design (see Figure 5) consisting of a cube with the section points, a menu, a toolbar, and a *cube opacity* slider.

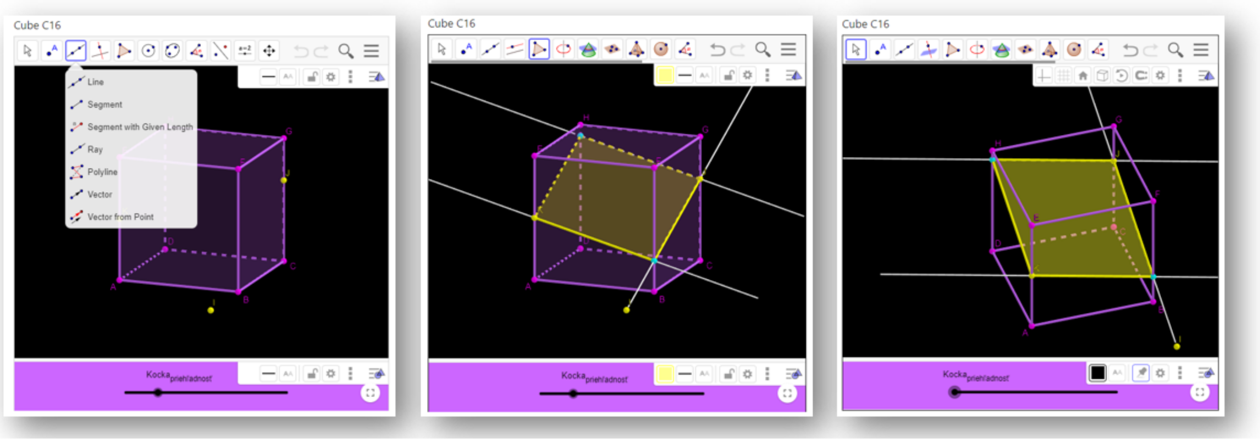

Figure 5: Illustration of applet Cube C16

When students enter the DigE, the 3D digital representation of a cube opens. The digital image corresponds to the cube image in the W3 worksheet and Category A of 3D prints. In this way, students perceive three different external representations of the same situation and from three different environments. Students can rotate the digital cube, move it and change the opacity by the *cube opacity* slider. The DigE is therefore considered as a "bridge" connecting the PhyE and the PaPE. The aim of the operative applets is that students cross-section a cube step by step. In the toolbar, they select a basic object (e.g. line) and using the menu, they can also change the type or colour of the objects. When students are cross-sectioning a cube, the 3D prints of category C can be used in a navigational or control sense. Students either cross-section a cube by following the 3D print and discover the context or they cross-section a cube themselves and compare it with the 3D print. Afterwards, by acquired experiences from the 3D printing manipulation and digital construction, students cross-section a cube on the paper worksheet.

DISCUSSION

In this paper, Lesson 3 was presented and discussed in detail. This lesson is one of the five cube cross-section lessons that were designed and carried out in our mixed methods research project. Altogether, three different resources were designed. A paper worksheet as the key resource and the physical and digital manipulatives are complementary. In addition, the digital environment is a "bridge" connecting the paper-and-pencil and physical environment. Every resource was designed strategically and each of them has its own advantages and disadvantages. The physical and the digital ones are considered to be artefacts which could support the development of students' spatial visualisation. Afterwards, this development could lead to a better understanding of paper-and-pencil representations and right cross-sectional drawings of a cube. The aim of the project was to design such multiple resources that would complement each other and offer a more holistic view on spatial arrangements in solids. It means the artefacts that would become instruments for students when manipulating them. Currently, we are in the process of qualitative and quantitative data analysis of

Phase 2. The following publication steps will lead to the remaining worksheets and the findings and results of this intervention.

REFERENCES

Bakker, A. (2019). *Design Research in Education.* Great Britain: British Library Cataloguing-in-Publication Data.

Cohen, L., Manion, L. and Morrison, K. (2018). *Research Methods in Education.* Great Britain: Bell an Bain Ltd, Glasgow.

Drijvers, P., et. al (2009). Integrating Technology in Mathematics Education: Theoretical Perspectives. In C. Hoyles & J. -B. Lagrange (Eds.), *Mathematics Education and Technology-Rethinking the Terrain, 13,* 89-132.

Faggiano, E., Montone, A. and Mariotti, A. (2018). Synergy between manipulative and digital artefacts: A teaching experiment on axial symmetry at primary school. *International Journal of Mathematical Education in Science an Technology, 49(8),* 1165-1180.

Gutiérrez, A. (1996a). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutierrez (Ed.), *Proceedings of the 20th PME International Conference, 1,* 3-19.

Gutiérrez, A. (2006). La investigación sobre enseñanza y aprendizaje de la geometría. In P. Flores, F. Ruiz, M. De la Fuente (Eds.), *Geometría para el siglo XXI* (pp. 13-58). Badajos, España: FESPM y SAEM Thales.

Hershkovitz, S. (2016). Concrete manipulatives and Virtual Manipulatives: Learning Fractiones in 4^{th} Grade. In *ICME13,* Hamburg, Germany.

Lieban, D. (2019). *Exploring opportunities for connecting physical and digital resources for mathematics teaching and learning.* (Doctoral dissertation), Johannes Kepler Universität Linz, Austria.

Rabardel, P. (2002). *People and Technology – A Cognitive Approach to Contemporary Instruments*. Francia: Université Paris. Retrieved December 12, 2019 https://hal.archives-ouvertes.fr/file/index/docid/1020705/filename/people_and_technology.pdf

Sinclair, N. et al. (2016). Recent research on geometry education: an ICME-13 survey team report. *ZDM Mathematics Education, 48,* 691-719.

Vágová, R. (2019). Plane, physical or dynamic representations of 3D objects. Which do you prefer when teaching solid geometry? In *Aplimat 2019: Proceedings of the 18^{th} conference on applied mathematics,* 1237-1251.

Vágová, R., Kmetová, M. and Lieban, D. (2020). The interplay of physical and digital resources to explore cube cross-section. In *Aplimat 2020: Proceedings of the 19^{th} conference on applied mathematics,* 1067-1081.

Outdoor photography: a resource in teacher training

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Several researchers refer the importance of outdoor photography for the understanding of math contents. However, we do not have enough knowledge about how preservice teachers pose problems in real life contexts. This paper describes a study where elementary preservice teachers capture photographs in their environment that allow task design. In particular, we want to identify what features of the environment were privileged and to understand what are their main difficulties and reactions when designing mathematical tasks. Results suggest that participants favoured elements related to buildings and expressed difficulties in the design of high-level cognitive tasks. This experience had a positive impact on them, providing a "closer look" at everyday objects, developing their "mathematical eye".

Keywords: Photography, Task design, Problem posing, Problem solving, Preservice teachers.

INTRODUCTION

Nowadays we are experiencing deep changes in different areas of society, in particular in mathematics education. So, school mathematics requires effective teaching that engages students in meaningful learning through individual and collaborative experiences, giving them opportunities to communicate, reason, be creative, think critically, solve problems, make decisions, and make sense of mathematical ideas (NCTM, 2014). In this context, we must stress the importance of complementing learning with other mathematical learning experiences as the outdoors. The process of acquiring information and the development of knowledge by students can occur in many ways and in many places, because the classroom is just one of the "homes" where education takes place (Kenderov et al., 2009). The use of the surroundings as an educational context can promote positive attitudes and additional motivation for the study of mathematics, allowing learners to understand its applicability and its connections (Kenderov et al., 2009; Vale & Barbosa, 2019). In this scope, we consider that seeing through photos (digital images), captured through a digital camera, the connection between the mathematics discovered in and outside the classroom, and not viewed as separate entities, can be a good learning strategy. On the other hand, several authors (e.g. Silver, 1997) say that along problem solving teachers must propose problem posing tasks, as an important mathematical experience that students must develop, as it allows them to apply their mathematical knowledge, while allowing the teacher to realize what mathematics students know. Furthermore, research findings show that mathematical problem solving and problem posing are closely related to creativity, thus being a possible pathway for students to develop this ability (e.g. Leikin, 2009; Silver, 1997).

This work intends to promote contact with contextualized mathematics, focusing on everyday life features, walking through and analyzing the place where we live, connecting some of its details through mathematical problem solving tasks, designed by preservice teachers. The main purpose is to foster positive attitude towards mathematics, through the observation and exploration of the urban environment. It is important that future teachers are aware of mathematics around them, with all the complexity but also beauty and challenge that it encloses. On the other hand, it is an opportunity for students to formulate problems, which implies making decisions about what to consider and what to ignore in the situation under study, applying and mobilizing personal mathematical knowledge in the face of a situation, specifically a realistic one. In this context, images of a real situation, captured through a digital camera, have fundamental importance in solving and formulating problems, playing an important cognitive role in mathematics teaching and learning (Arcavi, 2003).

So, our challenge was to characterize how preservice teachers pose problem solving tasks, in particular when they have to use photography in real life contexts. This paper intends to give some insights in this regard. We focus on the role of capturing photos in the environment for the purpose of formulating mathematical tasks, establishing connections between mathematics and reality. Thus, we aim to answer the following questions: 1) What features of the environment were privileged by the preservice teachers' mathematical eye? 2) What difficulties did they show in the tasks design? and 3) What reactions did they evidence during this experience?

THEORETICAL FRAMEWORK

Task design

Students must have mathematical experiences outside the classroom, observing everyday life, natural and architectonic heritage surrounding their schools, to discover connections of school mathematics with buildings, gardens, streets and so on, creating and solving tasks in real contexts (Bragg & Nicol, 2011; Kenderov et al., 2009; Meier, Hannula & Toivanen, 2018). Hence, it is important to create opportunities for preservice teachers to apply their knowledge about problem posing and problem solving to design tasks outside the classroom, for their own students. Formulating problems helps beginning teachers (and students) to consolidate problem-solving skills and to strengthen their mathematical knowledge and skills. Also, by doing it in the environment, allows seeing the applicability of mathematics in everyday life, as well as developing their own creativity.

Silver (1997) considers problem posing as being either the generation (creation) of a new problem or the reformulation of a given problem. Stoyanova (1998) considers problem posing as a process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. Brown and Walter (2005) propose two strategies for formulating problems. The *Accepting the given* strategy starts from a static situation, which can be an expression, a table, a condition, an image, a diagram, a phrase, a calculation or a set of data, from which we formulate questions to have a problem, without changing the starting situation. The *What-if-not* strategy extends a given task by changing what is given. From the information contained in a problem, we identify what is known, what is asked for and what limitations the response to the problem involves. By modifying one or more of these aspects or questions, new and more questions can be generated (Barbosa & Vale, 2018). For Sullivan and Liburn (2002) three main features underlie the creation of good questions: they require more than remembering a fact or reproducing a skill; students can learn by doing the tasks, answering the questions and teachers learn from each students' attempts; and there may be several acceptable answers. So, posing good questions makes good tasks. The authors propose practical and accessible methods for posing open-ended questions using a three-step process. Method 1 - *Working Backward*, includes identifying a topic; thinking of a closed question and writing down the answer; making up a question that includes (or addresses) the answer. Method 2 - *Adapting a Standard Question*, includes identifying a topic; thinking of a standard question; adapting it to make a good question. These methods can provide also information about the way we choose a mathematical photo. Either we look for the mathematical potential of an object (or phenomena) in a photo or we look for an object that matches a predefined subject. Any of these methods can generate tasks of different cognitive levels of demand (Smith & Stein, 2011).

Photography and mathematical eye

Several researchers (e.g. Meier et al., 2018; Munakata & Vaidya, 2012) work on photography outside the classroom as a way to motivate students, increase interest and understanding of content, through the connection between mathematics and everyday situations. In addition, this type of approach gives students the opportunity to conduct their own transformative and aesthetic experience. This type of photograph, that we call mathematical photo or problem picture, according to Bragg and Nicol (2011), is a photo of a real object, phenomenon, activity or situation that is accompanied by one or more questions or a mathematical problem based on the context of the photo. According to these authors, an image-based question can stimulate students' curiosity in answering the question and their engagement in the process of creating immediate questions or a problem. Gutstein (2006) argues that good tasks do not necessarily reside in the task itself but rather in the relationship between the task and the solver (student or teacher), related to students' interests and lives, aspect that reinforces the use of photos (digital images), because they are chosen by the user. Taking a photo creates an affective connection between everyday situations and mathematical concepts, which engages students with the tasks (Meier et al., 2018; Vale & Barbosa, 2019).

Developing a mathematical eye is a competence that students must acquire, because we live in a world where visual features are a crucial component in the society and in many professions. We apply the common term "mathematical eye" to refer to the use of mathematics as a lens to see and interpret things/elements that surround us. It means to see the unseen, interpret things in the world as a boundless opportunity, and discover mathematics involved by seeing the world around us with new eyes, *eyes that are open to the beauty of mathematics and its relation to the beauty of nature* (Stewart, 1997). We can also use the term “geometrical eye”, coined by Godfrey (1910) as the power of seeing geometrical properties detach themselves from a figure. For most people mathematics that surround them often remain "invisible" to their untrained or inattentive eye. We have to educate their mathematical eye, so that they can identify contexts and elements that can support rich mathematical tasks (Vale & Barbosa, 2019). Saying that students must develop their mathematical eye means that we have to discover new ways of looking and consider familiar things either in daily life, work or inside/outside the classroom. It means seeing common objects from a new perspective, whose level of detail varies with each individual's knowledge and experience. Barnbaum (2010) uses the metaphor of a detective when observing a crime scene. The detective will see a lot more details than an ordinary person. He also claims that the art of re-seeing must be taught. According to Arcavi (2003) visualization must become more visible in the teaching of mathematics. He discusses mathematical visualization in a more figurative and deeper sense, as *seeing the unseen*, not only what comes *within sight* but also what we are unable to see. It becomes a tool for students to learn mathematics (Vale, Pimentel & Barbosa, 2018). Using photos provides opportunities to use real world as a starting point to develop mathematical eye and build mathematical problems, affording teachers with knowledge about students' visual attention. Furthermore, according to Meier et al. (2018) the use of photos motivates students, increases creativity and provides that “everyday life outdoors and science/mathematics can be connected in a meaningful way through the experience of photography” (p.147).

METHODOLOGY AND SOME PRELIMINARY RESULTS

An exploratory qualitative methodology (Erickson, 1986) was adopted with a group of 13 elementary preservice teachers of a teacher training course conducted in a school of education in a Didactics of Mathematics curricular unit. Throughout the classes these pre-service teachers were provided with diversified experiences, distributed in curricular modules, focusing on problem posing and solving. We privileged learning outside the classroom, creativity, and the establishment of connections, particularly between mathematics and daily life. The preservice teachers were asked, in pairs to:

1) explore the surroundings, taking a city tour analyzing the rich architectonical urban area, where they had to capture, with their mobile phone camera, a set of life shots with potential to formulate mathematical tasks;
2) choose some of the photos. The choices resulted from the analysis of the mathematics underlying each photo and the group discussion;
3) formulate tasks and present the respective solutions. In order to create a task using photos we used the respective digital image, applying the *accepting the given* problem posing strategy (Brown & Walter, 2005) and then the future teachers used *method 1/method 2* to pose questions (Sullivan & Liburn, 2002);
4) create a poster including the photo, the formulated tasks and their solutions;

5) present, discuss and assess the posters by all students who participated in this experience, using an assessment grid that focused on the assessment of the tasks and the poster in global terms. The future teachers also made a written report describing their reaction to the experience. Figure 1 illustrates some of the different moments of this activity.

Figure 1: Examples of the different moments of the activity

Data was collected in a holistic and interpretive way, including observations of the whole experience, the set of photos chosen, the written reports (describing their reaction throughout the different phases of the experience, including how they chose and created the tasks) and the assessment grid applied to the posters. Data was crossed and analyzed in an inductive way, according to the nature of the data and the research questions. Thus, we organized the analysis according to the following categories: features of the environment and photos; problem posing and its difficulties; reactions to the experience.

The photos chosen by the teachers showed that their gaze focused on elements such as buildings/facades, flower boxes and prices. The choice of photos was based on "possibilities for good questions", as assumed by the participants. Only one group sought for photos that fit what they had already thought to propose. Based on the photos, they managed to build proposals suitable to the contents already in mind, being able to naturally highlight connections between mathematics and the environment. The participants supported this selection by referring to the mathematical content suggested by the captured images. They mentioned that these photos were the ones that most inspired them to formulate the tasks. The objects of reality were transformed into mathematical objects, having aroused, for the most part, the mobilization of contents in Geometry and Measurement, followed by Numbers and Operations. The level of demand of most of the tasks was of low level, using the application of basic concepts and procedures. For example, Figure 2 shows two of the tasks created by the participants. We consider that the first task has a low level of demand and can be solved without the solver being present on the spot. The other task has a high level of demand and the solver needs to be on the spot to collect the necessary data to solve the task.

	Margarida received money for her birthday totalizing €100,40. With this money, she		Watch the Avenue closely. For the Medieval Fair, the Avenue will be decorated with ribbons of

decided to go shopping at a fashion store that had the price table shown in the image. Help Margarida decide what she can buy in the store with the money she received.	colored handkerchiefs placed in a zigzag pattern supported on the lamps along the Avenue (1 ribbon for every two lamps). How many ribbons will it take to decorate all the lamps?

Figure 2: Examples of two tasks

A group of students choose a photo of a flower pot and proposed a routine problem of geometry (Figure 3). However, despite of having the opportunity to contact with the object in the real context they used unrealistic data, when formulating the task: they set the radius of the flowerpot to 3 meters, but the real measurement was about 20 cm. It would be more interesting if the solver had to actually do the measurement on the spot, instead of accessing the data through the task.

	The gardener wants to build a fence around the flower pot. Find out the minimum length of the fence to be placed around it, knowing that the radius of the pot has 3 meters length.

Figure 3: Example of a task with unrealistic data

All these future teachers were engaged in this experience; however, they said that the formulation of the tasks was not an easy process, mainly in diversifying the nature and the contents of the tasks. They also stated that they will use photography with their own elementary school pupils.

SOME CONCLUSIONS

We synthesize the main ideas taking into consideration the research questions that guided the study, and the data that emerged from the empirical work. The main features of the environment privileged by the preservice teachers' mathematical eye were buildings. The architectural details caught their attention in terms of possibility for mathematical exploration (Barbosa & Vale, 2018). The choice of photos was mainly based on the possibilities for good questions (Bragg & Nicol, 2011). The use of photos as a means for promoting mathematics learning had a positive impact on students, providing a "closer look" at everyday objects, looking for the underlying mathematics in a more conscious and intentional way (Meier et al., 2019; Vale & Barbosa, 2019). Task formulation was not an easy process for the future teachers, which can be explained by the fact that they did not have much experience with task design. This was one of the reasons for the expression of difficulties in going beyond the problems of direct application, formulating tasks that lacked originality. This implies a regular work so that there is a positive impact on the quality of the proposals. In agreement with Barnbaum's (2010) ideas, the more knowledge, training and experience we have, the more detailed and deeper the mathematical eye will be. We however observed that some of the students were able to formulate challenging tasks. All managed to build proposals with suitable contents for elementary school students, being able to naturally highlight connections between mathematics and the environment. Environment engages students to capture photos that inspire them in different ways influencing students' motivation for learning in the extent to which they relate their school learning to their daily life (Gutstein, 2006).

This study adds to our understanding that outdoor photography can help students in task design as a significant aspect of mathematics curriculum and of our practice as teacher educators. But we need more studies to help us how to include an instruction for (preservice) teachers to develop their mathematical eye as well as to create rich tasks to be proposed to their pupils in the scope of outdoor mathematics education. We believe, as Bragg and Nicol (2011), that through creating problem photos, teachers and students will see mathematics through a new lens.

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REFERENCES

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215-241.

Barbosa, A. & Vale, I. (2018). Math trails: a resource for teaching and learning. In V. Gitirana, T. Miyakawa, M. Rafalska, S. Soury-Lavergne, & L., Trouche (Eds.) *Proceedings of the Re(s)sources 2018 International Conference* (pp. 183-186). Lyon: ENS de Lyon.

Barnbaum, B. (2010). *The art of photography: An approach to personal expression*. San Rafael, CA: Rockynook.

Bragg, L. & Nicol, C. (2011). Seeing mathematics through a new lens: using photos in the mathematics classroom, *Australian Mathematics Teacher*, *67*(3), 3-9.

Brown, S. & Walter, M. (2005). *The art of problem posing*. Mahwah, NJ: Erlbaum.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York, NY: Macmillan.

Godfrey, C. (1910). The Board of Education Circular on the Teaching of Geometry, *Mathematical Gazette*, *5*, 195-200.

Gutstein, E. (2006). *Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice*. New York: Routledge.

Kenderov, P., Rejali, A., Bartolini Bussi, M., Pandelieva, V., Richter, K., Maschietto, M., Kadijevich, D., & Taylor, P. (2009). Challenges Beyond the Classroom—Sources and Organizational Issues. In E. Barbeau & P. Taylor (Eds.), *Challenging*

Mathematics In and Beyond the Classroom – New ICMI Study Series 12 (pp. 53-96). Springer.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129-145). Rotterdam, Netherlands: Sense Publishers.

Meier, A., Hannula, M. & Toivanen, M. (2018). Mathematics and outdoor photography experience – exploration of an approach to mathematical education, based on the theory of Dewey's aesthetics, LUMAT. *International Journal on Math, Science and Technology Education, 8*(2), 146-166.

Munakata, M. & Vaidya, A. (2012). Encouraging creativity in mathematics and science through photography. *Teaching Mathematics and its Applications, 31*, 121-132.

National Council of Teachers of Mathematics (2014). *Principles To Actions: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 3*, 75-80.

Smith, M. & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions.* Thousand Oaks, CA: Corwin Press.

Stoyanova, E. (1998). Problem posing in mathematics classrooms. In A. McIntosh & N. Ellerton (Eds.), *Research in Mathematics Education: a contemporary perspective* (pp. 164-185). Edith Cowan University: MASTEC.

Sullivan, P. & Liburn, P. (2002). *Good questions for math teaching*. Sausalito, CA: Math Solutions publications.

Vale, I. & Barbosa, A. (2019). A fotografia na aula de matemática: uma experiência promotora de conexões [Photography in math class: an experience that promotes connections]. In N. Amado, A.P. Canavarro, S. Carreira, R. Tomás Ferreira, I. Vale (Eds), *Livro de Atas do EIEM 2019* (pp. 183-186). Lisboa: SPIEM.

Vale, I., Pimentel, T. & Barbosa, A. (2018). The power of seeing in problem solving and creativity: an issue under discussion. In N. Amado, S. Carreira, K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect* (pp. 243-272). Cham, CH: Springer.

‘Kombi’ – A digital tool for solving combinatorial counting problems: theoretical funding of and empirical results on central design principles

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In the topic-specific Didactical Design Research project ‘PAZ-digital’, the tablet app ‘Kombi’ is being developed. ‘Kombi’ is a digital tool for iOS and Android tablets to solve combinatorial counting problems in primary and lower secondary school. It aims at supporting the development of sustainable structuring and counting strategies. This article outlines the project, discusses the challenges of solving counting problems analogously and presents digital potentials to overcome these. Moreover, it outlines basic design principles of ‘Kombi’. To determine to what extent the implemented principles and potentials are targeted, we carried out an exploratory interview study with experts and students. The empirical results of the 1st design cycle are presented and discussed with a view to the further app development.

Keywords: combinatorics, design principles, digital tool, tablet app, development.

INTRODUCTION AND RELEVANCE

One of the four main themes at the ERME Topic Conference on MEDA refers to issues related to task design in the digital age. This paper is a contribution to this theme as it focusses on the development and evaluation of a digital tool, designed for primary and early secondary school in order to solve combinatorial counting problems. The need for developing such a tool is based on theoretical and empirical requirements and challenges in solving combinatorial counting problems as well as special potentials of digital media, which – in our opinion – can counteract these challenges.

THEORETICAL BACKGROUND

Combinatorics and the development of a conceptual understanding

The field of combinatorics is of great importance not only in the context of probability, but also in computer science. It deals with combining elements into (new) objects: “The aim of combinatorial questions is to determine all permissible combinations […] and their quantity” (Höveler, 2018, p. 82) in an economical way. However, various studies have shown that solving combinatorial counting problems is particularly difficult for learners of different ages (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; English, 2007), supposedly due to a lack of conceptual understanding (Lockwood, 2014).

From a mathematical perspective, it is possible to use three different approaches to solve combinatorial counting problems (1) systematic listing, (2) counting principles and (3) combinatorial operations (Höveler, 2018). In order to develop a conceptual understanding, a propaedeutic thematization in primary school is recommended, which focusses on the structured notation of the objects (Höveler, 2014), referred to as ‘set of outcomes’ (e.g., Lockwood, 2014). Such a “set-oriented perspective [which focuses on the specific set of objects and its structuring] may help students to find more meaningful ways of understanding and articulating issues that arise as they solve counting problems” (Lockwood, 2014, p. 36). Therefore, Lockwood (2014, p. 36) highlights the need for teachers to be “more conceptually and less procedurally focused”, and to relate the set of outcomes with possible counting strategies. Höveler (2018) demands a focus on this relationship already in primary school.

Challenges in the development of a combinatorial understanding and counteracting them with digital potentials

With regard to the focus on the development of a conceptual combinatorial understanding in primary school, organizational and content-specific challenges were identified. When considering the challenges, the potential of digital media quickly became apparent (Höveler & Winzen, accepted).

In order to develop a set-oriented thinking, the structured listing of the set of outcomes is essential but usually requires *time and flexible materials*: Studies with elementary school students show that after an initial non-structured approach, children become more structured in the process (e.g., English, 2007; Höveler, 2014). In this context, English (2007) indicates the need of hands-on, moveable materials. In her study with 7- to 12-year-old children, the students who had access to these materials, “were able to develop and modify their solution strategies, detect and correct their errors, and develop generative procedures on their own” (English, 2007, p. 154). Working with movable materials seems to be more helpful than making notations on a sheet of paper. However, due to the combinatorial explosion (e.g. Höveler & Winzen, accepted), providing hands-on movable materials poses a particular challenge for regular classroom activities (Winzen & Höveler, 2020). The digital potential ‘dynamization and flexibility’ (e.g., Clements & McMillen, 1996) can be a possible solution as it offers to change arrangements of objects or their representation easily.

Furthermore, we identified a *lack of fitting between mental operations described by the learners and the possible analogue actions* (Höveler & Winzen, accepted): For example, learners verbalize the idea of the strategy ‘exchange pairs’ which is characterized by duplicating all created objects and changing the arrangement of the elements in the new objects (Höveler, 2014). Others pursue the idea of the ‘odometer strategy’ (e.g., English, 2007), where in a first step a constant “item is repeatedly selected until all possible combinations containing that item have been formed. [In a second step on] […] exhaustion of this item, a new constant item is chosen and the process repeated” (English, 2007, p. 145). In reality, however, in both strategies each object has to be created individually. The potential ‘fitting between virtual representations and mathematical ideas’ (e.g., Schulz & Walter, 2019) may help to overcome this obstacle as using digital tools may offer new actions which can lead to a better fitting between mental operation and action (Clements & McMillen, 1996). Furthermore, during the problem-solving process children articulate the use of a systematic strategy, but in some cases forget to create all possible objects with this strategy. The digital potential 'offloading' (e.g., Schulz & Walter, 2019) may help to reduce the extraneous cognitive load: Since appropriate functions are embedded within the app, the learners no longer need to create each outcome individually.

The *joint exchange about strategies and their viability* also proved to be a great challenge: Besides the missing possibility to restructure objects in order to modify strategies, the *volatility of the problem-solving process* is problematic so that the procedures are often difficult to understand (Huhmann, Höveler & Eilerts, 2019). The digital potential 'documentation process' (e.g., Parnell & Bartlett, 2012), for example by video or screen recording, may help to overcome this problem. In addition, the digital storage of different solution strategies can simplify the structural comparison and also offers to use these solutions again in the later learning process.

THE PROJECT 'PAZ-DIGITAL'

Aims and research framework

For the reasons given above, the 'PAZ-digital' project aims at developing and researching a subject-appropriate software for solving combinatorial counting problems, namely the tablet-app 'Kombi'. This software should offer different functions in three special areas: Firstly, the individual virtual solution of counting problems in a *play space* (Wollring, 2006). This play space offers to create and structure sets of outcomes spatially on the basis of different elements. A distinction is made between a flexible, virtual manipulative, named *free play space*, which does not provide specific tasks and a *task-based play space* which offers specified counting problems. Secondly, a virtually shareable *document space* (Wollring, 2006) allows sharing and working with sets of outcomes from other students in order to facilitate the joint exchange about structuring and counting strategies and the development of sustainable strategies in general. Thirdly, the central findings should be screened and sound recorded in an additional *research diary* and could be taken up again in later teaching sequences. The latter will prospectively counteract the volatility of the problem-solving process and enable a better exchange about the solution strategies.

To develop the 'Kombi' app, the project follows the approach of topic-specific Didactical Design Research (e.g., Prediger & Zwetzschler, 2013), which cyclically interlocks processes of development and research. During the cyclic development process there are different priorities: In the 1st and 2nd cycle the focus is on the development of the *play space*, whereas in the 3rd and 4th cycle, the *document space* and the *research diary* are worked on primarily. This article focuses on the 1st design cycle and therefore presents the theoretically and empirically based development of the *(free) play space* and the already implemented design principles.

Functionality of the free play space and implemented design principles in the 1st design cycle

In the current version of the *free play space*, two design principles have already been implemented: *dynamization and interactivity of actions* (design principle 1) and *strategy macros* (design principle 2). In the following we will outline how the *free play space* can be used to solve combinatorial counting problems in three steps to explain these already implemented design principles (Figure 1). We use the block tower problem ("Here are three blocks in different colors. How many two-tall towers can you build?") – a typical combinatorial problem in elementary school – as an example.

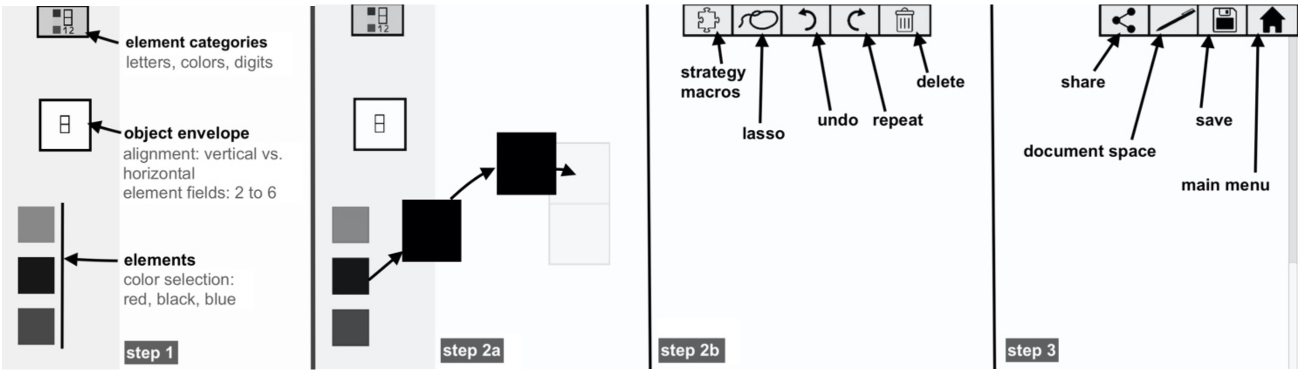

Figure 1: Solving a combinatorial problem in three steps in the free play space

Step 1: Selection of task variables

The challenge of *time and flexible materials* is faced with the implementation of design principle 1 and the digital potential 'dynamization and flexibility'. A part of this design principle becomes apparent in the first step: In the *free play space* the player can choose between three available element categories to create objects depending on the given task (Figure 1, step 1, light gray field). For the given task we pick the element category 'colors'. In the first place, the height and the alignment of the object envelope need to be set. It is currently possible to select between a horizontal and a vertical alignment of the object envelopes and between two to six element fields. For the two-tall towers, we pick a vertical orientation and the height two. Finally, it needs to be chosen how many and which elements out of the previously chosen element category are given to create the objects; in this case three different colors of 14 possible colors were picked.

Step 2: Creation and structuring of the objects

In the second step the colored two-tall towers can be created by pulling the object envelope into the white surface and filling it with selected elements; for instance, with the color black (Figure 1, step 2a). In this step the potential 'dynamization and flexibility' is considered: Like blocks on a table, the created objects can be moved freely on the surface. Furthermore, they can be reordered, shifted and deleted individually or as a group by drawing a lasso around several objects (Figure 1, step 2b). For better visibility, a zoom option is planned.

Furthermore, *strategy macros* (design principle 2) can be used. With their implementation we aim to overcome the *missing fit between mental operations described by the learners and the appropriate possible material actions*. A basic and already existing strategy macro is the 'duplicating of existing objects'. In the context of the block tower task, it is possible to duplicate a two-tall tower with a black block on the top as often as required, so that only the lower blocks need to be colored. This corresponds to a sub-step of the essential structuring strategy 'odometer strategy' (English, 2007). In this respect, the strategy macro considers the digital potentials 'fitting between virtual representations and mathematical ideas' and 'offloading', theoretically.

Step 3: Saving and sharing

In order to simplify the *joint exchange about strategies* the solution can be saved and in future also be recorded, shared with other app users or be analyzed and reflected in the *document space* (Figure 1, step 3).

Research questions and methodology of the 1st design cycle

The main goals of the empirical study in the 1st cycle were to check the manageability of the basic functions already available in the *free play space* and to ascertain the test subjects' wishes to expand the *strategy macros*. The following research questions were leading:

(1) What potentials and challenges does the current app version show when experts and learners are dealing with the basic functions? (2) Which additional functions are desired by experts in mathematics education and third graders within the scope of the play space, especially with regard to the strategy macros?

The data collection was based on clinical, guideline-based interviews (e.g., Hunting, 1997) with a total of five experts in mathematics education and eight third graders. As experts are more elaborate when it comes to solving combinatorial problems, they were included in the study and it was expected that they could think about possible extensions of the app on a meta level. The third graders were important as subjects in order to see how intuitive the use of the app is for the target group. In the interviews, a guideline-based introduction to solving combinatorial counting problems was given first, followed by an exploration phase in which the functions of the app for creating and structuring the set of outcomes could be explored freely. In the third phase, experts and learners were requested to solve a specific task before they were finally asked to express their wishes for further developments of the app with a special focus on the *strategy macros*. For the data analysis, the videotaped interviews were analyzed by sequence analysis (Dinkelaker & Herrle, 2009): The test persons' ways of use were compared. Aspects which were identified in several interviews, were analyzed in more detail in order to draw conclusions for the further app development.

RESULTS

Design principle 1: Dynamization and interactivity of actions

Regarding the selection of elements and the creation of the objects, it should be noted that neither the experts nor the students had any difficulty in selecting the elements and the object envelopes, nor in describing their function. In contrast, putting together an object turned out to be a challenge for all interviewees: They tried to push individual elements together into an object within the surface. Without the interviewer's help, only a few of them discovered that they had to pull the previously created object envelope into the surface and then fill it with elements in order to create an object. This may be due to the fact that the students' analogue approach differs from this digital procedure: Firstly, there is no object envelope and secondly, elements are directly combined to form an object. Once the object envelope was in the surface, the interviewees could fill in the objects without any problems. With regard to creating the whole set of outcomes, functions such as deleting, repeating and undoing, and changing an element in an existing outcome were used intuitively by everyone. Other functions needed a brief introduction: In order to perform actions like the above on several objects, they had to be circled with the lasso first. It became apparent that the interviewees used the simple option of dynamic arrangement for restructuring and often showed an increasing systematization in their own procedures. During the interviews, additional requests for further development became apparent: zooming in and out, alignment of the objects to a grid in the sense of a structuring and simultaneous addition of several object envelopes. It became apparent that the digital potential 'dynamization and flexibility' has been implemented well so far, but further development is necessary to ensure flexibility in solving the problems.

Design principle 2: Strategy macros

The results regarding the *strategy macros* show that the imaginable digital potentials have not been fully exploited yet: Results with regard to the 'duplicating of existing objects' function, which was found to be theoretically helpful, were diverse. On the one hand, three of the five experts used the duplicate function, reasoning that duplication is strategically wise, because it can outsource actions such as the repeated, individual creation of objects. This justification coincides with our theoretically assumed potential. Yet on the other hand, the experts pointed to a hurdle of the current app version: When duplicating, the repeated pasting of the duplicated objects is not possible so far. This means that the duplicating process must be carried out repeatedly to implement the digital potentials 'fitting between virtual representations and mathematical ideas' and 'offloading'. Furthermore, two experts pointed out that duplicating in combination with swapping the arrangement in the objects would be a helpful function. This procedure corresponds with the already mentioned sub-strategy of forming 'exchange pairs' and represents the idea of permuting objects. As three children and another two experts tried intuitively to swap the arrangement by twirling fingers, this function shall be implemented as a new strategy macro.

CONCLUSION

The results of the empirical investigation show that the current version of the *free play space* can largely be used intuitively by experts and learners to solve combinatorial counting problems. Using the basic object envelopes and selecting elements seems to be self-explanatory. However, the creation of an object is not. When using the app in class, it is therefore essential to discuss this function. Design principle 1 *dynamization and interactivity of actions*, in particular the options of shifting, deleting and the lasso function, had a significant contribution to the increasingly structured approaches of the learners. In this respect, the current app version already contains significant advantages of analogue, moveable materials (cf. English, 2007) without including their additional challenges in everyday teaching (e.g., lack of availability). With regard to design principle 2 *strategy macros* there is a need for further development: Thus, the *strategy macros* are to be expanded by further functions which make it possible to carry out analogue (sub-)strategies such as 'exchange pairs' or the 'odometer strategy'. Concerning the further development of the app, the main focus is on the extension of the *strategy macros* in the *play space*. In addition, a first version of the *document space* and its functions is to be created, so that based on the sets of outcomes, counting strategies can be derived and additionally a set-oriented thinking can be developed. Regarding the basic development of apps and their use, two central conclusions can be derived: (1) A one-sided theory-based app development is not sufficient as only empirical surveys provide information about the actual use of theoretically worked out app potentials. (2) Virtual manipulatives (as well as analogue tools) are not self-explanatory. In this respect, they are 'learning material' and need to be addressed in classroom, before they can be used as 'learning aid' (Schipper, 2009).

REFERENCES

Batanero, C., Navarro-Pelayo, V., & Godino, J.D. (1997). Effect of the Implicit Combinatorial Model on Combinatorial Reasoning in Secondary School Pupils. *Educational Studies in Mathematics*, *32*(2), 181–199.

Clements, D.H., & McMillen, S. (1996). Rethinking concrete manipulatives. *Teaching Children Mathematics, 2*(5), 270–279.

Dinkelaker, J., & Herrle, M. (2009). *Erziehungswissenschaftliche Videographie: Eine Einführung*. Wiesbaden: VS.

English, L.D. (2007). Children's strategies for solving two- and three-dimensional combinatorial problems. In G.C. Leder, & H. Forgasz (Eds.), *Stepping Stones for the 21st Century. Australasian Mathematics Education Research* (pp. 139–156). Rotterdam: Sense Publishers.

Höveler, K. (2014). *Das Lösen kombinatorischer Anzahlbestimmungsprobleme: Eine Untersuchung zu den Strukturierungs- und Zählstrategien von Drittklässlern* (Doctoral Dissertation). Retrieved from http://dx.doi.org/10.17877/DE290R-15563

Höveler, K. (2018). Children’s Combinatorial Counting Strategies and their Relationship to Conventional Mathematical Counting Principles. In E.W. Hart, & J. Sandefur (Eds.), *Teaching and Learning Discrete Mathematics Worldwide: Curriculum and Research* (pp. 81–92). Springer International Publishing.

Höveler, K., & Winzen, J. (accepted). Design-Prinzipien zur Entwicklung eines digitalen Arbeitsmittels zur Kombinatorik. In *Beiträge zum Mathematikunterricht 2020*. Münster: WTM-Verlag.

Huhmann, T., Höveler. K., & Eilerts, K (2019). Counteracting representational volatility in geometry class through use of a digital aid: results form a qualitative comparative study on the use of pentomino learning environments in primary school. In A. Novotná, & H. Moraová (Eds.) *International Symposium Elementary Maths Teaching: SEMT’19: Opportunities in Learning and Teaching Elementary Mathematics* (pp. 211–220). Prague: Charles University, Faculty of Education.

Hunting, R.P. (1997). Clinical Interview Methods in Mathematics Education Research and Practice. *Journal of Mathematical Behavior*, *16*(2), 145–165.

Lockwood, E. (2014). A Set-Oriented Perspective on Solving Counting Problems. *For the Learning of Mathematics, 34*(2), 31–37.

Parnell, W.A., & Bartlett, J. (2012). iDocument: How Smartphones and Tablets are Changing Documentation in Preschool and Primary Classrooms. *Young Children, 67*(3), 50–59.

Prediger, S., & Zwetzschler, L. (2013). Topic-specific design research with a focus on learning processes: The case of understanding algebraic equivalence in grade 8. In T. Plomp, & N. Nieveen (Eds.), *Educational design research. Part B: Illustrative cases* (pp. 407–424). SLO.

Schipper, W. (2009). *Handbuch für den Mathematikunterricht an Grundschulen*. Braunschweig: Schroedel.

Schulz, A., & Walter, D. (2019). ‘Practicing place value’: How children interpret and use virtual representations and features. In U.T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the CERME 11* (pp. 2941–2948). Utrecht: Utrecht University and ERME.

Winzen, J. & Höveler, K. (2020). Die iOS App „Notizen“: Ein digitales Arbeitsmittel zum Lösen von Kombinatorikaufgaben. *Die Grundschulzeitschrift*, *320*, 28–32.

Wollring, B. (2006). Zur Kennzeichnung von Lernumgebungen für den Mathematikunterricht in der Grundschule. In Kasseler Forschergruppe (Eds.), Lernumgebungen auf dem Prüfstand. Zwischenergebnisse aus den Forschungsprojekten (pp. 9–26). Kassel: Kassel University Press.

Theme 2

Mathematics Curriculum Development and Task Design in the Digital Age

Posters

Smartphone math-apps in learning environments (SMiLE): a project focussing on the development and evaluation of teacher training concepts

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THE PROJECT SMiLE

The poster provides a brief overview of the project structure. General facts include:

- SMiLE is a part of the so-called 'Global Teacher Research and Education Exchange Program Passau' funded by the German Academic Exchange Service (DAAD program: "Lehramt.International").
- Core of the project SMiLE: Collaboration with the Bavarian Ministry of Education on their ongoing project 'CAS in Exams'.
- Main goal of SMiLE: To develop and to evaluate teacher training sessions on how to use GeoGebra Apps in math classes in Bavarian high schools.
- Further project participants of SMiLE: University of Pedagogical Sciences (UCPEJV), Havana, Cuba and Södra Latins Gymnasium, Stockholm, Sweden.

THE PROJECT CAS IN EXAMS

The 'CAS in Exam' project is carried out in several Bavarian high schools (Gymnasium) by the Bavarian Ministry of Education. Students of the participating schools are allowed to use GeoGebra Apps on mobile devices in the so called 'Exam Mode' in exams from the 8th grade onward. The aim is to officially allow GeoGebra Apps on mobile devices in combination with the 'Exam Mode' in exams in all Bavarian high schools. The Professorship for Didactics of Mathematics at the University of Passau is in charge of the scientific supervision of this project. As part of the project SMiLE, appropriate teacher training courses are to be developed to facilitate the introduction of GeoGebra on mobile devices in exams across Bavarian high schools. The objectives of the scientific supervision are amongst others:

- To identify key elements for the training of students at university and further education of teachers.
- To develop a training concept to strengthen the use of GeoGebra in classrooms and thus to increase the appreciation of GeoGebra on mobile devices in mathematics.

THEORETICAL FRAMEWORK

The scientific basis for the project SMiLE is a design-based research (DBR) approach. Bakker (2014) describes a major aspect of DBR as follows:

> A key characteristic of DBR is that educational ideas for student or teacher learning are formulated in the design, but can be adjusted during the empirical testing of these ideas, for example if a design idea does not quite work as anticipated (p. 3).

The DBR approach fits the project structure because the adjustment and further development of teaching content during the project is essential. The research cycles are entered via two parallel designs. For this purpose, a workshop on the use of GeoGebra in exams and e-learning courses corresponding to the mathematics curriculum starting in 8th grade will be elaborated. The organisational structure for the implementation across Bavaria is provided by the Bavarian Teacher Training Academy. The poster presents first results and further steps in the project that serve as the basis for discussion and debate.

POSSIBLE IMPLICATION FOR RESEARCH IN THE AREA

As the use of Apps on mobile devices like smartphones and tablets in exams is new terrain for research, there is few literature on this specific topic. Nevertheless, the ICMI Study 17 (Hoyles & Lagrange, 2010) provides a summary of a range of efforts to examine the use of digital technologies (DT) in mathematics education. Referring to this study, Weigand (2014) concludes in his article "Looking back and ahead—didactical implications for the use of digital technologies in the next decade" that this study can be understood "as a request and as a challenge to develop new ideas—visions—in order to advance the integration of DT in mathematics education" (p. 4). Considering these results and results of various other studies on the use of digital technologies, the challenge that arises is to train teachers and students in a way that math classes can be meaningfully and sustainably enriched by the use of digital technologies. The employment of digital technologies can train a wide variety of student competencies. A prerequisite for the successful use of digital technologies is that students and teachers can rely on appropriate digital resources and training. Furthermore, teachers have to be trained in a way that the teacher-student interaction with curricular content is effectively guided. The project SMiLE seeks to contribute to this aim.

REFERENCES

Bakker, A. (2014). *An Introduction to Design-Based Research with an Example From Statistics Education.* New York: Springer. doi: 10.1007/978-94-017-9181-6_16

Hoyles, C. & Lagrange, J.-B. (Eds.). (2010). *Mathematics Education and Technology – Rethinking the Terrain. The 17th ICMI Study*. New York: Springer.

Weigand, H.-G. (2014). Looking back and ahead–didactical implications for the use of digital technologies in the next decade. *Teaching Mathematics and Its Applications*, *33(1),* 3-5. doi:10.1093/teamat/hru006

Fostering creativity through design of virtual and tangible manipulatives

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THEORETICAL GROUNDING AND CURRENT RESEARCH STATE

"How does creativity interact with expertise in problem solving and problem posing when mathematics meets real objects" has already been investigated (Singer & Voica, 2017, p. 75). According to Flores, Park, & Bernhardt (2018) "creativity is fostered, promoted and developed when […] learners pose and solve problems" also with the use of technology. Beyond problem solving, which we examine in a narrow and a wider sense by combining physical and digital manipulatives in (Donevska-Todorova, 2020; Donevska-Todorova & Lieban, 2020), the activities discussed in this work show new paths for students to become more creative and innovative mathematics learners and users. Recent studies (Lee & Carpenter, 2015; Leikin & Sriraman, 2017; Sánchez, Font, & Breda, 2019) call attention to the importance of nurturing creativity in school environments and the needs to introduce teaching practices that could foster creative processes in regular school activities. In addition, schools allow development of creativity skills for contributing to critical thinking, problem solving, autonomy and collaboration. Nevertheless, it seems that there are still only a few initiatives in teacher training programs related to fostering creativity and due to this there are numerous calls to develop such programs to change this situation (Sánchez, Font, & Breda, 2019). In their study with prospective teachers in Spain, the authors identified connections between the use of manipulatives and the development of creativity. The teachers explained activities where students' creativity could be fostered by the use of digital tools and other physical resources. However, pre-service teachers may not always recognise mathematical creativity, but a plastic or artistic creativity that students practice in making a certain object. Yet, even such exercises may contribute to involvement in a new mathematical activity later. This perspective reinforces the importance of introducing the use of manipulatives in digital and palpable formats and may further encourage students in designing their own pieces of artwork, logical games or two and three-dimensional visualizations and puzzles. Creativity in that sense relates to guided creations from scratch or adapting and redesigning existing materials by bringing new ideas into play. Regardless of where students obtain the inspiration from, they can use mathematical background knowledge and plenty of integrated skills for their brainstorm processes, constructions and designs.

RESEARCH QUESTION AND METHODOLOGY

Our research calls the fostering of creativity in mathematics education through the design of tangible and virtual manipulatives into question and we initially approach it through a first cycle of design research (Kelly, Lesh, & Baek, 2008).

RESULTS AND FURTHER PERSPECTIVES

As designing and evaluating resources is one of the themes of the ERME Topic Conference MEDA 2020, the poster shows authentic designs of virtual DGS and tangible 3D printed manipulatives that can support development of creative and divergent mathematical thinking during posing and solving construction problems, and tessellations offering a stable base for further discussions.

REFERENCES

Donevska-Todorova, A. & Lieban, D. (2020). Fostering Heuristic Strategies in Mathematical Problem Solving with Virtual and Tangible Manipulatives. In the Proceedings of the tenth ERME TC Mathematica Education in the Digital Age (MEDA) 2020.

Donevska-Todorova, A. (2020). Problemorientiertes Lernen zur Leitidee Raum und Form mit Hilfe von DGS und 3D Druck Technologie. *Beiträge zum Mathematikunterricht. Vorträge auf den 54. Jahrestagung der Gesellschaft für Didaktik der Mathematik* vom 09. bis 13. März 2020 in Würzburg.

Flores A., Park J., & Bernhardt S.A. (2018). Interactive Technology to Foster Creativity in Future Mathematics Teachers. In: Freiman V., Tassell J. (eds) *Creativity and Technology in Mathematics Education. Mathematics Education in the Digital Era,* vol 10. Springer, Cham.

Lee, S. & Carpenter, R. (2015). Creative Thinking for 21st Century Composing Practices. Across the Disciplines, 12.

Leikin, R. & Sriraman, B. (2017). *Creativity and giftedness: interdisciplinary perspectives from mathematics and beyond.* New York, NY: Springer Berlin Heidelberg.

Kelly, A. E., Lesh, R. A., & Baek, J. Y. (2008). *Handbook of design research methods in education. Innovations in Science, Technology, Engineering and Mathematics Learning and Teaching,* Routledge Taylor & Francis Group, the United States of America.

Sánchez, A., Font, V. & Breda A. (2019). Secondary school preservice teachers' references to the promotion of creativity in their master's degree final projects, 8. Proceedings of CERME 11.

Singer, F. M., & Voica, C. (2017). When Mathematics Meets Real Objects: How Does Creativity Interact with Expertise in Problem Solving and Posing?. In *Creativity and Giftedness* (pp. 75-103). Springer, Cham.

A conceptual approach on mingling Augmented Reality, 3D printing and ancient architectural modelling using GeoGebra

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In this poster we are introducing a hybrid concept of new technologies and investigating its effects on teachers/students learning while applying them on ancient architectural models. We are addressing technologies as augmented reality, 3D printing/scanning and 3D modelling using GeoGebra.

Keywords: data transformation, augmented reality, 3D scanning/printing, architecture modelling, mathematical education.

INTRODUCTION

In this conceptual poster the main idea is to mingle most recent technologies to help on reflecting on ancient architectures to interpret mathematical concepts during the modelling process. As (Rossi, 2018) shows in her articles the clear relation of ancient architectures and mathematics as proved in the ancient Egyptian architecture. Most of the ancient architecture were based on mathematical concepts although the theories and rules applied belong to an old version of mathematical knowledge and in our modern mathematics it is different. But by this study we are bridging between past mathematical theories and trying to apply it on today's modern mathematical concepts. This is by replicating, enhancing or even reconstructing ancient architecture. Ancient Architecture is the only link we can consider reliable to understand the past mathematical and geometrical theories accompanied by ancient documents. As for the ancient Egyptian architecture we found the Rhind Mathematical Papyrus which was dated from around 1650 B.C., which prove and elaborate their mathematical knowledge. We are adopting new technologies to bridge the gap between past and present with the teachers and the students. We are applying data transformation to show the models in many forms, digital by modelling using GeoGebra, augmented reality, 3D scanning and physical by using the 3D printing technologies. Now we will take some samples and findings from other researchers showing the importance of using the technologies we proposed and their impact on mathematical studies. The importance of 3D scanning comes with many promising technologies as laser or structured light scanners, but they are expensive so other alternatives for this could be computational photography and photogrammetry techniques. They provide 3D models from real existing models by using dense images combined together and captured from different angles as elaborated by (Martínez-sevilla et al., 2018). These technologies can be applied to architectural models resulting in 3D representations which can help the students to reflect on their mathematical understandings. Moreover, combining augmented reality with 3D scanning in the digital form or relying on the AR GeoGebra option will give students the opportunity to experience more technologies. AR has an impact in the science classroom as shown in the review of (Gopalan et al., 2018) they found that instead of using text and images only, the assistance of audio or three-dimensional models as in AR will give students a better understanding. 3D Printing is a potential technology in education as(Szulżyk-Cieplak et al., 2014) believe that 3D printers have direct effect on the processes of teaching. As printed physical models help in delivering to the students a better understanding of the creation process. They believe that it enhances the students' involvement during the classrooms and that it gives them an edge in transferring their ideas into reality. As all these practices aim is to increase and foster the creative thinking of both teachers and students in finding and expressing their mathematical knowledge using different technologies and applying them to architecture. Lastly this will bring us to combing all these ideas in a research question that we will find an answer too during this research journey: What are the best practices that help teachers and students in modelling ancient architecture using AR,3D printing/ scanning and using GeoGebra? what are the practices direct/ indirect impact on their mathematical knowledge?

FUTURE WORK

As a future step to this emerging research ideas is to develop methods for the teachers and students in the classroom and playing around with these various technologies presented in this research. The data that will be collected at each stage of the research using different data collection methods like interviews, insights and surveys will all end up in formulating the methodology of this research. And the methodology findings at each stage of the project will help in formulating the project stages and paths as well as giving answers to the research question.

REFERENCES

Gopalan, V., Bakar, J. A. A., Zulkifli, A. N., & Alwi, A. (2018). A review of augmented reality elements in science learning. *Journal of Telecommunication, Electronic and Computer Engineering*, *10*(1–10), 87–92.

Martínez-sevilla, Á., Ureña, C., & Recio, T. (2018). *Augmented Reality , Maths Walks and GeoGebra*.

Rossi, C. (2018). Egyptian Architecture and Mathematics. *Handbook of the Mathematics of the Arts and Sciences*, 1–12. https://doi.org/10.1007/978-3-319-70658-0_57-1

Szulżyk-Cieplak, J., Duda, A., & Sidor, B. (2014). 3D Printers – New Possibilities in Education. *Advances in Science and Technology Research Journal*, *8*(24), 96–101. https://doi.org/10.12913/22998624/575

Montessori materials created in the maker spirit

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We connect the principles of manipulatives in Montessori education to the maker movement. A manipulative is an object designed to let the learner perceive a mathematical concept by manipulating it. Existing examples were used, and students were encouraged to explore, digitalize, adapt, and produce them in makerspace-like facilities. This combines benefits from such manipulatives and the possibility to develop new versions of them with additional features such as being more inclusive.

Keywords: Manipulatives, Montessori, digital fabrication, maker movement.

MONTESSORI METHOD AND MAKER MOVEMENT

In classical Montessori education, manipulatives are often used to support constructivist reasoning. Many originate from teachers such as Montessori herself (Laski et al. 2015). While such manipulatives are usually created or prepared by teachers, it can have benefits to encourage students to recreate and adapt them. In our approach we suggest a combination between the creative spirit and the digital tools found in the maker movement and manipulatives developed for Montessori education. We describe a manipulatives-adaptation-lifecycle from an idea to a creation of a physical manipulative enriched with additional ideas and return to a digital representation. These recreated manipulatives should follow the Montessori spirit but can encourage students to add additional attributes and new ideas.

Manipulatives in Montessori education have general attributes such as defined by Laski et al. (2015). They found four principles: First, manipulatives should be sustainable. Second, a concrete and transparent representation of a concept should slowly move to abstraction over time. Third, manipulatives should differ from everyday objects and avoid distracting features. And fourth, connections between the manipulative and the mathematical concept should be explained to the students, for example, arranging triangles into a hexagon. A digital preparation on, for example, GeoGebra and a digital production such as laser cutting, 3D printing, or CNC milling lets students discover intrinsically how mathematics and a manipulative are connected. We assume that a digital component can foster the other three principles.

Schools often lack the possibility to acquire and maintain machines for digital production. The maker movement offers open makerspaces using free and open source software and machines to encourage and enable as many people as possible including students to produce their own ideas. Usually makerspaces provide digital production technologies.

DO-IT-YOURSELF OPEN RESSOURCES MONTESSORI

In a do-it-yourself spirit in combination with makerspaces, the maker movement allows students to develop and produce their own artefacts. In 2020, over 1700 FabLabs and even more maker- and hackerspaces exist worldwide. They give support for teachers and workshops especially designed for children. Using digital fabrication in makerspaces, Montessori puzzles and manipulatives can be turned into computer games. With open source plans, they can be recreated as classic manipulatives. For example, in the Montessori four colors game, students have to move around color plates to the designated position that corresponds to the assigned game card. The process of creating manipulatives complies with the aforementioned principles. If objects are created by students, they include their personal preferences and if created using the mentioned technologies, support a long lifespan. This leads to higher sustainability. Skills to identify connections between a mathematical concept and the manipulative are trained during the creation process.

Lutz created the above-mentioned Montessori four colors game using GeoGebra, which allows students to playfully find their first sorting algorithms. Campuzano, Lutz, and Lieban added downloadable SVG instructions for laser cutting to the game on the GeoGebra platform (https://www.geogebra.org/m/ddtky6by#chapter/502856). Lieban recreated physical versions of the manipulatives with his classes in Brazil to stress the physical and digital connection. In digital development, the main motivation was exploring Boolean logic for giving automated feedback on the material and introducing basic coding practices to math teachers. The games shown in the poster can be played digitally and downloaded to be created physically. When doing this, students are able to adapt materials towards inclusivity. So not only are materials more durable but also open to adaptation for a wider group of students.

OUTLOOK

A makerspace-like facility is currently set up in Lieban's school to investigate manageability of such maker-oriented workshops and we also intend to establish some abroad connections in the maker culture. Connecting schools with the maker movement will allow the creation of their own varieties of Montessori materials. Completely new manipulatives could be developed and tested against Montessori principles. Creating more materials on GeoGebra for students to explore and create them will be the next goal to gain more data for further research.

REFERENCES

GeoGebra. https://www.geogebra.org/.

Laski, E. V., Jor'dan, J. R., Daoust, C., & Murray, A. K. (2015). What makes mathematics manipulatives effective? Lessons from cognitive science and Montessori education. *SAGE Open*, *5*(2), 2158244015589588.

Towards implementing computational thinking in mathematics education in Austria

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INTRODUCTION

When Austria introduced the new mandatory subject "Digital Education" in September 2018, computational thinking finally made it into the curriculum. Schools can choose to offer specific subjects or dispense the content into already existing subjects where seen fit. Computational thinking (CT) tools and methods found their way into multiple scientific fields (Orton, et al., 2016) but overall, research shows that it is better to implement CT in other subjects than to teach it as a stand-alone subject because it tends to be separated from real-world problems (Weintrop, et al., 2015). Therefore, Weintrop et al. (2015) published a set of ten core CT skills, which we will refer to in this poster as computational thinking skill (CTS-#) 1 to 10 (see Table 1).

Set of computational thinking skills (CTS)	
- CTS-1: Ability to deal with open-ended problems - CTS-2: Persistence in working through challenging problems - CTS-3: Confidence in dealing with complexity - CTS-4: Representing ideas in computationally meaningful ways - CTS-5: Breaking down large problems into smaller problems	- CTS-6: Creating abstraction for aspects of problem at hand - CTS-7: Reframing problems into a recognizable problem - CTS-8: Assessing strengths/weaknesses of a representation of data/representational system - CTS-9: Generating algorithmic solutions - CTS-10: Recognizing and addressing ambiguity in algorithms

Table 1: Set of computational thinking skills (Weintrop, et al., 2015)

ANALYSIS

This poster presents our interpretation as to where Weintrop's 10 CTS can be implemented and trained in the subtopics of the current curriculum of Austria's lower secondary mathematics education. Topics in the Austrian mathematics curriculum in lower secondary education are divided into four areas: (1) working with numbers and units, (2) working with variables, (3) working with geometric shapes and bodies, and (4) working with models and statistics (RIS, 2020). There are 33 subtopics in grade 5, 25 in grade 6, 26 in grade 7, and 19 in grade 8, giving us a total of 103 subtopics.

CTS-1 could be found in solving and interpreting equations or formulas, comparing different models, working with linear functions, calculating approximations and bounds and in justifying the Pythagorean Theorem. In total we found 19 applications for this skill. As CTS-2 is a very opened skill, it is possible to match it to any topic of the curriculum. 28 topics could be matched to CTS-4, whereas most of them were found in grade five. Each topic in the mathematics curriculum of years five to eight deals with breaking down large problems into smaller ones. Therefore, CTS-5 is also one of the leading skills. On the contrary, there were only 7 topics that could be matched to the skill "creating abstraction for aspects of problem at hand". CTS-7 could be found in each lesson, as it represents one of the core aspects of mathematics, while CTS-8 is present only in the topic "working with statistics and models". Moreover, CTS-9 "generating algorithmic solutions" is part of an everyday mathematics lesson. Recognizing ambiguity in algorithms (CTS-10) is a skill that is not found very often in grades five to eight.

An interesting fact is that subtopics with CTS decrease from grade 5 (60%) to grade 8 (53%) but the overall implementation is very stable. It needs further investigation if this is related just to the topics or to the matching of CTS. Of course, the matching of CTS witch the single subtopics is subjective and needs further review of more than just three teachers.

CONCLUSION AND OUTLOOK

In this poster we have examined possible applications of CT in the current mathematics curriculum. We found out that already lots of CTS are implemented without adding extra content, whereas some of them are highly represented and some are very special and rare. In the upcoming months, we will concentrate on the further investigation of matching the CTS to the subtopics.

REFERENCES

Orton, K., Weintrop, D., Beheshti, E., Horn, M., Jona, K., & Wilensky, U. (2016). Bringing Computational Thinking Into High School Mathematics. In C.-K. Looi, J. Polman, U. Crest, & P. Reimann (Eds.), *Proceedings of the 12th International Conference of the Learning Sciences* (pp. 705-712). Singapore: International Society of the Learning Sciences.

RIS. (2020). *Rechtsinformationssystem des Bundes*. Bundesrecht konsolidiert: Lehrplan neue Mittelschule: https://www.ris.bka.gv.at/Dokumente/Bundesnormen/NOR40207228/NOR40207228.pdf

Weintrop, D., Behesti, E., Horn, M., Orton, K., Kemi, J., Trouille, L., & Wilensky, U. (2015). *Defining Computational Thinking for Mathematcis and Science Classrooms*. New York: Springer Science+Business Media.

Introduction to algebra via image processing

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Image processing is an interesting application of mathematical functions and can even be used as a context for the introduction of algebra. This poster introduces a webpage that can be used for image processing and algebra learning.

Keywords: image processing, algebra curriculum, variable, function.

CONVENTIONAL APPROACH TO ALGEBRA IN SCHOOL

There much more ways to introduce algebra to students than can be discussed in this poster. However, results are often far from optimal and a hypothesis of my project is that this is due to lack of motivation. The research question is thus: How to improve on this? One of the most prominent approaches uses pattern sequences (e.g. P. Drijvers: Secondary Algebra Education, chapter 4, Sense publishers) that yield number sequences that are described in general by an expression. This pattern based approach dominates e.g. in the papers submitted to the algebraic thinking TWG of CERME. Kohanová and Solstad (2019) report a study with Norwegian mathematics teacher students who showed low performance in the following task in Fig 1. Only 50% were able to produce a sensible algebraic expression for the general case in part d). My hypothetical explanation is that such linear pattern generalization tasks like this lack relevance and provide no intellectually challenging insights (e.g. how to derive a formula for 1+2+…+ n). In contrast, this poster introduces an approach based on the theory of constructionism (Papert) and abstraction in context (Hershkowitz et al. 2001) that has been shown to be highly motivating.

1. Monika began designing the pattern with short sticks. Each day she continues the pattern.

a) Describe how she would proceed to make her design on 5th day.

b) How many sticks will she need to make her design on 8th day?

c) How do you calculate how many sticks she will need to make her design on day number 100? Write down the corresponding formula.

d) Write down the general formula for total number of sticks she will need to make her design on day number *n*. Explain how you got the formula and how do you know that it is correct.

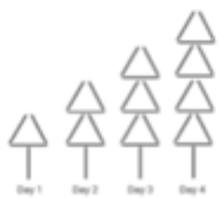

Fig. 1: Task from Kohanová and Solstad (2019)

IMAGE PROCESSIG APPLET

The idea of using image processing as a context for learning algebra has been developed in a sequence of German language papers starting with Oldenburg (2006), but it has never been exposed to the English-speaking didactics community. It allows tasks that are completely opposite to the figural pattern tasks, namely authentic, relevant, and aesthetically appealing. The idea of the applet (which is the most elementary one out of a sequence of applets) is very simple: On the left there is a grey scale image. Brightness of each pixel is encoded as a number from 0 (black) to 100 (white). A transformation can be achieved by changing the brightness according to a rule (function) specified by an algebraic expression in the variable b (for brightness). The resulting image (the image of the function, a nice language association) is shown on the right. Some transformations: b+20 (lighter), b/2 (reduce contrast), 100-b (negative), 10000/b (nonlinear negative).

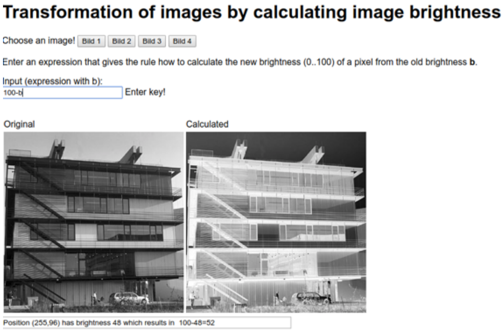

Fig. 2: The applet http://myweb.rz.uni-augsburg.de/~oldenbre/webBV/Statisch/onepix5_en.html

The didactical conception includes the following considerations: The introduction of an expression involving a symbolic variable solves a real problem (namely to tell the computer how to compute brightness for each pixel for a huge number of pixel), the calculation is transparent (it can be checked for each pixel), algebraic action gives attractive results, and the concept of function as a mapping is introduced. Moreover, the question is triggered, if different expressions can have the same effect. This puts expression equivalence in the reach of the teaching unit.

EXPERIENCE

The applet has been used in a limited number of classrooms and with a larger group of 5th graders (approx. 11 years old) on the occasion of public presentation of the university activities. Due to severe restrictions of data collection in public schools no evaluation in regular classrooms could be conducted (and this poster is part of the effort to collaborate on this project). Subjective impressions from the many occasions are extremely positive. Usually the whole episode takes 45 minutes and virtually all students engage in the activities and a lot of discussions between students is stirred up, e.g. if some expression did not show the effect they expected.

REFERENCES

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. Journal for Research in Mathematics Education 32, 195-222.

Kohanová, I., Solstad, T. (2019). Linear figural patterns as a teaching tool for preservice elementary teachers – the role of symbolic expressions. 11th CERME, Utrecht University, Utrecht, Netherlands.

Olenburg, R. (2006). Die Mathematik der Bildverarbeitung. In Meyer, J, and Oldenburg, R. (Eds.) Materialien für einen realitätsbezogenen Mathematikunterricht 9: Computer-Anwendungen. Franzbecker: Hildesheim.

Digitally increasing the qualitative understanding of the derivative

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Students difficulties with interpreting and using graphs in a meaningful way are well documented in the literature. In the Netherlands students are required to be able to sketch the graph of the derivative of a function given only the graph of that function. In this explorative study we investigate on the performance of a digital environment designed to enhance the qualitative understanding of the derivative.

Keywords: derivative, feedback, digital environment, learning with technology.

AIM AND RATIONAL OF THE STUDY

It is a known fact that students often struggle with making a sketch of a derivative in the absence of a formula that can be computed. This is also collaborated by Stahley (2011). Previous research (e.g. Sari, Hadiyan and Antari, 2018) shows that digital environments provide opportunities for students to explore graphs in a qualitative way.

In our research we investigate how students learn to sketch the graph of a derivative within a digital environment developed by Van der Hoek (2019). Furthermore, we investigate on a GeoGebra applet as a possible addition to the environment. We use an adapted version of the framework of Vos, Braber, Roorda, and Goedhart (2010) to investigate students' understanding of the derivative. Within this framework we distinguish five levels of qualitative understanding where a student exhibiting a certain level also possesses the previous levels: *no operable knowledge (L0), knowledge of some connection* between the slope of the function and the derivative *(L1), knowledge of the location* of points on the graph of the derivative relative to the horizontal axes *(L2), understanding the derivative as slope* that is, the derivative represents the course of the slope of a graph *(L3), understanding the derivative as gradient*, that is a point on the graph of the derivative represents the gradient of a tangent *(L4)*. In this paper we present the results of an explorative study which involved 4 students.

THE DIGITAL ENVIRONMENT AND GEOGEBRA APPLET

The environment presents an animation video explaining the relation between a graph and its derivative followed by four tasks to sketch the derivative of a given graph digitally. Feedback on the sketch can be obtained using a button. This feedback is designed to increase the level of understanding. Since it proves difficult to design feedback to increase to L4 we also developed a GeoGebra applet in which the gradient of a tangent is plotted as the student moves it along a graph.

METHODOLOGY

We used individual semi structured interviews to investigate students learning. Four students (age 16, 17) participated in this study. First, they were given a preexamination. Two of the students (students 1 and 2) then had the opportunity to practice using the digital environment while the other two students (3 and 4) practiced using a textbook. Directly after that, a post-examination was conducted. A third interview took place three weeks later. During the last interview all the students also interacted with the GeoGebra applet.

RESULTS

Analyses of students interviews shows that students 1 and 2 have risen two levels after three weeks. However, students 3 and 4 had the same rise in understanding. Furthermore, we found that student 3 achieved a rise from L3 to L4, the highest recorded level, after interacting with the GeoGebra applet.

PRELIMINARY CONCLUSIONS AND IMPLICATIONS

Based on these results we may carefully conclude that the contribution of the environment to the qualitative understanding is roughly the same as the contribution of a textbook. Which begs the question: What may we add to the environment to make a difference? Interacting with the GeoGebra worksheet did give student 3 a more profound understanding of the relationship between a function and its derivative. Why does this interaction lead to better understanding? And how may we implement this interaction in a digital learning environment? These are questions that we want to discuss at MEDA and incorporate in our future work

REFERENCES

Sari, P., Hadiyan, A., and Antari, D. (2018). Exploring derivatives by means of GeoGebra. *International Journal on Emerging Mathematics Education*, 2(1), 65-78.

Stahley, J. R. (2011). Students' Qualitative Understanding of the Relationship Between the Graph of a Function and the Graphs of Its Derivatives (Doctoral dissertation, University of Maine).

Van der Hoek, G. (2019). Hellinggrafiek schetsen. Wiswise, Retrieved from https://app.dwo.nl/vo/

Vos, P., Braber, N. D., Roorda, G., & Goedhart, M. J. (2010). Hoe begrijpen en gebruiken docenten van de schoolvakken natuurkunde, scheikunde en economie het wiskundige concept ‘afgeleide’. Tijdschrift voor Didactiek der Bètawetenschappen, 27, 37-62.

Theme 3
Assessment in Mathematics Education in the Digital Age

Papers

Evaluating educational standards using assessment "with" and "through" technology

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This paper reports on a feasibility study of creating a standardised assessment instrument to evaluate students' competencies found in the German national standards. The study aimed at combining widespread tools in math-classes, such as dynamic geometry and spreadsheets, in an integrated and computer-driven way. We report on the mathematical and technical feasibility: What limits were reached, and which opportunities have appeared? The report provides indications that a development process is feasible but that an attention to the task description is required, as the student may be unaware of the manipulations to perform tasks.

Keywords: assessment, educational standards, affordances, dynamic geometry, spreadsheets

INTRODUCTION

The possibilities of using digital media in (mathematical) learning processes increases more and more. Therefore, large-scale assessments have to be adapted to new ways of learning and to new competencies. The current technical state of the art is to focus on accepting a final answer from students (Pelkola, Rasila & Sangwin, 2017) though the collection of log-data is considered to be enhancing assessment.

In Germany a large-scale assessment called VERA is conducted based on a paper-pencil-test once a year in grade 8 (14 y old) (IQB, n.d.). Turning it into a digital test environment could bring many advantages: assessing mathematical digital competency (Csapo, Molnar & Toth, 2009; Geraniou & Jankvist, 2019), using rich and dynamic items, or the integration of automatic scoring (Drijvers, 2018). This paper is based on a first feasibility study with an evolved test instrument that combines key features of the existing paper-pencil-assessment with innovative ideas.

BACKGROUND

Standardized competency assessment is a form of testing that aims to measure the competencies reached by all testees in a comparable way. The results can inform on the attainment of teaching (as is the case of TIMSS or PISA studies, but also as any examination), on the general competency considered important (as is the case for the PIAAC study) or for other research purposes. Standardized assessment has often been made with paper and pencil and this remains the dominant practice for mathematics competency testing as the manipulation of mathematical objects on computers remains fragmented and isolated. Nevertheless, both theoretical considerations as well as studies demonstrate the strong potential of testing with a computer. This applies particularly to the higher objectivity of automatic scoring and the larger amount of information obtained through log-files (e.g. as described by Goldhammer & Zehner, 2017). Aspects such as the individual task solving behavior, the testee's self-confidence or the dependency on a particular expression media can be assessed. Among other aspects, this feasibility study aims at exploring how achievable it can be to obtain such extra information.

In mathematics education, assessment constitutes a large area. The notion of computer-based mathematical assessment becomes more commonplace by the fact that computers are often used to support mathematical learning activities. Stacey and Wiliam (2012) differentiate *assessment with digital technology*, where the computer-tools (even calculators) support mathematical processes but not assessment processes, from *assessment through digital technology*, where it is driven by computer activity.

Sangwin et al. (2009) propose ways to automatically evaluate certain answer types and create a competency model based on the use of particular assessment tools. Such methods are mostly applied with formative assessment in which results help to enhance the learner's competencies rapidly. In contrast, little best practice is known for summative evaluations of mathematical competencies, whereby the technical state of the art is to focus on accepting a final answer from students (Pelkola, Rasila & Sangwin, 2017). Fine-grained analysis on the manipulation of mathematical tools is an evolving domain: Multiple research around intelligent tutoring systems have given the rise to successful training tools such as the Algebra Tutor (Koedinger et al., 2008). They are largely specialized and almost impossible to integrate within a summative evaluation where the overarching goal is to evaluate the breadth of competencies and not to depend on specialized tools, each including specialized user-interfaces. Since this study aims at a broad competency evaluation, a versatile and generic tool to perform standardized assessment has been chosen: the product CBA-ItemBuilder (Rölke et al., 2012).

However, as noted by Drijvers et al. (2016), widespread digital tools for performing mathematics exist and are even part of learners' everyday lives. The ICILS studies showed that German schools are equipped below-average regarding to technology-related resources for both teaching and learning (Fraillon et al., 2019). The strived body of competencies, the national educational standards (NES), is based on a subject-related normative competence structure model that explains which competencies students should gain until the finalization of different grades. The NES are oriented at general educational aims. In the case of mathematics education, the NES describe a competence model with three dimensions: competencies, basic concepts (i.e., contents) and levels of requirements (i.e., difficulties) as shown in Figure 1 (KMK, 2003). Aiming at evaluating the educational system and investigating which competencies students have achieved by certain grade levels, the nationwide paper-pencil test *VERA*

(VERgleichsArbeiten [comparison test]) is carried out in Germany. VERA is based on the described normative model of competence. Though the subject-related use of digital tools and media is not mentioned directly in this model, the basic concepts can be divided into different facets where it is mentioned partly. Nevertheless, more and more teachers integrate digital media to enhance learning processes. Therefore, the integration of tools very similar to the learned tools appears to be a good approach to the paradigm of assessment with technologies while gaining the benefits of assessment through digital technology: Using technology with affordances (in the sense of Kaptelinin, 2013) that are well-known to learners such as radio buttons, or that are either part of familiar digital tools for doing mathematics or very similar to them. In the case of the NES, this includes calculators, spreadsheets and dynamic geometry systems.

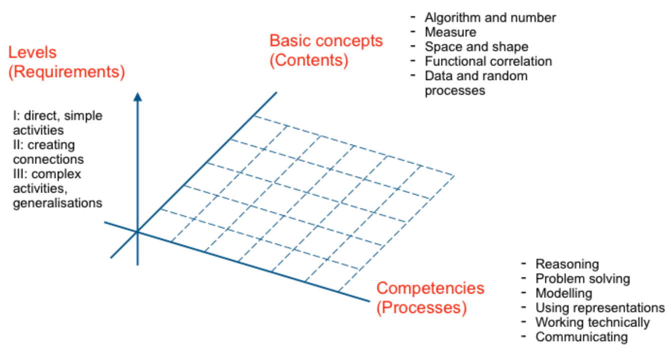

Figure 1. The normative competency structure model of the NES for the subject mathematics (see KMK, 2003).

The analysis of student answers using these technologies is only partially widespread. While the use of dynamic geometry systems for learning is common, the analysis of the correctness is not: Pioneering works such as in the ThEdu workshops, of Kovács, Recio and Vélez (2018), or of Kortenkamp and Richter-Gebert (2004) have not yet yielded a widespread applicability (Pelkola, Rasila & Sangwin, 2017) even though they demonstrate elementary validation strategies such as the use of simple predicates to indicate when two points are close to each other. Finally, there seems to be a need of analyzing the use of spreadsheets for learning purposes.

Our feasibility study explores the creation of a standardized assessment tool that can be deployed under general school conditions to assess mathematical competencies, including the application of digital tools and, therefore, even assessing mathematical digital competency (Geraniou & Jankvist, 2019), by means of a computer-based tool. Since we aim at using the schools' infrastructure, this exploration implies certain technical challenges: Some parts of the infrastructure may break because of the computer resources used by several actors (e.g. the bandwidth consumption, but also the installation of incompatible software). Moreover, the intent to combine *assessment with technology* with *assessment through technology* raises several validity concerns (Drijvers, 2018). In contrast, the combination also offers the opportunity to reach educational validity compared to using technically simple MC-items only (Sangwin & Jones, 2017).

RESEARCH DESIGN

Taking into account the described opportunities and limitations, we have set forth in this project to investigate the feasibility of assessing the dimensions of the NES using digital means despite the under-average equipment. The special demands of this as well as the associated paper-pencil test instruments played a special role in developing the test. They had an impact on the item design process, the technical realization and its technical deliveries. Besides exploring which contents of the NES can be assessed at all, we intended to investigate technical opportunities of embedding dynamic geometry and spreadsheet software. Because of its many opportunities, automatic scoring and a log-data-collection empowered by computer-based-assessment were included. The following question defines the focus of the first feasibility study: To what extent can competencies of the NES be assessed by digital means and integrating a variety of digital tools that are common to the testees?

To answer this question by means of an explorative study, we developed a construction process spanning from the design of items to the delivery. During the process we constantly investigated technical possibilities and limits. The first step was to analyze the contents of the NES that could be assessed at all or could be assessed better than with paper-pencil. In contrast to comparable studies (see Csapo, Molnar & Toth, 2009; Pelkola, Rasila & Sangwin, 2018), the focus was not laid on designing items that can be assessed in a paper-pencil test as well or using items that already exist. Instead, innovative items with integrated digital tools or enhanced tasks through embedding video or audio material should be designed. Concluding, this study aimed at increasing the sophistication: dimension F should be reached in the sense of the assessment possibilities proposed by Hoogland and Tout (2018). On this basis, the item authors, who most work as mathematics teachers in Germany, have designed tasks that – in terms of structure – are mainly inspired by the paper-based VERA tasks, but which incorporate the innovative possibilities mentioned above. The item authors did not use tools of constructing tasks completely in a digital format but designed documents with descriptions of the items and prepared files for the digital tools. Once the documents were developed, two rounds of reviews were conducted by experts in mathematical didactics. Between the two rounds a detailed discussion on the items took place and the items were revised. This process of item construction followed partly the established development process of paper-based items (see Rupp & Vock, 2007). It was repeated three times and the subsequent selection followed a few criteria: Firstly, diverse media with a broad spectrum (dynamic geometry, spreadsheets, video, audio, picture, calculator) should be included so that technical possibilities and limits as well as students' usage of different embedded media can be evaluated; secondly, different facets of the NES (different contents, competencies and difficulties) should be assessed.

Following the motivation of assessing with technology (Drijvers et al., 2016), several digital tools for mathematics have been considered. One of each category was expected. Because the brief testing time, test-specific learning should be limited to a minimum. It was, thus, important that the tool affordances (see Kaptelinin, 2013) are similar to those of the tools known by the testees. Moreover, to avoid the need for a technique-oriented workflow, the tools were to be integrated in a webpage.

For dynamic geometry, the strong diffusion of GeoGebra at schools and its abilities to be used on the web made it a de facto candidate. It is important to note that a GeoGebra construction embedded in a webpage, just as an activity that can be found among the GeoGebraTube server, is not necessarily including all functions of a desktop application. Multiple parameters allowed to restrict the set of actions.

For spreadsheets, the dominant tool is Microsoft Excel, a tool that only lives in its entire function set on desktops of Windows and macOS computers. Desktop alternatives often used at school include OpenOffice and LibreOffice. While multiple web-based spreadsheet services exist such as Collabora or Office365, we have either evaluated their incompleteness, bandwidth demand, license or incompatibility with regards to privacy. Only two tools remained with an open-source license and with an almost complete runnability on the client: EtherCalc or OnlyOffice. The latter was chosen for its greater visual and functional similarity to the dominant tool.

Using the item authoring tool CBA-ItemBuilder the chosen tasks were converted from design sketches into the digital format by item-implementors embedding all the external resources. This was followed by an internal reviewing process focusing on the technical problems: loops of revisions followed, so that a test instrument was designed for a 45-minute test period and could be conducted in nine classes. In total, 229 students took part. Beforehand, a system check was able to show that the schools were suitable for participation by checking technical requirements and carrying out test-like scenarios on randomly selected computers within the participating schools. Both system check and testing were observed and documented by the test leaders.

Automatic scoring was applied (with MCQs, with GeoGebra predicates). The logfiles generated by the CBA-ItemBuilder were converted into Excel-files and GeoGebra- and OnlyOffice-snapshots were extracted to visualize the final stage.

RESULTS

On the basis of the described research process, especially two strands of first results can be presented: concerning the assessment and the technical realization.

Assessment: First of all, it can be stated that this study was able to assess elements from all the basic concepts (see Figure 1): carry out targeted measurements in their environment, take measurements from source material, use them to carry out calculations and evaluate the results and the methods in relation to the situation; operate mentally with lines, surfaces and solids; draw and construct geometric figures using appropriate tools such as compass, ruler, triangle ruler or dynamic geometry software; use percentage calculation for growth processes (for example, interest calculation), also using a spreadsheet; systematically collect data, record it in tables and present it graphically, also using appropriate tools such as software. Those contents were spread over the different competencies and levels of requirements.

It could be observed that students have been comfortable using the web-browsers and the standard input affordances such as plain text fields or radio buttons. However, several students were unclear how to enter simple mathematical formulae such as the multiplication sign in a regular text field with the keyboard (normally written with the special characters · or ×). We assume that providing a symbol bar may be enough but the need to input more complex formulae (e.g. roots or fractions) might appear. Based on the analysis of GeoGebra-snapshots, difficulties can be extracted: In cases where a construction was presented without specific buttons (e.g. a cube in a 3D space), some learners have simply not discovered the possibilities to explore a solid from all its facets. In others, testees moved provided elements such as points without any discernable mathematical activity. In cases where a construction was required to be done either by using the adequate tool directly or by constructing the different steps (a perpendicular bisector), some testees simply moved provided elements or did something not appropriate for the task. However, the assessment seems to have been successful with dynamic geometry constructions when the operations were simple and explained by small sentences (e.g. "drag the dots according to the cube" or "move the girl to estimate the height"). Therefore, the difficulties may sometimes be attributed to a lack of appropriate usage of dynamic geometry tools, but sometimes also to missing (not digital) mathematical competencies or mathematical knowledge. As for the spreadsheet, even simple tasks such as entering `=4+5` could not be executed correctly; instead the testees entered `4+5` or `4+5=` which lead them to ask the test leader why the calculation was not executed.

Technical Realization: Among the biggest technical challenges was the use of the schools' infrastructure because of the reported equipment limitations in Germany.

Thanks to the new runtime technology for the CBA-ItemBuilder concluding its development (using contemporary frameworks for JavaScript), the delivery technology could be refined with lower dependency on the network: Ensuring that most web-resources are stored in cache prior to starting and delivering the measured assessment data (results and logs) in an asynchronous way. This study has provided good signs that this approach is doable in schools: In classes where preparation in advance was done only a little significant lag was perceived. In a few cases, web-browsers became unstable; changing the computer was then the go to solution.

The web-embedding nature has shown to be viable. In this study, it was based on the principle of iframes (webpages in webpages) which communicate to the CBA-ItemBuilder to send and receive their data. As long as an introspection of the tools' state was technically feasible, it was possible to gather the changes of interesting objects (e.g. movements of points); this was the case for GeoGebra but not yet for OnlyOffice. However, the storage of state was possible in both cases, thus, enables evaluators to view the last created state (a geometry construction or spreadsheet).

CONCLUSION AND FUTURE WORKS

Concluding the feasibility study on evaluating the NES, a first analysis showed that an assessment of several facets seems doable as a large-scale assessment in future. Though technical realizations (e.g. the input of formulae) have to be developed further and affordances should be adapted to student habits, first the Item Response Theory is used to investigate the scalability. This would be a step in ensuring the quality criterions and therefore making the tests comparable (Drijvers, 2018). In order to estimate why students were not familiar with all integrated tools, it is planned to conduct a survey on the use of digital media and tools in mathematics education in connection with an upcoming study. Further, for this upcoming study a greater amount of participating schools (20) is planned. Moreover, the workflow described above is going to be adapted, so that the item authors directly design tasks using the CBA-ItemBuilder and embedding the digital tools. We expect to make the workflow more effective and problems regarding the affordances or the technical realization faster to detect and handle in this way; this enriches the ongoing ItemBank conceptual development (Chituc et al., 2019), but requires more item-authoring capabilities. In summary, the goal to assess the NES for mathematics education with and through technology requires further development on technical aspects as well as on considerations and studies about how to ensure the essential quality criterions for a nationwide standardized competency assessment. Nevertheless, the opportunity of assessing students' competencies in mathematics education was demonstrated. As Hoogland and Tout (2018) claimed, this study did not tend to reduce contents or competencies, but instead focused on enhancing assessment and innovative items.

REFERENCES

Chituc CM., Herrmann M., Schiffner D., Rittberger M. (2019) *Towards the Design and Deployment of an Item Bank: An Analysis of the Requirements Elicited*. In: Herzog M., Kubincová Z., Han P., Temperini M. (eds) ICWL 2019. LNCS 11841.

Csapó, B., Molnár, G., Toth, K. (2009). Comparing Paper-and-Pencil and Online Assessment of Reasoning Skills. A Pilot Study for Introducing Electronic Testing in Large-scale Assessment in Hungary. The Transition to Computer-Based Assessment: Official Publications of the EU.

Drijvers, P. (2018). Digital assessment of mathematics: Opportunities, issues and criteria. *Mesure et Évaluation En Éducation*, *41*(1), 41–66.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of Technology in Lower Secondary Mathematics Education: A Concise Topical Survey*. Springer. https://doi.org/10.1007/978-3-319-33666-4

Fraillon, J., Ainley, J., Schulz, W., Friedman, T., & Duckworth, D. (2019). *Preparing for Life in a Digital World: IEA International Computer and Information Literacy Study 2018 International Report*. IEA.

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics*, *102*(1), 29–45.

Goldhammer, F., & Zehner, F. (2017). What to Make Of and How to Interpret Process Data. *Measurement: Interdisciplinary Research and Perspectives*, *15*(3–4), 128–132. https://doi.org/10.1080/15366367.2017.1411651

Hoogland, K., & Tout, D. (2018). Computer-based assessment of mathematics into the twenty-first century: pressures and tensions. *ZDM Mathematics Education*, 50, 675-686.

IQB. (n.d.). *VERA - An overview*. Institute for Educational Quality Improvement. Retrieved March 12, 2020, from https://www.iqb.hu-berlin.de/vera

Kaptelinin, V., Affordances, in Lowgren et al., The Encyclopedia of Human-Computer Interaction, 2nd Ed, access on 2020-03-12 from https://www.interaction-design.org/literature/book/the-encyclopedia-of-human-computer-interaction-2nd-ed/affordances.

KMK. (2003). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss*.

Kortenkamp, U., & Richter-Gebert, J. (2004). Using Automatic Theorem Proving to Improve the Usability of Geometry Software. *Proceedings of MathUI*, 13.

Kovács, Z., Recio, T., & Vélez, M. P. (2018). Using Automated Reasoning Tools in GeoGebra in the Teaching and Learning of Proving in Geometry. *International Journal for Technology in Mathematics Education*, *25*(2), 33–50.

Pelkola, T., Rasila, A., Sangwin, C. (2018). Investigating Bloom's Learning for Mastery in Mathematics with Online Assessment. Informatics in Education, v17 n2, 363-380

Rölke, H. (2012). The ItemBuilder. A graphical authoring system for complex item development. In T. Bastiaens & G. Marks (Eds.), *Proceedings of world conference on E-Learning in corporate, government, healthcar, and higher education 2012* (pp. 344–353). AACE. https://www.learntechlib.org/p/41614/

Rupp, A., & Vock, M. (2007). National educational standards in Germany: Methodological challenges for developing and calibrating standards-based tests. In *Making it comparable. Standards in science education* (pp. 173–198). Waxmann.

Sangwin, C., Cazes, C., Lee, A., & Wong, K. L. (2010). Micro-Level Automatic Assessment Supported by Digital Technologies. In C. Hoyles & J.-B. Lagrange, *Mathematics Education and Technology-Rethinking the Terrain: The 17th ICMI Study*. Springer US. https://doi.org/10.1007/978-1-4419-0146-0

Sangwin, C. J., Jones, I. (2017) Asymmetry in student achievement on multiple-choice and constructed-response items in reversible mathematics processes. *Educational Studies in Mathematics Education*, 94, 205–222.

Stacey, K., & Wiliam, D. (2013). Technology and Assessment in Mathematics. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third International Handbook of Mathematics Education* (pp. 721–754). Springer Science+Business. https://doi.org/10.1007/978-1-4614-4684-2

Challenges encountered in mathematical problem-solving through computational thinking and programming activities

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This paper aims at exploring the challenges students face when engaging in mathematical problem-solving through computational thinking (CT) and programming by a combination of theoretically derived insights and task-based activities. The main method used is a semi-structured interview of one undergraduate student who was presented with a mathematical task to solve while responding to questions in a dialog with the teacher on the mathematical problem solving process through CT and the programming language MATLAB. Conclusions are drawn from the results to promote CT and programming in mathematics education.

Keywords: Algorithm, computational thinking (CT), MATLAB, mathematical problem-solving, usability

INTRODUCTION

Mathematics students are expected to have basic CT skills in parallel to emerging programming languages (Wing, 2014). Moreover, CT as a competency for future work in society should be acquired by all university mathematics students. It is argued that CT can improve mathematical problem-solving by benefitting from the power of computational processes and programming languages (Shute, Sun, & Asbell-Clarke, 2017). This study explores the challenges students face when engaging in mathematical problem-solving through CT and MATLAB.

THEORETICAL BACKGOUND

CT, or similar designations such as algorithmic thinking, is becoming an important learning goal at all levels of mathematics education. According to Misfeldt and Ejsing-Duun (2015), CT is described as the ability to work with algorithms understood as systematic descriptions of problem-solving and construction strategies. Similarly, Wing (2014) describes CT as "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer—human or machine—can effectively carry out". More precisely, algorithmic thinking can be defined as the process of solving a problem step-by-step in an effective, non-ambiguous and organized way that can be translated into instructions to solve problems of the same type by an individual or a computer (Filho & Mercat, 2018). The main commonality between CT and mathematical thinking is problem-solving processes (Wing, 2008). CT is also quite similar with engineering thinking in terms of design and evaluation of processes (Pérez-Marína et al, 2018). Moreover, CT and algorithmic constructs such as variables and flow statements (if-then-or-else, for, repeat, etc.) are closely connected to arithmetical and mathematical thinking (Lie, Hauge, & Meaney). This close connection might provide opportunities for mathematical problem-solving by means of CT and programming. Thus, students' mathematical background and problem-solving skills are critical for building efficient algorithms for problem-solving rather than trial-and-error and getting the program to run (Topallia & Cagiltay, 2018). Moreover, CT requires students to be engaged in a continuously changing, problem-solving process in interaction with the computer.

Drawing on these research studies, the paper proposes a three-step approach to solve mathematical problems by means of CT and programming languages. *Firstly*, students should have a good mathematical background to benefit from CT and programming languages. More specifically, they should be able to benefit from their knowledge to make sense of a mathematical task and have a good understanding of it before formulating an algorithm and programming the solution. *Secondly*, CT, in turn, should enable students to analyse and decompose the mathematical task and design an algorithm and how to perform it step-by-step before programming it. Engaging students in mathematical problem-solving through CT may enable a better understanding of mathematics beyond textbook mathematics and paper-pencil techniques. *Thirdly*, students should be able to translate the mathematical problem and the associated algorithmic solution to the constructs of the programming language. This presupposes that the language is usable. Performing programming activities in mathematics education may provide opportunities to gain knowledge that is otherwise difficult to acquire without experimenting with the program and thinking algorithmically. However, this might be difficult to achieve unless the mathematical tasks are well-designed, and the programming language is usable.

When referring to the term "usability", the research literature focuses on educational software such as GeoGebra, CAS, SimReal, etc. (Artigue et al., 2009; Bokhove & Drijvers, 2010; Hadjerrouit, 2019). However, programming languages are different from educational software and how they are used to implement mathematical problems. Hence, evaluating the usability of programming languages might not be as straight forward as it may seem. Still, three usability criteria can be applied to programming languages with slight modifications. Firstly, the extent to which the language is easy to use and allows a quick familiarization with it. The second criterion aims at whether the constructs of the programming language (variables, if, for, while, etc.) are difficult to grasp. The third criterion is the feedback provided by the language in terms of error messages, and whether these are useful to foster a successful implementation of the mathematical problem through correcting and improving the program.

Finally, engaging students in mathematical problem-solving through CT and programming languages should be placed in a pedagogical context to enable a good degree of autonomy so that the students can work on their own and have a sense of control over their mathematical learning. Clearly, students should be able to acquire knowledge without being completely dependent on the teacher. Moreover, CT and programming languages should be a motivational factor for learning mathematics and should support students' engagement in problem-solving by means of motivating tasks that are tied to the students' mathematical activities. Another pedagogical issue concerns students' interactions with the language and the feedback it provides to foster computational thinking when solving mathematical problems.

THE STUDY

Context of the Study and Research Question

This work is a single case study conducted in the context of a first-year undergraduate course on programming with applications in mathematics. The participant was one student from one class of 8 students enrolled in the course in 2019. The student had average knowledge background in mathematics, but no experience with programming languages. The course introduced the basics of algorithmic thinking and the core elements of MATLAB, that is variables, flow statements, e.g. loops, if-then-or-else, for, repeat, etc. MATLAB suits well mathematical problem-solving because the solutions are expressed in familiar mathematical notation. The research question is: *What challenges do students face when engaging in mathematical problem-solving through CT and MATLAB?*

Mathematical Task

The mathematical task presented to the student is: The length of a curve may be approximated using Pythagoras' theorem by positioning a triangle adjacent to the curve (Fig. 1, left, below). The length of the green line between A and B may then be approximated as $\sqrt{x^2+y^2}$. The task is to write a MATLAB function approximating the curve length of $f(x)=2^x$ between two given x-values (Fig. 1, right, below).

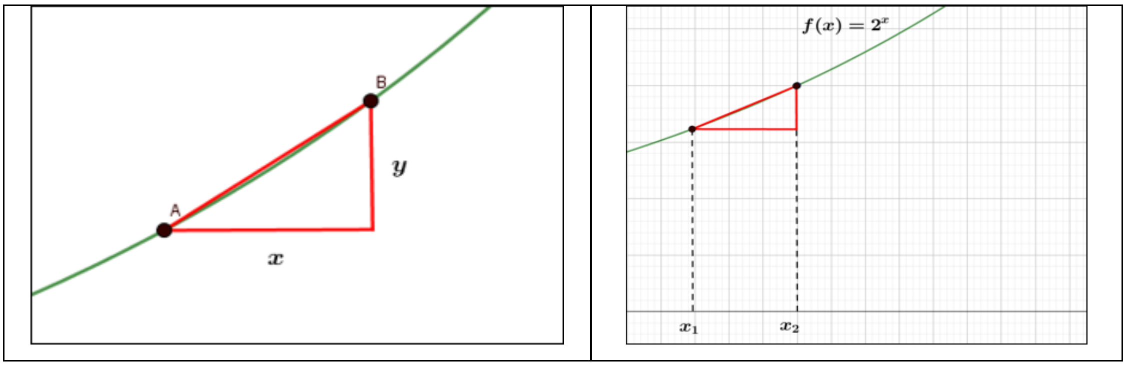

Figure 1: The mathematical task

The student was then presented with the following skeleton of a MATLAB function: function length=length estimate (x1, x2), length=? The student was asked to enter the formula, based on x1 and x2, and replace the question mark. The MATLAB function sqrt(x) can be used to calculate the square root √x.

Data Collection and Analysis Method

The main data collection method is a task-based semi-structured interview of one student who was presented with a mathematical task to solve, while responding to questions in a dialog with the teacher on the mathematical solving process by means of CT, algorithms, programming activities with MATLAB, and teacher assistance.

The order of the interview questions follows roughly the three-step approach to problem-solving presented in the theoretical framing, sometimes moving back and forth depending on the student's responses. Likewise, the data analysis method relies on the three-step approach to problem-solving, that is:

a) Understand the mathematical problem
b) Analyse and decompose the mathematical task, and then design an algorithm and how to perform it step-by-step before programming it
c) Finally, translate the algorithmic solution to the programming language code

More specifically, the student is expected to solve the task in three steps as follows:

a) Understand the task, that is using Pythagoras' theorem to calculate the hypotenuse
b) Formulate an algorithm, that is find the lengths of the triangle hypothenuse using the function $f(x) = 2^x$ and relate it to x_1 and x_2
c) Translate the algorithm into MATLAB code, corresponding to: function length = lengthEstimate (x1, x2); length = sqrt((x2-x1)^2 + (2^x2 - 2^x1)^2)

The analysis of the results seeks indications of problem-solving through CT by means of MATLAB according to this three-step approach. This is not the same as analysing and coding in the sense of grounded theory without theoretical background. Rather, the analysis tries to address the research question about the challenges encountered when solving a mathematical problem through CT and MATLAB. The interview data were transcribed and analysed according to an inductive strategy based on the interplay between the three-step approach to problem-solving and the empirical data collected by means of the semi-structured interview (Patton, 2002).

RESULTS

The results describe how the participating student engages in mathematical problem-solving through CT, MATLAB and teacher assistance. The student was given the task described above, and paper and a pen to use. It took some time before the student made sense of the mathematical task. The teacher asked the student to develop a skeleton of the solution. The student did, and started thinking about the length, but suggested an incorrect solution, based only on the values of x1 and x2. After some calculation trials, the student noticed that the attempted solution was wrong. Then, the teacher encouraged the student to think computationally.

T: But before you start using MATLAB, are you going to make an algorithm (..), for problem-solving before you start using MATLAB?

S: I just have to sit and think about it.

But still, the student continued guessing and calculating without thinking computationally. After an attempt to make sense of the task and calculate the length of the curve connecting it to the hypotenuse with a trial and error approach, the teacher provided a hint.

T: Now you say you know the hypotenuse and calculate y. But the hypotenuse is the unknown numeral here, the one you are supposed to calculate.

S: Yes.

T: So now you have turned the problem around (…). It is just that that thought was a little backwards, maybe.

S: Yes, it is quite possible.

After this short dialogue, the student started using MATLAB without developing an algorithmic solution and a clear strategy for solving the problem. The teacher then engaged in a discussion to guide the student step by step towards an algorithmic solution. Afterwads, the teacher tested the function *"on zero and one and then I got 1.4*". Likewise, the student tested the formula and found 1.4142. The dialogue continued:

T: (…). Do you have anything to say about (…) like that afterwards?

S: No, I am, I was a little bit in doubt about how to (…) First, it was the task you asked about (…) and then it was (…) and then I thought (…) f(x) is the function in x2 would be that point minus the function of that point (…) that it would be the length. But that is where I was wrong, I felt (…) Because you meant it to be here (…) and I understand that now.

T: Yes, that is the point, (…). That is why you have to use Pythagoras to find (…) Did you think (…) There was a hint here, wasn't there? Square root?

S: Yes, yes, yes, the square root (…). I knew it was probably wrong, but I just didn't quite understand what it was.

T: Well, because there was a clue there that you couldn't use, wasn't there. Then you realize that there is something (…)

This excerpt shows there is little indication that the student was following a problem-solving strategy based on a clear understanding of the problem before formulating an algorithm and starting programming. A few minutes later, the teacher asked the student if there is a tendency to favour pen and paper to solve the task algorithmically before starting using MATLAB since developing an algorithm does not automatically require using the computer.

S: If I have it in my head, sometimes I start with MATLAB, and then I write some sort of sketch before going through it carefully. If I am not sure, I will start with paper.

T: Maybe the task was not quite clear?

S: Yes, so far, but I had probably forgotten some of the principles there.

T: Principles related to MATLAB or to the mathematical assignment?

S: To the mathematical problem.

Again, this excerpt shows that the student does not have a good understanding of the mathematical task in order to develop an algorithmic solution before programming it. Therefore, the teacher reminded the student about the importance of algorithmic thinking before translating the solution into MATLAB code.

T: Now, the point of the assignment is that you should be able to translate the mathematical solution into MATLAB code. That is really the point here.

S: Yes, I felt that when I understood the mathematical solution, I had no trouble putting it into MATLAB. It was simply that I had (…) forgotten a bit the length thing there. That f of that minus f of that is delta y, then.

Despite the challenges in understanding the task, the student admitted that the assignment was not about advanced mathematics such as calculus. Nevertheless, the student pointed out that the task is related to a logical way of thinking, which is implicitly associated with CT, but not to calculus or algebra as this excerpt shows:

S: It is not exactly (…) very advanced mathematical functions that we have been working on. It is not quite calculus. That is a lot of plus and minus and logical stuff that isn't (…)

T: Which is not directly related to mathematics?

S: Yes, yes, it is related to very basic mathematics. As many people know, and so it is related to a (…) logical way of thinking that, yes, as one might find (…) may find some of it in mathematics, but it is (…) it does not recall very much, so purely mathematical, about calculus or algebra. Although one can put some formulas into it too, then.

Then the teacher asked the student to elaborate on this issue and how to connect mathematics and the programming constructs of MATLAB.

S: Let us see. Yes, I feel it is on two levels, calculus learning and MATLAB, somehow (…) if I was just working with MATLAB (…) I do not feel like I am getting any better at calculus, because these are two different things.

T: We probably should have had two courses (…) First a basic course in programming, and then a course in (…) because it is very difficult (…) to teach high level mathematics and low-level programming.

S: Yes, I think high-level math, then you need some knowledge in pretty good programming. But our basic understanding of programming is far from it.

The excerpt shows the student considers programming very different from mathematics. There is also no indication that CT could help to bridge the gap between the two subjects. On possible explanation of the disconnection between mathematics and programming is that the student does not clearly see the connection between the mathematical task and the programming solution (length = sqrt((x2-x1)^2 + (2^x2 - 2^x1)^2)). Another explanation is the lack of CT skills which makes it difficult to connect the task with the language constructs of MATLAB. As a result, it seems that there is such a large difference in level between the two subjects that the student was challenged to connect the mathematics task to MATLAB.

DISCUSSION AND PRIMILILARY CONCLUSIONS

The research question addressed in the paper is: *What challenges do students face when engaging in mathematical problem-solving through CT and MATLAB?* The first challenge is the lack of mathematical knowledge, which hindered the student to make sense of the task, have a good mathematical understanding of it, and then develop a problem-solving strategy that can be translated into an algorithm, and implemented using MATLAB. The second challenge is related to the implementation of the algorithmic solution in MATLAB. This was a challenging task as the student struggled to become familiar with MATLAB constructs due to lack of background knowledge and experience with programming. This shows that the minimum requirement to engage in mathematical problem-solving through computational thinking (CT) and programming is a combination of good background knowledge in mathematics and familiarity with the programming language in question in terms of usability and effective implementation of the solution. A third challenge is directed towards the integration of mathematical and programming skills to a coherent whole. To make mathematics interact better with programming, the pedagogical context around first-year undergraduate mathematics courses should be well designed to ensure a smooth integration of CT and MATLAB into the courses in terms of varied and intrinsically motivating tasks that are suited to the students' knowledge level. Moreover, as this study shows, the role of the teacher is still important to assist students in designing algorithms and implementing computational solutions. Clearly, student autonomy cannot be fully expected for novices without good knowledge background in mathematics and familiarities with programming. Hence, the acquisition of CT skills for mathematical problem-solving should consider pedagogical modalities.

In conclusion, the outcome of the study can be summarized as that both lack of mathematical skills and programming experience have led to problems with completing the task. Even though the participating student is representative for the average student enrolled in the course, the study is limited to be generalized from an empirical point of view, but it forms an hypothesis that can be explored in subsequent studies whether mathematical skills really form a prerequisite for programming mathematical problems, perhaps with students from computer science. Other relevant questions are: To what extent is CT compatible with mathematical thinking? What is the importance of CT in bridging the world of mathematics and programming? Nevertheless, two preliminary conclusions can be drawn from the study. Firstly, the relationships between mathematics, CT, and programming languages are quite complex in educational settings. Secondly, engaging in mathematical problem-solving through CT and programming seem to require both good background in mathematics and algorithmic thinking. Future work will use both quantitative and qualitative methods, and a theoretical approach that helps to analyse in more depth the interactions between mathematics, CT, and programming.

REFERENCES

Artigue, M. et al. (2009). Connecting and integrating theoretical frames: The TELMA contribution. *International Journal of Computers for Mathematical Learning 14*, pp. 217-240.

Bokhove, K., & Drijvers, P. (2010). Digital tools for algebra education: Criteria and evaluation. *International Journal of Mathematics Learning 15*, pp. 45-62.

Filho, P., & Mercat, C. (2018). Teaching computational thinking in classroom environments: A case for unplugged scenario. *Conference: Re(s)sources 2018 - Understanding Teachers' Work Through Their Interactions with Resources for Teaching*. Lyon, France.

Hadjerrouit, S., & Gautestad, H.H. (2019). Evaluating the usefulness of the visualization tool SimReal+ for learning mathematics: A case study at the undergraduate level. In: D. Sampson et al. (Eds.), *Learning Technologies for Transforming Large-Scale Teaching, Learning, and Assessment*, (pp. 71-89). Springer Nature Switzerland AG.

Misfeldt, M., & Ejsing-Duun, S. (2015). Learning mathematics through programming: An instrumental approach to potentials and pitfalls. In K. Krainer, & N. Vondrová (Eds.). *Proceedings of CERME9* (pp. 2524-2530). Prague, Czech Republic:

Lie, J., Hauge, I. O., & Meaney, T. J. (2017). Computer programming in the lower secondary classroom: mathematics learning. *Italian Journal of Educational Technology*, *25*(2), pp. 27-35.

Patton, M. Q. (2002). *Qualitative research & evaluation methods*. London: Sage Publications

Pérez-Marín, D., Hijón-Neira, R., Bacelo, A., & Pizarro, C. (2018). Can computational thinking be improved by using a methodology based on metaphors and scratch to teach computer programming to children? *Computers in Human Behavior,* pp. 1-10.

Topalli, D., & Cagiltay, N. E. (2018). Improving programming skills in engineering education through problem-based game projects with Scratch. *Computers & Education*, *120*, pp. 64-74.

Shute, V.J., Sun, C., & Asbell-Clarke, J. (2017). Demystifying computational thinking. *Educational Research Review*, pp. 1-10.

Wing, J. M. (2008). Computational thinking and thinking about computing. *Philosophical Transactions of the Royal Society A*, *366*(1881), pp. 3717-3725.

Wing, J. M. (2014, January 9). Computational thinking benefits society. *Social Issues in Computing,* 40th Anniversary Blog, University of Toronto. Retrieved from http://socialissues.cs.toronto.edu/index.html%3Fp=279.html

Towards a shared research agenda for computer-aided assessment of university mathematics

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In this article we describe our plan to develop a shared research agenda for computer-aided assessment of university mathematics, drawing on input from the community of mathematics education researchers and university teachers interested in this topic. Such an agenda will help to establish a programme of research aligned with practical concerns, which would contribute to both theoretical and practical development. As well as describing the process that we will follow, we provide three illustrative examples of use-inspired research questions that have arisen in our own teaching of university mathematics.

Keywords: computer-aided assessment, university mathematics, research agenda.

INTRODUCTION

The past two decades have seen growth in the technical sophistication of computer-aided assessment of mathematics (Sangwin, 2013) and in its widespread use. Indeed, a recent survey of university mathematics departments in the UK found that many were introducing computer-aided assessment since increases in student numbers had made "previous methods of assessment unsustainable" (Iannone & Simpson, 2012, p13). This presents a need for practitioners (i.e. university teachers) to be informed by existing research on computer-aided assessment, and conversely for researchers to direct attention at emerging practical concerns. What is needed is a programme of "use-inspired basic research" that contributes to both improved theoretical understanding and improved practice (Lester, 2005). In this paper, we set out our plan for a collaborative approach to establish an agenda for such a programme of research.

A model for our approach is provided by a recent project to establish a research agenda in numerical cognition (Alcock et al., 2016). In that project, 16 researchers from a variety of relevant disciplines undertook a systematic process to identify important open questions in the field, modelled on similar exercises in other fields (Sutherland et al., 2011).

In the next section we describe the qualitative process to achieve a similar outcome for computer-aided assessment of university mathematics. Following that, we give three examples of ways that teaching and research have interacted in our own recent work, to suggest some possible directions for the shared agenda.

COLLABORATIVE PROCESS

Our planned process is set out in Table 1. Following the model of Alcock et al. (2016), we will begin with an online phase to gather and prioritise an initial set of questions

(Stages 1 and 2). We anticipate around 30 researchers will collaborate on the project, sourced from the authors' professional contacts and relevant conference proceedings (e.g. MEDA, EAMS, BSRLM). Researchers will be asked to suggest research questions online (Stage 1) and then review one another's questions (Stage 2) according to the below scope and criteria.

This will be followed by a series of in-person meetings to discuss and further prioritise the questions (Stages 3-5). A key forum for these discussions will be a new Working Group of the British Society for Research into Learning Mathematics, which will meet in June and November. Between July and October, input from the broader community will be sought at international conferences, including MEDA 2020 in Linz. Discussions will take place in groups of collaborators, each group led by one of the authors, to focus on clarifying, refining and winnowing questions. This will include identifying any questions that are already addressed in the literature.

Stage	Description	Purpose
1	Online form	Gathering suggested research questions
2	Online survey	Inviting participants to rate the importance of each question, to focus attention on the most important questions in the next stage
3	BSRLM Working Group (13 June)	Discussing the questions; suggest refinements and possible grouping into themes
4	Conference discussions	Discussing the questions and prioritising, with input from a broader range of participants
5	BSRLM Working Group (14 November)	Using the priorities identified in Stage 4 to produce a focused list of questions

Table 1: Summary of the collaborative process

Following this process, the project leads will prepare a manuscript summarising the process and the resulting set of themes and research questions. The manuscript will be shared with participants for comments, before it is submitted for publication.

Scope. The scope of possible research questions is broad, but has limits. In particular, questions should directly relate to computer-aided assessment of mathematics in universities around the world:

- Computer-aided – relying on technology in a fundamental way;
- Assessment – concerning formative or summative assessment, rather than teaching or learning tools (though of course the line between these and formative assessment is blurry);
- Of university mathematics – based on any and all topics or task-types which are relevant to mathematics as studied at university.

Question criteria. When filtering and prioritising questions, participants will be asked to bear in mind the selection criteria used by Alcock et al. (2016, p24-25):

> "the eventual questions should:
>
> - address an important gap in knowledge;
> - be formulated specifically (not as a general topic area);
> - be clear, where appropriate, about specific interventions and outcome measures;
> - be answerable specifically (not by 'it all depends');
> - have a factual answer that does not depend on value judgements;
> - be answerable through a realistic research design;
> - be of a scope that could reasonably be addressed by a research team."

The criterion "be answerable through a realistic research design" is important because we seek to set a research agenda that will be acted upon. (We are grateful to a reviewer of an early draft of this paper for making this suggestion.) For example, it might be enlightening to randomise students into a computer-aided assessment or control group for the duration of their higher education study, but this would not be practical and a research question that required such a method would be excluded by this criterion.

Themes. We do not wish to pre-judge the grouping of questions into themes (Stage 3), but it is worth noting that the scope of possible research questions is broad enough that they could reasonably be grouped into themes. One example would be simply delineating cognitive and affective issues, while another is given by the tentative "onion model" shown in Figure 1, where research questions are concerned with different levels of generality. Of course, once we have gathered participants' questions, we will be in a better position to define and delineate themes.

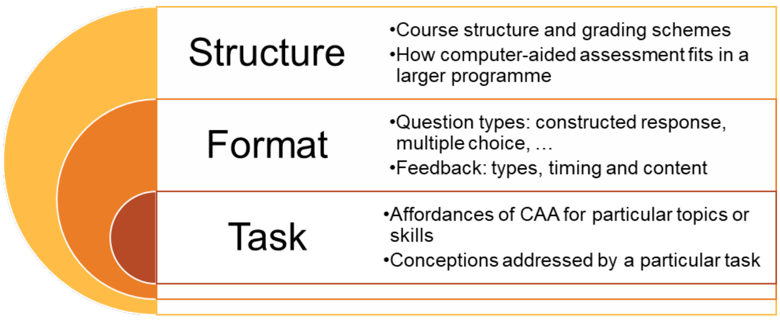

Figure 1: A possible theme structure for the research questions.

CASE STUDIES

In the following sub-sections, we describe three ways that research questions about computer-aided assessment of university mathematics have naturally arisen from our work teaching university mathematics. These serve as examples of possible research questions which could arise though the process described above. These examples also illustrate how a programme of "use-inspired basic research" (Lester, 2005) could inform both theoretical and practical development.

Example 1: Feedback on common errors

Modern computer-aided assessment tools, such as STACK, have the ability to interpret a student's response and offer feedback accordingly (Sangwin, 2013, section 6.10). This presents the opportunity to identify errors based on common misconceptions, and give corrective feedback. In particular, the question author may anticipate answers that would follow from the students holding a certain mathematical misconception or, in Fischbein's (1989) language, a "tacit model", which is an understanding held by the student that will "influence, tacitly, the interpretations and the solving decisions of the learner" (p9). Existing work on e-assessment has found that "elaborated feedback" giving more detail than just the correct answer leads to improved performance (Attali & van der Kleij, 2017; Shute, 2008), but it is not known whether such feedback addressing tacit models is effective in adjusting the models that students use.

This arose as a practical concern when using STACK to deliver a fully online course in introductory university mathematics (Kinnear, 2019). For the topic on integration, several questions have the potential to expose the tacit model that "definite integral = area" without proper regard for areas above and below the x-axis. These questions were programmed to detect and give specific feedback on this misconception, as can be seen in the example question shown in Figure 2.

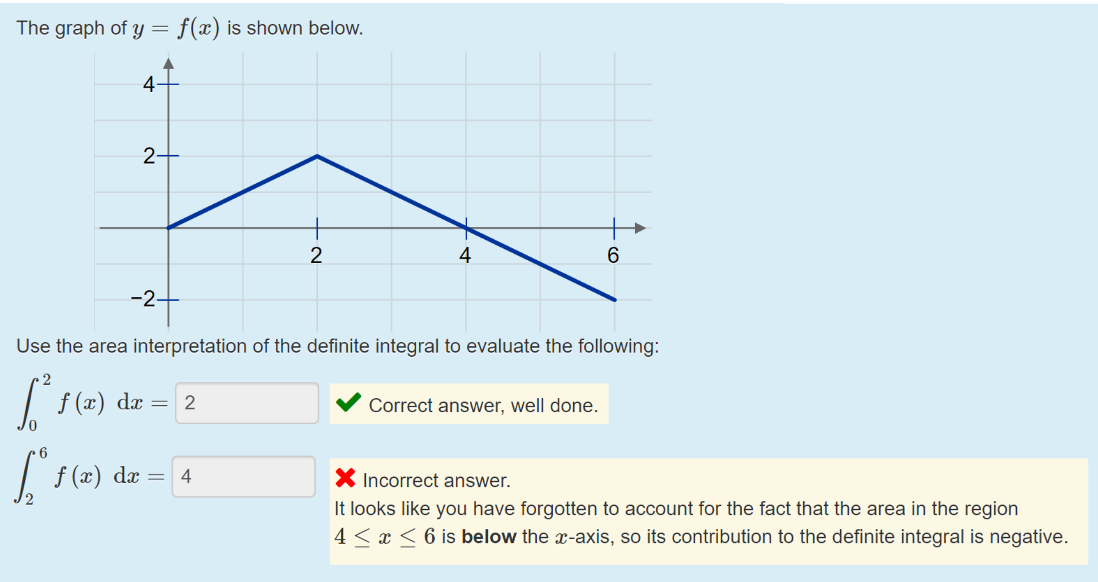

Figure 2: Example of a STACK question giving feedback on a common error.

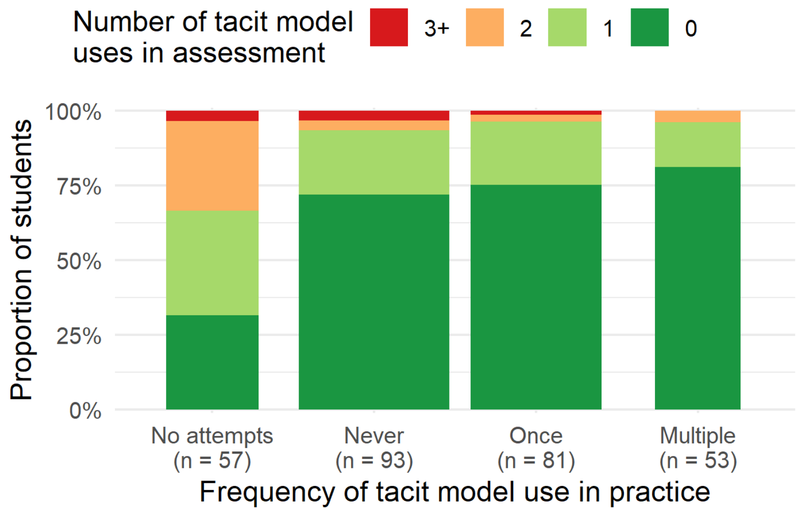

Figure 3: Association of feedback on tacit model, or misconception, use during initial practice with its use during subsequent assessments.

Analysis of the student response data shows that this feedback is frequently triggered; for the example question shown in Figure 2, 50/528 of responses triggered the feedback. Furthermore, there is some observational evidence that the feedback is helpful: Figure 3 shows that students who use the tacit model (and receive targeted feedback) multiple times when working through the course learning materials go on to use it less frequently in the topic assessment than students who used the tacit model just once or not at all during learning.

This motivates the broad research question: what are the features of feedback addressing specific misconceptions that help students to resolve those misconceptions? This could be addressed through experimental studies, testing different versions of the specific feedback across different university mathematics topics, and with qualitative investigation of students' solution strategies.

Example 2: Learner-generated examples

The pedagogical approach of prompting students to generate examples has been suggested as an effective way to help students engage actively with mathematics (Watson & Mason, 2006). Computer-aided assessment could be used to underpin wider use of this approach, making use of the ability to evaluate the properties of many student responses and give appropriate feedback (Sangwin, 2003). However, there is currently a lack of empirical support for the efficacy of promoting example generation (Iannone et al., 2011).

Example generation questions were a feature of the online course described by Kinnear (2019). The use of such questions with a large number of students presents an opportunity to contribute to the theoretical and empirical basis for this pedagogical approach. However, questions about the approach's efficacy arguably fall outwith the scope of our planned process, as they do not inherently rely on computer-aided assessment. That said, there are many possible questions about example generation that do specifically relate to computer-aided assessment; here, we outline two.

One question is: to what extent can current computer-aided assessment tools meaningfully judge typical student responses? For example, asking for an example of a function which has a limit of 0 as $x \to \infty$ would require using the underlying computer algebra system (CAS) to evaluate the limit of whatever function is supplied by students, and it is not clear *a priori* that the CAS will be able to do this for the range of examples that might be offered by the students. Answering this question would rely on both design work to identify suitable topics and questions, and empirical work to gather typical student responses.

A second question is: how does the use of computer-aided assessment affect students' example generation strategies and success, relative to the same tasks on paper or orally? Students' strategies have been studied in previous work (e.g. Iannone et al., 2011) but computer-aided assessment brings additional constraints that warrant further investigation. Returning to the example of the function with a given limit, students may immediately be able to offer a sketch of such a function, but be unable to write the corresponding expression. They may think of an example which has a piecewise definition, but not know how to enter this in proper syntax (if it is even possible in a given computer-aided assessment system). Answering this question would likely require in-depth qualitative investigation of students' strategies, e.g. through observations or clinical interviews.

Example 3: Assessing proof comprehension

Proof is a hallmark of the discipline of mathematics, and differentiates mathematics from other subjects. We know mathematical proof is difficult to learn. The ability to accept mathematical expressions and manipulate them enables computer-aided assessment to advance well beyond multiple choice questions. However, most current systems are still a long way from being able to accept a complete mathematical argument from a student in free-form text.

There is currently a lot of practical development of computer-aided assessment, much of it instinctive and practitioner-based rather than theory-based. This research is based on the following pragmatic theoretical epistemological position:

> "to successfully automate a process it is necessary to understand it profoundly. It therefore follows that successful, or even partial, automation of a process necessitates the development of a certain kind of understanding." (Sangwin, 2019, p314)

This leads to two directions for research questions. First, we are likely to learn much about students' understanding of proof through online tests, e.g. tests developed to the standard of the proof reading-comprehension tests of Mejia-Ramos et al. (2012). As an example of the kinds of proof-based misconceptions we have explored with online assessment, we asked our students to read a proof by induction for the formula for the sum of the squares of the natural numbers from 1 to n, where $P(n)$ is the statement that

$1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$. Students were then asked to write out $P(3)$. Of the 350 attempts online, 26% of students wrote only 3(3+1)(2*3+1)/6, confusing the *equation* $P(3)$ with the value of the sum of the series. If students really do confuse the formula for the value of the sum of the series with the equation expressing a complete induction hypothesis, then there is little hope of them correctly completing a proof by induction.

This observation arose by asking the kind of free-entry answer current computer-aided assessment facilitates, and examination of such common mistakes. Questions which separate concerns associated with proof are likely to lead to productive research questions about how students learn proof.

A second direction for research questions is to use students' work to shed light on the nature of the subject itself. By its very nature, students' work is often incomplete, incorrect and/or inconsistent. This is neither pejorative, nor following a deficit model of learning. Indeed, the attempt to assess such work automatically throws interesting light on the forms of reasoning used and what professionals will accept as criteria for acceptable proofs. For example, Sangwin & Köcher (2016) examined questions from specimen examination papers and identified "reasoning by equivalence" as the most important single form of reasoning in elementary mathematics. The attempt to produce automatically assessed examinations identified an important form of reasoning, finding common ground in different areas of the subject.

CONCLUSION

In this paper, we have outlined three examples of fruitful bi-directional interactions between university mathematics teaching and mathematics education research. We have also outlined our plans for a collaborative process to gather, collate, refine and prioritise a set of research questions, to establish a shared research agenda for the community interested in computer-aided assessment of university mathematics. Such an agenda would help to drive forward both theoretical and practical developments.

REFERENCES

Alcock, L., Ansari, D., Batchelor, S., Bisson, M.-J., De Smedt, B., Gilmore, C., … Weber, K. (2016). Challenges in mathematical cognition: A collaboratively-derived research agenda. *Journal of Numerical Cognition*, *2*(1), 20–41. https://doi.org/10.5964/jnc.v2i1.10

Attali, Y., & van der Kleij, F. (2017). Effects of feedback elaboration and feedback timing during computer-based practice in mathematics problem solving. *Computers and Education*, *110*, 154–169. https://doi.org/10.1016/j.compedu.2017.03.012

Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the Learning of Mathematics*, *9*(2), 9–14. https://doi.org/10.2307/40247948

Iannone, P., Inglis, M., Mejía-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics*, *77*(1), 1–14. https://doi.org/10.1007/s10649-011-9299-0

Iannone, P., & Simpson, A. (2012). *Mapping University Mathematics Assessment Practices*. ISBN 978-1-870284-01-1

Kinnear, G. (2019). Delivering an online course using STACK. In *Contributions to the 1st International STACK conference 2018*. Fürth, Germany: Friedrich-Alexander-Universität Erlangen-Nürnberg. https://doi.org/10.5281/zenodo.2565969

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM - International Journal on Mathematics Education*, *37*(6), 457–467. https://doi.org/10.1007/BF02655854

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3–18. https://doi.org/10.1007/s10649-011-9349-7

Sangwin, C. J. (2003). New opportunities for encouraging higher level mathematical learning by creative use of emerging computer aided assessment. *International Journal of Mathematical Education in Science and Technology*, *34*(6), 813–829. https://doi.org/10.1080/00207390310001595474

Sangwin, C. J. (2013). *Computer aided assessment of mathematics*. Oxford University Press. https://doi.org/10.1093/acprof:oso/9780199660353.003.0003

Sangwin, C. (2019). Reasoning by Equivalence: The Potential Contribution of an Automatic Proof Checker. In G. Hanna, D. Reid, & M. de Villiers (Eds.), *Proof Technology in Mathematics Research and Teaching. Mathematics Education in the Digital Era, vol 14* (pp. 313–330). Springer, Cham. https://doi.org/10.1007/978-3-030-28483-1_15

Sangwin, C. J., & Köcher, N. (2016). Automation of mathematics examinations. *Computers & Education*, *94*, 215–227. https://doi.org/10.1016/J.COMPEDU.2015.11.014

Shute, V. J. (2008). Focus on Formative Feedback. *Review of Educational Research*, *78*(1), 153–189. https://doi.org/10.3102/0034654307313795

Sutherland, W. J., Fleishman, E., Mascia, M. B., Pretty, J., & Rudd, M. A. (2011, June 1). Methods for collaboratively identifying research priorities and emerging issues in science and policy. *Methods in Ecology and Evolution*. John Wiley & Sons, Ltd. https://doi.org/10.1111/j.2041-210X.2010.00083.x

Watson, A., & Mason, J. (2006). *Mathematics as a Constructive Activity*. Routledge. https://doi.org/10.4324/9781410613714

From paper-and-pencil to computer-based assessment: an example of qualitative comparative analysis

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Comparative studies on paper-and-pencil and computer-based test principally focus on statistical analysis of students' performances. In educational assessment, comparing students' performance does not imply a comparison between the solving processes followed by students. The purpose of this study is to present a comparative qualitative analysis based on observing and subsequently comparing the behaviour of students involved in solving tasks in different environments. This analysis allows to highlight similarities and differences in the solving process with regard to the administration environment.

Keywords: Computer based assessment, Comparative study.

INTRODUCTION

The use of new technologies into teaching and learning processes has opened new and wide frontiers of study in the field of mathematics education. Regarding assessment, on the one hand research in computer based tests concerns the validity of these tests, on the other it focuses on their comparability with existing paper tests. In these two perspectives, large-scale surveys were conducted; they involve students from different educational levels, from primary to secondary instruction (Drasgow, 2015; Way, Davis, & Fitzpatrick, 2005).

The first studies conducted on the topic involve the *National Assessment of Education Progress* (NAEP). Russell & Haney (1997) carry out a study to compare the effects of administering a test in two environments (paper and pencil vs computer) on performance (in terms of scores) of secondary school students. The findings reveal differences concerning the type of response: no substantial differences are identified in the case of multiple-choice items, while some differences are found regarding open-ended items.

Other research is conducted in the *Texas statewide tests in mathematics, reading/English language arts, science and social studies* (Way, Davis, & Fitzpatrick, 2006). Differently from previous results, these studies show that the scores obtained in computer-based tests are higher than those with paper and pencil. Contrasting results are observed for example in *Florida State Assessment in high school reading and mathematics* (Nichols & Kirkpatrick, 2005). In fact, significant differences in students' performance are found, with scores measured on the paper test resulting slightly higher than digital test results.

The research studies listed represent a very small part of those developed on the topic; from these and many others, it is clear that surveys mainly follow a statistical approach to comparison. In particular, they focus on measuring performance, referring to scores in terms of correct or wrong answers, and then on a purely quantitative analysis. In addition, there is no shared perspective on whether computerised tests can be compared with those adopting a paper and pencil format. Such contradictions can be reasonable also because investigating computer-based assessments need to consider their characteristics in terms of affordances and constraints. Ripley (2009) defines two approaches: *migratory and transformative*. Migratory approach is the use of technological support as a tool of administration; it consists in a transition of tasks conceived in paper format into digital format. Transformative approach involves the transformation of original paper tests integrating new technological devices which support interactive tools (graphs, applets, etc) that enhance new affordances.

In this perspective two important questions arise: is it possible to define a certain level of comparability between *tranformative tasks*? Is it possible to design categories for comparing the behaviour of students who are involved in comparable *transformative tasks*? The first research question is described and discussed in previous studies (Lemmo & Mariotti, 2017). The main purpose of this paper is to define categories to observe, describe, analyse and subsequently compare the behaviour of students engaged in mathematics tasks in the computer and paper and pencil environment using a transformative approach.

THEORETICAL FRAMEWORK

In order to define appropriate categories to describe a situation of migration from paper to computer, we start from the theoretical framework of problem solving defined by Schoenfeld (1985). In the first pages of *Mathematical problem solving* (1985), the author explains and describes the aims of his research:

> […] the goal of the research that has generated this book is to make sense of people's mathematical behaviour – to explain what goes on in their heads as they engage in mathematical tasks of some complexity (Schoenfeld, 1985, p.5)

In these pages, Schoenfeld refers to *mathematical behaviours* rather than to processes, approaches, strategies, heuristics… his goal is to observe and then describe as accurately as possible, what happens during a solution process. The term *behaviour* seems to fit in with the aim of our analysis; we adopt this term to refer to what takes place when students deal with a task. The qualitative observation of the protocols presented by Schoenfeld is based on four categories related to knowledge and behaviour of a possible solver in front of a generic task. This categorization allows an analysis of the solving process through elements that can be studied separately. Schoenfeld defines *Resources* as the knowledge of certain disciplinary facts that the solver uses to solve a task. *Heuristics* are the set of general rules for problem solving. They are suggestions that help a solver to better understand a problem or achieve progress towards a solution. In other words, Heuristics can be interpreted as the set of strategies that can be adopted to solve the task. The third category, called *Control*, includes a series of practices that the solver adopts in order to: choose which of the available information to use; plan the solution process… Finally, the *Belief systems* is linked with solver metacognitive aspects (for a detailed description, see Schoenfeld, 1985, chapters 2, 3, 4 e 5). We define behaviour profiles referring to the 4 categories in order to highlight students' behaviour during the task solution process and to reveal possible differences and similarities that may depend on the administration environment. The profiles necessarily depend on the characteristics of the tasks selected in the study. For this reason, we will describe the profiles in the next paragraph.

METHODOLOGY

The tasks

For the experimentation we choose to administer tasks designed starting from the applet created by Freudenthal Institut Research group in Mathematics education [1]. The applet named *the broken calculator* (Fig. 1), is one of the tasks administered to students and it is the one that we explain as an example.

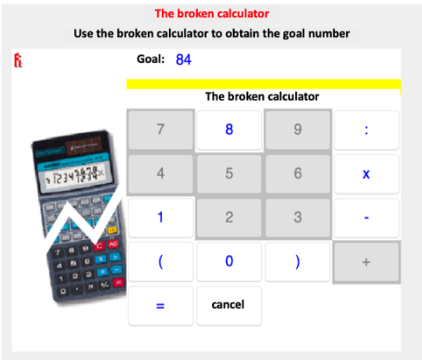

Figure 1: Screenshot of the application: The broken calculator

The task asks students to determine a procedure to obtain the *goal number* using only the working keys. The interface of the applet looks very similar to a real calculator (fig. 1); the darkened keys represent the buttons not working while the blue ones are the only ones that the solver can use.

Task design in the two environments

The purpose of our study is to conduct a qualitative research on the behaviour of students involved in the solution of tasks assigned within the two environments: paper and computer. Therefore, defining tasks that can be administered in the two environments and can be considered comparable between them is essential. In line with these purposes, we built two tasks, one in the computer environment and one in the paper and pencil environment, starting from the broken calculator applet. In Figure 2 we present the task in the paper and pencil environment. The task in the digital format appears equal, the difference is that the user can use the calculator as a real calculator.

"You have a broken calculator; the functioning buttons are only the blue ones that you see in the picture. It could be possible to obtain the number written near the word ***goal*** *only using the functioning buttons?"*

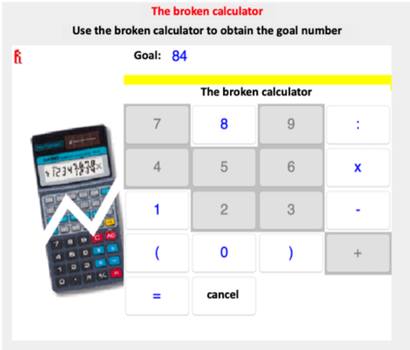

Figure 2: The broken calculator in the paper and pencil environment

Analysis profiles

The main interest of the experimentation is to highlight students' behaviours during the task solution process with the aim of highlighting possible differences and similarities that may depend on the administration environment. In order to do this, it is essential to define profiles that can guide us in the analysis of the students' solving processes, as they will appear in the collected protocols. In this perspective, we outline a priori hypotheses with reference to the behaviours that students might adopt in solving the task. Starting from these hypotheses, we define the profiles.

The task focuses on the use of a calculator, for this reason, the Resources that the student should involve are those evoked by the use of the calculator according to the requests of the task. In particular, disciplinary contents concerning formalisms and properties of arithmetic calculation can be recalled. Like all traditional calculators, the solver can use the numeric keys to construct numbers with two or more digits. The positional representation of numbers is therefore an additional resource that may be available to the student although not necessary. These two hypotheses can be schematised by identifying three possible solver profiles concerning Resources (R):

- R0: the solver does not recall the content and procedures necessary to solve the task;
- R1: the solver recalls the contents and procedures necessary to facilitate the resolution of the task; for example, those referring to the positional decimal representation of numbers.
- R2: The solver recalls the content and procedures strictly necessary to solve the task; for example, only those related to formalisms and properties of operations and not those related to the positional decimal representation of numbers.

In the process of finding a solving expression to obtain the goal number, the Control category plays a crucial role but is difficult to identify. One aspect that we can observe is the choice to keep memory of the operations activated:

- C1: the solver remembers the activated procedures by writing them down, step by step;
- C2: the solver remembers mnemonically activated procedures;

Concerning Heuristics category, there is no single procedure to achieve the solution. In general, it can be assumed that there are two main heuristics and consequently we can identify two profiles referring to the students who adopt them (E):

- E1: the solver analyses the goal number in terms of the result of one or more operations;
- E2: the solver proceeds with a try-and-error approach based on available keys.

In particular, the student identified with E1, starts from the analysis of the goal number in order to identify possible operations that give as a result this number. After identifying these operations, he/she looks for those that are possible according to the restrictions imposed by the task and identifies the resolving expression. On the other hand, students identified with E2, start from the analysis of the available keys and try to find an increasingly accurate approximation of the goal number.

The category that guides the solution process in all its phases is the Beliefs System. This is a strictly personal and subjective category that depends on the experiences of individuals. It is not easy to outline a priori profiles to describe this dimension. For this reason, it is a category not explored in our study.

The experimentation

The experiment was carried out in 2013 and involved a sample of grade 6 and 8 students from an Italian Secondary School in Bologna. The sample consists of 16 students, respectively 4 couples of grade 6 students and 4 of grade 8. Each mathematics teacher selected in her class two pairs of students considered for her equivalent by level of knowledge and abilities in mathematics. One pair solve the task in paper and pencil environment while the other in the computer environment.

In both environments, a test consisting of 5 tasks was administered to the students: 2 of them of the broken calculator. The test was administered between December 2013 and January 2014. No time limit was imposed. The entire experiment was videotaped.

PROTOCOL ANALYSIS

In the follow we discus data collected; we highlight similarities and differences that we found between students' behaviour in the solution of the task. These similarities and differences are first discussed in the same environment and then in the two environments.

We describe the students' solution process of the first task of the broken calculator: the task asked students to obtain the number 58 using the keys: $\boxed{“4”, “3”, “0”, “\times”, “-“, “+”}$

Analysis of students' behaviour in the digital environment

In Table 1 there are students profiles elaborated by the protocol analysis. The name "P_cbt" refers to pair of students involved in the digital tasks. Couples 1 and 2 are composed of grade 6 students; couples 3 and 4 are made up of grade 8 students.

	P_cbt_1	P_cbt_2	P_cbt_3	P_cbt_4
Resources	R2➔R1	R2➔R1	R1	R1
Control	C2	C1	C2	C2
Heuristics	E2	E2	E2	E2

Table 1: students' profiles in the digital environment

As we can see from Table 1, there are differences and similarities between the 4 couples of students with respect to profiles. Common traits are highlighted in grey.

In facing the task, all the couples recognise the Resources necessary to solve it and also those that can facilitate its resolution, i.e. the positional representation of the numbers. Going into detail, the students have two different attitudes. The first two couples recognise the possibility of using two-digit numbers thanks to the interaction with the calculator after few minutes (they pass from R2 to R1), while the other two immediately recognise this possibility (R1).

All couples choose to adopt a trial-and-error strategy starting with the working keys (E2). However, students differ according to the choice of operations to be used: pair 1 and 3 use multiplication as preferred operation while pair 2 and 4 use all operations without any particular preference. Regarding the Control, only one couple choose to use the calculator to keep track of the calculations (C1). The others do not use the calculator and use mental calculation. This causes several interruptions in the solution process also caused by the lack of feedback on the correctness of the calculations.

We continue the discussion considering the couple of students who face the task in paper and pencil environment. In Table 2 there are the profiles through which we describe the students. The name "P_ppt" refers to students involved in the paper and pencil tasks. The numbers refer to couples of the same class of the ones in cbt.

	P_ppt_1	P_ ppt _2	P_ ppt _3	P_ ppt _4
Resources	R2	R2	R2	R1
Control	C2	C2	C2	C2
Heuristics	E1➔E2	E2	E2	-

Table 2: students' profiles in the paper and pencil environment

Observing Table 2, we can notice that the first three couples are characterised by the same profiles; the only isolated case is the couple 4 which we describe separately. This fact might suggest that if students are involved in a task in a familiar environment, they act on common behaviour patterns: R2-C2-E2. All students of the three couples choose an exploratory strategy (E2) using a predominantly mnemonic control system (C2). None of the three pairs recognised among the numeric keys the possibility to use them as digits but only as numbers (R2).

Regarding Resources, the first three couples of students interpreted the numbers presented on the keys as the only ones available to identify the solving expression. No couple use paper to do calculations or keep track of procedures; the use of the pen is limited to writing down the solving expression of the task. In general, the three couples choose an exploratory approach; only one couple starts to look for the factorisation of the goal number but gives up the strategy in a few minutes (they pass from E1 to E1).

Couple 9 significantly differs from the other: after 30 seconds, the students identify the solving expression. Such a fast solution does not allow us to hypothesise the type of approach adopted and for this reason, we cannot make inferences regarding Heuristics. The main difference between this couple and the others is that they are the only that use two-digit numbers (R1).

Let us consider the comparison of the profiles in the two environments. Concerning Control and Heuristics, the most frequent profile among students is C2-E2 in both environments. Therefore, for these categories the environment does not reveal correlation with the students' choices. However, it has a considerable influence in terms of the Resources. In fact, the possibility of using the calculator seems to have an effect on the Resources mobilised: all the students who engage the task in the digital environment identified themselves with the R1 profile; on the contrary, in the paper environment, the presence of only an image seems to limit this awareness linked to the positional representation of the figures (R2).

CONCLUSION

In the last decades, there is no shared perspective on whether computerised tests can be compared with those adopting a paper and pencil format. Such contradictions can be reasonable if these studies have been carried out on the assumption that comparing student performances can provide information about the problem-solving processes performed to provide the answers.

In this paper we propose categories for qualitative comparative analysis of student behaviour in the two environments. The categories of analysis are inspired by the framework developed by Schoenfeld (1985).

The discussion of the results highlights differences and similarities in relation to the profile patterns identified in the two environments. In particular, students who face the task in paper and pencil environment are categorised by the same profile pattern. This does not happen in the digital environment. These aspects would not be observable comparing performance (in terms of correct/wrong answer) because all couples of students determine the solution of the task regardless of the environment. Starting from equal performance, we notice that Resources activated by students are distinct in the two environments. Such difference necessarily influenced the procedures activated with similar heuristics.

The analysis currently has a strong limitation; in particular, it is not possible to ignore the task because the profiles are built ad hoc. A possible future development could be to design other studies with other tasks or with the same but on a larger scale. In this way it would be possible to validate these results for even wider use and to outline generic indicators that would allow the study of correlations also in a more general perspective regardless of the task being considered.

The use of categories to identify analysis profiles can open perspectives for assessment in mathematics. Categories allow to isolate particular variables of students' behaviour in order to identify where students have difficulties in solving a task. These difficulties may not necessarily strictly depend on mathematical aspects but may be transversal, for example, related to the text comprehension of the task or to the environment in which it is administered. In addition, the attention to some variables based on profiles allows to improve formative assessment and to propose specific didactic situations focused on processes and not on products.

NOTES

1. http://www.fi.uu.nl/wisweb/applets/mainframe_en.html

REFERENCES

Drasgow, F. (Ed.). (2015). *Technology and Testing: Improving Educational and Psychological Measurement*. Abingdon, Oxon, United Kingdom: Routledge.

Lemmo, A., Mariotti, M., A., (2017). From paper and pencil- to Computer-based assessment: some issues raising in the comparison. *Proceedings of the 10th Conference of European Research in Mathematics Education*, (pp. 3540-3547) Dublin (Ireland).

Nichols, P., & Kirkpatrick, R. (2005). Comparability of the computer-administered tests with existing paper-and-pencil tests in reading and mathematics tests. *Annual Meeting of the American Educational Research Association.* Montreal, Canada.

Ripley, M. (2009). Transformational computer-based testing. In F. Scheuermann, & J. Björnsson (Eds.), *The transition to computer-based assessment* (pp. 92-98). Luxembourg: Office for Official Publications of the European Communities.

Russell, M., & Haney, W. (1997). Testing writing on computers: An experiment comparing student performance on tests conducted via computer and via paper-and-pencil. *Education policy analysis archives, 5* (3), 1-18.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic press.

Way, W. D., Davis, L. L., & Fitzpatrick, S. (2005). Practical questions in introducing computerized adaptive testing for K-12 assessments. (pp. 1-20). Pearson Educational Measurement.

Semi-automated assessment: the way to efficient feedback and reliable math grading on written solutions in the digital age?

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Process-oriented feedback is powerful in math teaching yet highly labour-intensive. As a consequence, digital assessment with fully-automated feedback has received much attention. Despite this, digital assessment faces shortcomings concerning students' input: it only reads specifically-formatted answers, and learners solve higher-order questions more naturally when using paper-and-pencil. Therefore, we investigated the use of a semi-automated assessment (SA) method in which teachers write atomic feedback that can easily be reused for multiple students. SA is implemented in Moodle. During a lab study (n = 1800 corrections), we examined (1) whether SA saves time; (2) whether SA delivers more reliable scores compared to handwritten assessment; and (3) how teachers perceive SA. Mixed effect models were used for data analysis.

Keywords: digital assessment, feedback, semi-automated assessment, atomic feedback, reusable feedback.

INTRODUCTION

Process-oriented feedback is a crucial instrument in learning processes (Hattie & Timperley, 2007): it tells students which mathematical operations were appropriate (strengths), which were not (weaknesses) and how task solutions can be improved (strategies) (Rakoczy et al., 2013). To provide good process-oriented feedback, teachers need 'interpretative knowledge' (Mellone et al., 2020) of students' errors.

Fully-automated feedback (FA) versus paper-and-pencil assessment (PP)

Because it is time-consuming for teachers to provide such feedback, considerable research has been devoted to fully-automated assessment (**FA**) in mathematics education (Sangwin, 2013). FA provides extensive, immediate feedback for students, substantial time profits for teachers and often endless training possibilities as questions can be automatedly generated.

Less attention has been paid to overcoming the drawbacks of FA. First, preparation of FA questions is challenging as it is complex to create the accompanying correction schemes that give partial grades and adapted feedback. The central weakness is, however, the fact that not all mathematics questions can be easily automated; especially higher-order thinking questions are solved by students more naturally using paper-and-pencil (Threlfall et al., 2007). Furthermore, almost all digital test environments offer too little mathematical tools that allow students to express themselves mathematically, as they would with pen and paper. Answer possibilities are mostly limited to pre-defined response fields instead of free answering formats used in paper-and-pencil assessments (**PP**) (Kocher & Sangwin, 2016). Hoogland & Tout (2018) have shown that, as a consequence, a lot of FA-questions focus on lower-order goals, such as procedural skills. Besides, for higher-order questions, no difference between the effect of immediate and delayed feedback is found yet (van der Kleij et al., 2015), tempering the need of FA for that kind of questions. All in all, FA does not easily allow to assess open-ended, challenging problems triggering higher-order thinking.

In mathematics, students' answers will contain systematic error patterns, meaning that different students often make analogous mistakes (Movshovitz-Hadar et al., 1987) and teachers keep on noticing the same mistakes again. As paper-based assessment still has an essential place in mathematics teaching (Threlfall et al., 2007), it is surprising that using these error patterns to speed up the assessment process remains largely unstudied in the literature. In the present research, we want to bridge the gap between FA and PP and develop a new, semi-automated assessment method **(SA)**.

Semi-automated assessment with atomic feedback (SA)

SA is a method in which students work out their solutions using paper-and-pencil, but the teacher assesses them digitally, making it different from PP-tests who are handwritten assessed by the teachers.

Student's solution

Manipulate the formula:

$$A = 2\pi r h + 2\pi r^2 \quad \text{to } h$$

$$\frac{A}{2\cdot\pi\cdot r} = h + 2\pi r^2$$

$$\frac{A - 2\pi r^2}{2\cdot\pi\cdot r} = h$$

Classic feedback

Mind the fact that the dominant operation in the right-hand side of the equation is an addition! It is impossible to divide the left-hand side by $2\pi r$ because, in the first step, it is not handled as the common factor of the right-hand side. Your final answer is right, but written this way, it seems as coincidence. Going from the first to the second step, normally you would substract $2\pi r^2$ from both sides, meaning that it shouldn't be placed in the nominator. It is unclear of this is an additional mistake or a compensation of the previous mistake.

`Grade: 3/10`

Atomic feedback

- First step
  - Mind the fact that the dominant operation in the right-hand side of the equation is an addition!

    `Threshold: Max 50% of the points`
  - It is impossible to divide the left-hand side by $2\pi r$ because, in the first step, it is not handled as the common factor of the right-hand side.
- Second-step
  - Your final answer is right, but
    * Going from the first to the second step, you should substract $2\pi r^2$ from both sides.
    * $2\pi r^2$ shouldn't be placed in the nominator.

      `-2 points`
    * It is unclear of this is an additional mistake or a compensation of the previous mistake.

`Grade: 3/10`

Figure 1: An example of classic versus atomic feedback

In SA, teachers have to write *atomic* feedback items. These feedback items are all saved, so they can easily be reused when another student makes the same mistake. The system suggests relevant items to reuse. The teacher can see the solutions of the students on-screen or asses directly from the students' sheets. It is possible to assess test-by-test or question-by-question. SA generates a student report that can be printed or viewed online in an e-learning system.

To achieve high levels of reusability in SA, teachers must give *atomic* feedback: instead of writing long pieces describing lots of different mistakes at once, they must (1) identify the independent error occurring, and (2) write small feedback items for each error, independent of each other (see Figure 1). As such, SA can create point-by-point feedback only covering those items that are relevant to a student's solution. In addition, *clustering* of feedback is allowed, meaning sub-items can be added to feedback items. Clustering ensures that feedback can be written as atomic as possible and avoids teachers' need to write too specific items, which would compromise the reusability. It also allows related feedback to be orderly shown to students. Besides, the feedback cluster will be a decisive factor for the algorithm to decide which feedback will be suggested for reuse to the teacher. However, to support maximal flexibility, a feedback item can be part of different clusters.

If a question must be graded, the teacher can associate feedback items with partial scores to be subtracted. It is also possible to associate items with a threshold (e.g. 'if this feedback item is given to a student, a student can get at most 50% of the points'). The teacher can always still manually change the associated score of an item.

Solutions to assess handwritten students' tests digitally are available, like *Gradescope* (Singh, A. et. al., 2017), but the integration of reusable atomic feedback items has never been studied before.

Envisioned benefits of semi-automated assessment

SA might be a promising go-between for FA and PP, throwing off the current limitations of FA. First, SA gives rise to potentially significant time savings: solutions are assessed reusing already given feedback as much as possible. This might enable a faster feedback and grading process than PP, especially when a question has already been assessed many times, filling the database with lots of reusable feedback. Second, SA allows students to write down any mathematical expression, using the structure they prefer for their reasoning, and hence fully expressing themselves mathematically. Third, SA does not limit the use of open-ended, challenging higher-order thinking questions as there are no pre-defined response fields; the assessment work is in the hands of the teacher. Fourth, a teacher only gives feedback when a mistake occurs (no need for a crystal ball), omitting the need to develop complex correction schemes beforehand as is the case for FA (Sangwin, 2013). The loss of immediate feedback is a drawback, but remember that no significant difference in effect is found yet between delayed and immediate feedback for higher-order thinking questions (van der Kleij et al., 2015). In many cases, SA assessment might thus be a valid assessment method, combining the strengths of PP- and FA-assessment (see Table 1).

This study aims to verify these envisioned benefits experimentally. As we are currently collecting data, we have organised the rest of this proposal as follows: we introduce the guiding research questions, examine the followed methodology and conclude by looking at possibilities for further research.

Feedback & Assessment		
	Computer-assisted	
Paper-and-pencil based (PP)	Semi-automated (SA)	Fully automated (FA)
- delayed feedback	- delayed feedback	+ immediate feedback
+ natural mathematical expressions	+ natural mathematical expressions	- too little mathematical tools
+ no pre-defined response fields	+ no pre-defined response fields	- pre-defined response fields
+ questions are easy to develop	+ questions are easy to develop	- need for an automated correction scheme and anticipation on mistakes
+ high-order thinking questions possible	+ high-order thinking questions possible	- high-order thinking questions difficult
- time consuming	+ time profits	+ time profits

Table 1: Advantages and disadvantages of different types of assessment

RESEARCH QUESTIONS AND HYPOTHESES

(RQ1) Does SA-feedback lead to significant time savings compared to PP-feedback and can we predict them?

Hypotheses: *As SA-feedback and SA-grades are reusable, we hypothesise that SA will be faster than PP. It might be possible that SA only becomes faster when a certain threshold (cf. number of tests assessed) is reached, as the database first must be filled with reusable feedback. Before that threshold, we hypothesise that there will be no significant time difference between SA and PP. To predict possible time savings, we seek for reusability measurements of the used feedback items, e.g. the ratio of already used feedback items to all feedback items used to correct a student's solution. When a solution is assessed with exclusively new feedback items, this ratio will be 0. If assessing a solution requires 5 different feedback items, of which 4 have already been used before, the reusability factor equals 0.8.*

(RQ2) Is teachers' SA grading more reliable compared to PP-grading?

Hypotheses: *Reliability is the degree to which an assessment produces stable and consistent results (Feldt, 2004). We focus on intra-rater reliability: how consistent is a teacher's grading (for a particular set of tests) over time? Extensive research has shown that teachers' PP-assessments are biased in numerous ways as teachers tend to forget how they handled the same mistakes before (Parkes, 2012). Because SA remembers already given feedback and associated grades, we hypothesise that the SA-grading stability will be better. With respect to inter-rater reliability, we expect no significant differences between SA and PP-grading, because in the current experiment teachers only use their own atomic feedback items. However, a follow-up study with a group of assessors contributing to and sharing the same database of atomic feedback items, is planned.*

(RQ3) A) How do teachers perceive SA? B) What about the feedback quality in the different conditions?

Hypotheses: *We hypothesise that teachers will appreciate the way SA integrates in their classroom practice as opposed to FA, which can sometimes feel a bit alienating. However, learning to write atomic feedback and using the SA-tool might be difficult. We will also compare the quality of given PP- and SA-feedback. As teachers are constantly remembered of already given feedback under SA, this could increase their interpretative knowledge (Mellone et. al., 2020).*

METHODS & MATERIALS

Materials

Development of MathSA

We developed an SA-tool called 'MathSA', and integrated it as an advanced grading method in the open-source e-learning platform Moodle. The Moodle-framework contains a lot of features (e.g. a grade book, uploading assignments,…) and is the most popular e-learning platform.

Test on linear equations

In close cooperation with a math teacher, we developed a test on linear equations, consisting of three equally weighted questions: (1) solve an equation (easy/procedural), (2) manipulate a formula (complex/procedural, see Fig. 1) and (3) a modelling question consisting of a word problem (complex/problem-solving). The three questions combined form a representative, standard test on linear equations.

Survey based on the TAM-model

We will develop a short, validated survey based on the Technology Acceptance Model (Davis, 1989) in order to measure how teachers perceive SA.

Participants

60 students of Grade 9 in one secondary school in Flanders (Belgium) solved the test on linear equations. We gathered informed consents from all students.

30 Belgian secondary math teachers with at least 3 years of working experience will participate voluntarily in a lab study. They were contacted through announcements in math teaching magazines and subscribed via www.mathsa.uantwerpen.be. We aim for diversity among participating teachers in gender, experience and school type. We plan to organise a focus group on MathSA and atomic feedback with 8 of the participating teachers. They will be selected based on their answers in the survey (cf. diversity in terms of gender, experience and views on technology). We received ethical clearance.

Design

All students conducted the test on linear equations in an authentic context: the students had been studying linear equations during classes, they were accustomed to the test lay-out, and afterwards, the grades were incorporated in the students' grade reports.

During the lab study, each teacher will assess all 60 solved tests. For each teacher individually, a random selection of 30 tests will be assessed under the SA-condition. The remaining 30 tests will be assessed under the PP-condition. This yields 1800 test corrections in total and indicates a within-subject design. With respect to RQ2, we will ensure every test is marked the same number of times under each condition. To avoid bias coming from a growing familiarity with the test, exactly half of the teachers will start with assessing PP-tests; the other half will first handle their SA-tests.

During the whole study, participating teachers will not be informed about the research questions, to prevent bias in their grading style. To control for bias due to inexperience with MathSA (cf. SA-condition), we provide sufficient training opportunities during the lab study, before we start with the actual experiment.

In RQ1, the dependent variable is the time a teacher needs to assess a single question. The independent variable is the assessment condition (PP/SA). As the assessment time also depends on: the teacher (categorical), the quality of the student's answer (measured by the test score), and the familiarity the teacher has with the test items (number of 1 to 30, indicating how many tests the teacher has already corrected under the same condition), these are all included as moderating variables.

For RQ2 (reliability of SA-grading), at least one month after the lab study, teachers will be asked to grade the same tests again under the same condition. This period in between guarantees that teachers will largely have forgotten how they handled particular tests. We will calculate the differences in scores between both measurements (score lab study – score month after) and use this as the dependent variable. The assessment condition (PP/SA) will be used as the independent variable. We will include the teacher (categorical) and the quality of the student's answer (measured by the average of the test scores given by all the teachers during the lab study) again as moderating variables.

To answer RQ3a, we will survey the participating teachers and conduct a focus group. To compare the given feedback under both conditions and whether they show different levels of interpretative knowledge (RQ3b), text mining will be used.

Procedure

In February 2020, 64 students (9^{th} grade) solved the test on linear equations during their regular math class. It was conducted like every other test by their teacher and solved with paper-and-pencil. The researchers were not present during this test taking. Students were asked afterwards if the test could also be used for the research. All students agreed. We randomly deleted 4 tests to have exactly 60 tests.

We have been conducting the lab study on different moments in the months of July and August 2020. Due to the Covid-19 crisis, it wasn't feasible to gather all the 30 teachers at the same time in one place. During the lab study a presentation about useful process-oriented feedback in mathematics was given. Second, teachers were trained to get familiar with MathSA and the formulation of atomic feedback. Before the start of the actual experiment, all the teachers got half an hour to get familiar with the MathSA-tool. We offered some students' answers on entirely different math topics than the topic of the test of the experiment as training possibilities. During this training, teachers got tips to make good atomic feedback and had the opportunity to ask questions. Third, half of the teachers started assessing under the PP-condition: they wrote detailed feedback on each test and graded it. They were allowed to develop a personal correction scheme in advance. Every time they started to asses a test, they had to push the space button on the computer in front of them, so that the time needed to correct the test could be tracked. The other half of the teachers started assessing under the SA-condition, providing atomic feedback and scores with MathSA. The tool automatically keeps track of the time used for each test. In both conditions, participants were never allowed to return to an already corrected test. Fourth, after the break, the groups swapped conditions (SA/PP) and assessed the other, remaining tests. Finally, they were asked to fill in the survey.

After all the lab experiments on different dates are executed, we will organise the focus group with 8 participants online at the end of August.

In September 2020, a month after the lab study, the participants will receive the ungraded copies of their 30 PP-tests by post. They will be asked to re-grade them and will be invited to re-grade the remaining 30 SA-tests online. Their previous SA-corrections will have disappeared, but their feedback items of the lab study will have been saved. They will have one month to re-score the 60 tests under the same condition as during the lab study. Their PP-grades will be sent back to us through an online form. At the start of the experiment, all participants have been informed about this additional individual work (about 3 hours), but they do not know that they must re-grade the same tests.

Data analysis

We will construct mixed models (i.e. models containing both fixed effects and random effects) to examine the time differences (RQ1) and the consistency differences (RQ2) between SA and PP. In both models, the fixed effect is the condition (PP/SA). The random effects are the moderating variables mentioned in the design. The survey data will be analysed and cross-tabulated with teachers' characteristics (e.g. age, experience, technology acceptance score). We will also link the survey with the qualitative data from the focus group.

FURTHER RESEARCH

This paper describes the first study of this doctoral research. The goal of this first study is to explore SA as a new assessment method and get an indication on how SA behaves when teachers use it. The next step of the project is to focus on the students' point of view and conduct quasi-experimental studies to measure students' learning effects. We also plan an integration of SA-assessment with Bayesian networks for elaborate student tracking.

REFERENCES

Davis, F. D. (1989). Perceived Usefulness, Perceived Ease of Use, and User Acceptance of Information Technology. *MIS Quarterly*, *13*(3), 319–340.

Feldt, L. S. (2004). Estimating the Reliability of a Test Battery Composite or a Test Score Based on Weighted Item Scoring. *Measurement and Evaluation in Counseling and Development*, *37*(3), 184–191.

Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research*, *77*, 81–112.

Hoogland, K., & Tout, D. (2018). Computer-based assessment of mathematics into the twenty-first century: pressures and tensions. *ZDM*, *50*(4), 675–686.

Kocher, N., & Sangwin, C. (2016). Automation of mathematics examinations. *Computers & Education*, *94*, 215–227.

Mellone, M., Ribeiro, M., Jakobsen, A., Carotenuto, G., Romano,P. & Pacelli, T. (2020): Mathematics teachers' interpretative knowledge of students' errors and non-standard reasoning. *Research in Mathematics Education*, 1479-4802

Movshovitz-Hadar, N., Zaslavsky, O., & Inbar, S. (1987). An Empirical Classification Model for Errors in High School Mathematics. *Journal for Research in Mathematics Education*, *18*(1), 3–14.

Parkes, J. (2012). Reliability in Classroom Assessment. In J. H. McMillan (Ed.), SAGE Handbook of Research on Classroom Assessment. SZAGE, 107–124.

Rakoczy, K., Harks, B., Klieme, E., Blum, W., & Hochweber, J. (2013). Written feedback in mathematics: Mediated by students' perception, moderated by goal orientation. *Learning and Instruction*, *27*, 63–73.

Sangwin, C. (2013). Computer-aided Assessment of Mathematics. Oxford University.

Threlfall, J., Pool, P., Homer, M., & Swinnerton, B. (2007). Implicit aspects of paper and pencil mathematics assessment that come to light through the use of the computer. Educational Studies in Mathematics, 66(3), 335–348.

Singh, A., Karayev, S., Gutowski, K., & Abbeel, P. (2017). Gradescope: a Fast, Flexible, and Fair System for Scalable Assessment of Handwritten Work. *Proceedings L@S '17.*

van der Kleij, F. M., Feskens, R. C. W., & Eggen, T. J. H. M. (2015). Effects of Feedback in a Computer-Based Learning Environment on Students' Learning Outcomes: A Meta-Analysis. *Review of Educational Research*, *85*(4), 475–511.

Digital feedback design in the Heidelberger MatheBrücke

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The Heidelberger MatheBrücke is an online learning environment that contains digital randomized tasks and adaptive feedback for consolidating mastery and understanding of concepts and procedures that are central for secondary school mathematics. It is designed for use in summative learning contexts, which has lead to certain decisions regarding the design of the feedback and the used knowledge model. This paper outlines the underlying theoretical considerations and specifies the MatheBrücke's feedback design and knowledge model.

Keywords: assessment, feedback, basic knowledge

INTRODUCTION

There is a growing interest in digital feedback in mathematics education. In their analysis of the impact of technology on assessment in mathematics from 2012, Stacey and Wiliam made few explicit references to feedback. Of course, assessment without giving some information about the outcome and what to do about it is not helpful, hence feedback must be seen as a necessary part of assessment. An upcoming MEDA analysis of digital assessment systems in mathematics education, however, contains a separate section that explicitly focusses on the feedback design of the digital assessment systems (Fahlgren et al., in press). So why is there a specific interest in feedback when assessment and feedback can be seen as only two sides of the same coin?

Feedback is communication. While assessment concentrates on diagnosis, feedback focusses on the act of informing about the outcome of assessment and how to proceed from there. Furthermore, feedback needs to adapt to cognitive and individual traits of the learner to have the desired effects. Adaption, again, could show in several ways, for example language, content, and timing, all visible especially in feedback relating to single task performance. Yet to be valid, feedback needs a theoretical basis for pointing to differences between expected and actual knowledge in detail. Intelligent Tutoring System design (ITS) requires a knowledge model which serves as a basis for automated response analysis and feedback preparation. So two questions come into view when conceptualizing digital feedback in mathematics education. (a) What are the aspects of mathematical knowledge about which feedback aims to inform? And (b) how should feedback be designed so that it is effective for the learner? In this paper, both questions will be addressed with a focus on the specific demands by the online learning platform Heidelberger MatheBrücke. These will be outlined in the first of the following three sections, before proceeding to answering both questions.

THE HEIDELBERGER MATHEBRÜCKE

The Heidelberger MatheBrücke [1] is an online learning environment with digital randomized tasks and adaptive feedback for consolidating the mastery and understanding of concepts and procedures that are central for secondary school mathematics. The MatheBrücke is to be used in summative, not formative learning situations. This means that students who work with the material are expected to have already some experience with the concepts and procedures addressed. While the MatheBrücke addresses basic concepts and procedures, it does not address higher order competencies such as problem solving or reasoning. This accounts for the third objective of the MatheBrücke. It aims at developing a routine disposition of basic knowledge which can be adapted and applied to a variety of problem solving situations.

KNOWLEDGE MODELLING

An essential prerequisite for adaptive feedback in Intelligent Tutoring Systems (ITS) is a knowledge model against which the actual performance of the student can be assessed. A common approach to domain knowledge modelling requires identifying and defining knowledge components (Brusilovsky & Millán, 2007).

If made explicit at all, the knowledge models of existing assessment systems appear to be based on general cognitive constructs as declarative, procedural and conceptual knowledge (e.g. Tacoma et al., 2018). These are central parts of well-known transdisciplinary knowledge models as e.g. by Bloom, Anderson and Krathwohl. They are also often used in mathematics education (e.g. Hiebert & LeFevre, 1986, de Jong & Ferguson-Hessler, 1996). However, some researchers question the suitability of the procedural/conceptual dichotomy for modelling mathematical knowledge. Star and Stylianides (2013) have shown that mathematics teachers generally see procedural knowledge as being inferior to conceptual knowledge, thus overlooking the epistemological significance of symbolic language in mathematics. For Kent and Foster (2015) procedural fluency itself can be positive evidence of understanding mathematics. In fact, a process model of solving equations by Block (2015) identifies a complex interplay of mental actions that is far from merely applying procedures without reflection. Hence, there seems to be good reason for "abandon[ing] the conceptual/procedural framework entirely and select new words or phrases to describe knowledge outcomes of interest." (Star & Stylianides, 2013, p. 179)

The knowledge model of the MatheBrücke follows a genuine mathematics educational approach to conceptualizing mathematical knowledge. While aspects of declarative, procedural and conceptual knowledge are still, if only implicitly, present, its central constructs are specific forms of accessing or understanding mathematical concept that are well-established within mathematics education. Details will be presented in the following section.

The WiGORA frame of reference

Following the MatheBrücke's aims as outlined above, the knowledge model is meant to be a concise and summative view on what facets of knowledge of a given mathematical object a student needs to have at his or her disposal once it has been taught. Further, the facets of knowledge are considered normative, i.e. they are meant to cover mathematically sound ways of accessing a mathematical object. And last, this framework is for conceptualizing an "intelligent content knowledge base" (Klieme et al., 2007) for developing higher level competencies, it is not a framework for higher level competencies itself. The acronym WiGORA derives from the German labels of the five facets of knowledge that make up the frame of reference (Pinkernell, 2019):

Declarative knowledge ("Wissen") refers to the ability to recall or identify correct definitions, rules or characteristic properties of a mathematical concept or procedure as well as the necessary terminology associated with it. Declarative knowledge basically is knowledge about facts and information (Anderson, 1976). It also comprises prototypical knowledge that characterises, but not necessarily defines, the object (Rosch, 1983, Tall & Bakar, 1992).

Explanatory models ("Grundvorstellungen" or GV for short) refers to the ability to recall or identify conceptualisations of a mathematical object that "make sense" (vom Hofe & Blum, 2016). The concept GV is one of the key concepts of German Stoffdidaktik, which "should be able to, on the one hand, accurately fit to the cognitive qualifications of students and, on the other hand, also capture the substance of the mathematical content at hand" (vom Hofe & Blum, 2016, p. 227).

Representational flexibility ("Repräsentationale Flexibilität") refers to the ability to switch within and between representational forms or registers of a mathematical object. Following Duval (1999), this ability is specific to understanding higher level mathematics since a mathematical concept, being essentially abstract, can not be addressed otherwise.

Operational flexibility ("Operationale Flexibilität") refers to the ability to apply, adapt and modify mathematical procedures for situational needs. This facet refers to the cognitive construct of operations in the sense of Piaget and Aebli. Characterised for example by reversibility or transitivity of the mental operations involved (Fricke, 1970), corresponding tasks would require reversing procedures or selecting efficient procedures over routine ("strategic flexibility": Rittle-Johnson & Star, 2007).

Knowledge application ("Anwendung") refers to the ability to identify a mathematical concept or procedure as suitable for solving a problem. Here, the concept or procedure is considered a potential model for mathematising situations within or outside mathematics ("Mathematisierungsmuster": Tietze, Förster, Klika & Wolpers 2000). This facet, as all five facets do, focusses on meaning and use of a given mathematical object. It does not refer to the modelling process or parts of it, but it addresses the content knowledge base of modelling.

FEEDBACK DESIGN

To be used at the end of a teaching unit or school level, the Heidelberger MatheBrücke is not for summative purposes. Feedback here therefore needs to be addressed to users who have some experience with the mathematical concepts and procedures. The following gives an overview of relevant results from feedback research and then derives the specific feedback design presently being implemented.

Theoretical background

Defining Feedback: For educational purposes, a generally accepted definition of feedback is that of Hattie and Timperley (2007) according to which feedback is "information provided by an agent […] regarding aspects of one's performance or understanding." (p. 81) The addressee is an individual, who could be, for example, a student who is informed about his knowledge of teaching content, or a teacher about his knowledge of the effects of his teaching.

Adaptive feedback: Giving feedback also intends to activate the addressee to close the gap (Boud & Molloy, 2013). Thus, feedback must be perceived as "advice for action" (Ras et al., 2016), it needs to adapt to cognitive and personal traits of the learner (Brusilovsky & Millán, 2007).

simple components of feedback (SF)	knowledge of result (KR)	*i.e. stating whether the answer is correct or wrong*
	knowledge of performance (KP)	*i.e. stating the rate of correctness*
	knowledge of the correct result (KCR)	*e.g. naming or outlining a correct response*
elaborate components of feedback (EF)	knowledge about task constraints (KTC)	*e.g. hints or explanations on type of task or required processing rules*
	knowledge about concepts (KC)	*e.g. hints, explanations or visualizations of concepts, terms, properties*
	knowledge about mistakes (KM)	*e.g. hints or explanations of location and types of errors*
	knowledge about how to proceed (KH)	*e.g. hints, guiding questions or corrections, successful strategies, or worked-out examples*
	knowledge about metacognition (KMC)	*e.g. hints, guiding questions or explanations on metacognitive strategies*

Table 1: Content related classification of feedback components (Narciss, 2008)

Adaptivity manifests itself in content and amount of information and timing as two major parameters of feedback design (Mory, 2004). Timing refers to whether feedback is given immediately or after a specified delay. Content and amount of information refers to type and number of feedback components, for which Narciss (2008) gives a concise overview (cf.table 1).

In fact, research has shown that the use of worked out examples leads to comparably high learning gains (cf. Renkl, 2002). However, while novices profit from worked out examples, this might not hold for students on a higher expert level. Also, timing plays an important role, esp. when looking at the general achieving level as a persistent individual trait. Following Shute (2008), information for low achievers should be made promptly, those for high achievers should be delayed to allow time for revision. A metastudy by Van der Kleij et al. (2015) confirms the significance of elaborate feedback (EF, cp. Table 1) and timing for learning outcomes in digital learning environments. Their findings suggest that the effects of EF as compared with simple (SF) or no feedback at all are more substantial for higher order learning outcomes than for lower learning outcomes. However, timing does not seem to make a difference when looking at its effects on higher or lower order learning outcomes.

Levels of content related feedback The previous findings relate to feedback on performance in single tasks. Task level is one of four levels of the Hattie and Timperley model (2007) which together refer to the addressee's different cognitive and personal traits. When operating at task level or process level, feedback informs about knowledge and processes needed for task performance. When working at self-regulation level and self level, feedback informs about meta-cognitive or affective aspects. It seems that, with upcoming interest in Intelligent Tutoring Systems (ITS), yet another level is needed to locate effects of cognition related feedback. ITS design requires to address aspects of knowledge of a domain as a whole (Brusilovsky & Millán, 2007), so it is feedback at domain level that informs about performance and processes related to concepts and procedures or abilities and competencies that relevant for knowledge of the domain.

Feedback in the MatheBrücke

In accordance with the study results reported above, the MatheBrücke´s task level feedback aims to provide as little information as necessary to activate reflection on each level of expertise. For this, feedback here follows a multi-staged design, each new stage offering more detailed information if needed, which can be accessed only after a delay of a certain time and, additionally, after a deliberate click by the learner. Table 2 shows the outline of a two-staged feedback that has been object of a small-scale study on comparing two types of feedback in the MatheBrücke. The data indicated that even low-achievers could benefit from a two-stage feedback where worked out examples were not given immediately but delayed for 60 seconds (Pinkernell, Gulden, Kalz, in press).

	components		
appears without delay	KR: “Unfortunately, your answer is wrong.”		
	KM: “You probably made this error: …”	*or*	KTC: “The first step of the correct solution would be …”
	“Load another question of the same kind and try again!”		
appears after 60 sec.	KH: “Click here for a worked out example…”		

Table 2: Structure of a two-staged feedback in the Heidelberger MatheBrücke

SUMMARY AND OUTLOOK

The Heidelberger MatheBrücke is an online learning environment with digital randomized tasks and adaptive feedback for consolidating mastery and understanding of concepts and procedures that are central for secondary school mathematics. The underlying knowledge model of each domain of the MatheBrücke comprises (a) central concepts and procedures of the domain that are identified on a curricular basis and (b) facets of mastery and knowledge that are expected to show when engaging with those central concepts and procedures. It is primarily meant for summative learning situations, not formative, hence the feedback is designed to specifically address learners that have some experience with the concepts and procedures of the domain. To adapt to various levels of expertise, the feedback design follows a multi-staged approach, each step increasing the amount of information about correct answers, errors made and processing knowledge that is required for successful solutions. Presently, the knowledge model serves as a frame of reference for preparing and selecting tasks. It is not yet being used for feedback on domain level. Feedback presently operates on task level only, where further designs of multi-staged structure are being explored.

NOTES

1. mathebruecke.pinkernell.online

REFERENCES

Anderson, J. R. (1976). *Language, memory and thought*. Lawrence Erlbaum Associates.

Boud, D., & Molloy, E. (2013). Rethinking models of feedback for learning: The challenge of design. *Assessment & Evaluation in Higher Education*, *38*(6), 698–712.

Brusilovsky, P., & Millán, E. (2007). User Models for Adaptive Hypermedia and Adaptive Educational Systems. In P. Brusilovsky, A. Kobsa, & W. Nejdl (ed.), *The Adaptive Web* (Bd. 4321, p. 3–53). Springer Berlin Heidelberg.

de Jong, T., & Ferguson-Hessler, M. G. M. (1996). Types and qualities of knowledge. *Educational Psychologist*, *31*(2), 105–113.

Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning. In F. Hitt & M. Santos (ed.), *Proceedings of the 21th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education: Bd. I* (p. 3–26).

Fahlgren, M., Brunström, M., Dilling, F., Kristinsdóttir, B., Pinkernell. G, & Weigand, H.-G. (2019). *Technology-rich assessment in mathematics.* Manuscript submitted for publication

Fricke, A. (1970). Operative Lernprinzipien im Mathematikunterricht der Grundschule. In: Fricke, A., Besuden, H.: Mathematik. Elemente einer Didaktik und Methodik, Stuttgart 1970.

Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research*, *77*(1), 81–112.Kent, G., & Foster, C. (2015). Re-conceptualising conceptual understanding in mathematics. *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education*, 2656–2661.

Hiebert, J., & LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawre.

Kent, G., & Foster, C. (2015). Re-conceptualising conceptual understanding in mathematics. *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education*, 2656–2661.

Johnson, C. I., & Priest, H. A. (2005). The feedback principle in multimedia learning. In R. Mayer (ed.), *The Cambridge Handbook of Multimedia Learning* (p. 449–463). Cambridge University Press.

Kluger, A. N., & DeNisi, A. (1996). The Effects of Feedback Interventions on Performance: A Historical Review, a Meta-Analysis, and a Preliminary Feedback Intervention Theory. *Psychological Bulletin*, *119*, 254–284.

Mory, E. H. (2004). Feedback research revisited. In D. H. Jonassen (ed.), *Handbook of research on educational communications and technology* (2nd ed, p. 745–784). Lawrence Erlbaum.

Narciss, S. (2008). Feedback Strategies for Interactive Learning Tasks. In *Handbook of Research on Educational Communications and Technology* (p. 125–144). Lawrence Erlbaum Associates.

Pinkernell, G. (2019). Conceptualising knowledge of mathematical concepts or procedures for diagnostic and supporting measures at university entry level. *Proceedings of the 11th Congress of European Research in Mathematics Education*. https://hal.archives-ouvertes.fr/hal-02422666/document

Pinkernell, G., Gulden, L., & Kalz, M. (in press). Automated Feedback at task level: Error analysis or worked out examples—Which type is more effective? *Proceedings of the 14th International Conference on Technology in Mathematics Teaching*.

Ras, E., Whitelock, D., & Kalz, M. (2016). The promise and potential of e--assessment for learning. In P. Reimann, p. Bull, M. Kickmeier-Rust, R. Vatrapu, & B. Wasson (ed.), *Measuring and visualizing learning in the information-rich classroom* (p. 21–40). Routledge.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction*, *12*(5), 529–556.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, *99*(3), 561–574.

Rosch, E. (1983). Prototype Classification and Logical Classification: The Two Systemp. In E. K. Scholnick (ed.), *New trends in conceptual representation: Challenges to Piaget's theory?* (p. 73–86). L. Erlbaum Associates.

Sangwin, C. J. (2013). *Computer Aided Assessment of Mathematics.* Oxford University Press.

Shute, V. J. (2008). Focus on Formative Feedback. *Review of Educational Research*, *78*(1), 153–189.

Stacey, K., & Wiliam, D. (2012). Technology and Assessment in Mathematics. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Hrsg.), *Third International Handbook of Mathematics Education* (S. 721–751). Springer New York.

Tacoma, S., Sosnovsky, S., Boon, P., Jeuring, J., & Drijvers, P. (2018). The Interplay between Inspectable Student Models and Didactics of Statistics. *Digital Experiences in Mathematics Education*, *4*(2–3), 139–162.

Tall, D., & Bakar, M. (1992). Students' mental prototypes for functions and graphs. *International Journal of Mathematical Education in Science and Technology*, *23*(1), 39–50.

Van der Kleij, F. M., Feskens, R. C. W., & Eggen, T. J. H. M. (2015). Effects of Feedback in a Computer-Based Learning Environment on Students' Learning Outcomes: A Meta-Analysis. *Review of Educational Research*, *85*(4), 475–511.

vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a Category of Subject-Matter Didactics. *Journal Für Mathematik-Didaktik*, *37*(S1), 225–254.

Theme 3

Assessment in Mathematics Education in the Digital Age

Posters

Towards comprehensive technology-supported formative assessment in math education – a literature review

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INTRODUCTION

Formative Assessment (FA) aids to support the improvement of students' comprehension, and identify individual learning needs during the learning process with a variety of methods in the lesson (Wiliam, 2011). It evaluates students' level of competence gain to enhance the ongoing learning process. In math education, FA is particularly important because more than 80% of class time is spent working on tasks and solving problems (Hiebert et al., 2003), which are the main focus areas of FA. This potential has been recognized in research for several years and FA in math has been examined extensively (Rakoczy et al., 2019). The manifold of FA practices, however, has led to the development of a wide variety of tools with different foci and support approaches. This leads to challenges in tool selection and deployment in a given didactical setting. Scientifically, the conceptual and technical developments have largely followed disjoint support strategies and can hardly benefit from each other by adopting their learnings and the empirical findings about their impact. We aim at providing a comprehensive, yet structured, appraisal of existing technology-supported FA instruments in math education. We, therefore, propose a review-framework based on a conceptual foundation in FA and demonstrate its use by reviewing the state-of-the-art on technology-supported FA in math education. We believe this review will serve the interest of teachers, researchers, and technology developers. The former will benefit from this poster by learning about different tools, while the latter two will be able to compare their own approaches to those of peers.

METHODOLOGY

As mentioned above, the amount of existing literature on FA in math education exceeds the limits of what can be reviewed in-depth for the poster proposal. We thus have applied certain boundaries to limit the scope of our review. First, we only seek papers, which *explicitly cover FA topics in math education*. Second, we solely examine papers, which are *published after 2012*, as technological developments lead to the limited value of older contributions. Third, we only analyze papers that aim to *explicitly support FA via technological tools*. We queried the following term in Google Scholar/Web of Science and filtered papers which are not older than 2012.

("formative assessment" OR "formative feedback" OR "formative evaluation" OR "assessment for learning")("maths" OR "math" OR "mathematics")

LITERATURE REVIEW

Wiliam (2000) has identified five different classes of supporting activities that can be included in FA: sharing success criteria with learners (SSC), classroom questioning (CQ), comment-only marking (CM), peer- and self-assessment (PSA), and formative use of summative tests (FST). We analyze the 14 papers we have identified in our literature search with respect to their technical foundations, the types of FA they support, and the empirical evidence they provide on their effects in Table 1.

Literature	Technology[1]	Web	Platform	SSC	CQ	CM	PSA	FST	Scale[2]	Lvl[3]
Azmi & Kankarej, 2015	Various/MD	✓							90S	3
Barana & Marchisio, 2016	Maple/MD	✓	Moodle			✓		✓	2C 100T	2
Brunström & Fahlgren, 2019	CCT		Standalone	?	?	?	?	?		
Cusi et al., 2016	CCT		Standalone		✓				1C 18T	1
Faber et al., 2017	Snappet	✓	Standalone		✓			✓	1808S 24T	1
Gaona et al., 2018	WIRIS	✓	Moodle					✓	5507S	3
Isabwe & Reichert, 2012	P2PASS/MD	✓	Standalone	✓			✓		12S	3
Isabwe, 2012	P2PASS/MD	✓	Standalone				✓		27S	3
Isabwe et al., 2013	MD	?	Standalone	✓			✓		45S	3
Isabwe et al., 2014	A-PASS/MD	✓	Standalone	✓			✓		45S	3
Lee et al., 2012	CRS		Standalone		✓			✓	38T	2
Martin et al., 2016	AMC/MD	✓	Standalone	✓				✓	148T	1
Olsher et al., 2016	STEP/MD	?	GeoGebra		✓					2
Wünsche et al., 2019	CodeRunnerGL	✓	Moodle			✓		✓	300S	3

[1] MD: Mobile Device, CCT: Connected Classroom Technology, CRS: Classroom Response System [2] S: Student, T: Teacher, C: Classroom
[3] 1: Primary, 2: Secondary, 3: Tertiary level of education

Table 1: Concept matrix for reviewed papers.

We reach two major arguments about the use of technology in the FA of math subjects. First, there is no existing technology that spans across all five classes of FA methods. Especially, PSA and CQ results are not combined in a single tool. While this is not surprising given the different foci of support (mainly: presence-based vs. online), the trend towards large-scale blended learning settings makes it worthwhile to explore the potential of a comprehensive support system. Providing such technological support in this field could, in particular, be useful for large-scale classroom settings, where individual formative feedback can hardly be provided by teachers themselves. The review also shows that particular web-based tools are developed with different technologies and most of the systems do not offer any integration into widely-used platforms such as Moodle. Therefore, standardization and integration can be a future problem. This poster guides the reader on how to quickly compare available FA studies that utilize technological tools. We will further explore this direction of research because approaches addressing all identified types of FA methods have a strong potential to guide students in their learning processes based on more complete data and without the need to switch between systems.

REFERENCES

Aral, Nilay, & Oppl, Stefan. (2020). Towards comprehensive technology-supported formative assessment in math education – List of references. Zenodo. http://doi.org/10.5281/zenodo.3950047

Computer-aided assessment based on dynamic mathematics investigations

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In the poster, we will present a planned study focusing on the design of DMS tasks and elaborated feedback within a CAA system. The study will be conducted in a first year engineering mathematics course during autumn 2020.

Keywords: computer-aided assessment, feedback, engineering mathematics, dynamic mathematics software.

BACKGROUND

It is well established that the transition from secondary school mathematics to university mathematics is a major issue among mathematicians and mathematics educators. Larger student groups, and hence less teacher contact, and changes of study methods towards more independent study requires a greater responsibility among students. To tackle this issue, many educators in higher mathematics education have introduced continuing assignments to increase students' engagement (e.g. Rønning, 2017).

At Karlstad University, a developmental project to increase first year engineering students' learning in mathematics was initiated in 2015, based on experiences from research projects at upper secondary school (Fahlgren, 2015). The focus has been on the development of student activities designed for a dynamic mathematics software (DMS) environment, in this case GeoGebra. The intention behind these activities is to deepen students' understanding by providing learning environments where they can explore and communicate mathematics with peers. Course evaluations indicate that students appreciate this part of the course. For example, in the latest course evaluation when requested to answer the open question "What has been good with the computer-based activities?", 119 out of 193 students in some way expressed that it gave them increased understanding. Since the project turned out well, today these activities constitute mandatory parts of the first year engineering mathematics courses at Karlstad University. However, due to the limited time available to the course teachers, the feedback provided to students on their submitted answers has so far only been on correctness. Moreover, the feedback has often been delayed since it has been a challenge for the teacher to assess (in a short time) a large number of student responses. One way to reduce the workload of correction is to outsource it by using technology (Rønning, 2017).

COMPUTER-AIDED ASSESSMENT

The past decade has seen a rapid development of technology that facilitates assessment in mathematics, as well as in other subjects. A common name for these types of technology is computer-aided assessment (CAA) systems. One advantage of using these types of technology is that they are time saving by providing automated correction of student responses. Furthermore, besides providing students with immediate feedback on their submitted answers, CAA can also be used to provide feedback on students' ongoing work, e.g. appropriate hints and suggestions. In addition, there are CAA systems, e.g. STACK, in which it is possible to embed dynamic interactive environments, such as GeoGebra (Sangwin, 2015). However, it is a challenge to design CAA-tasks and elaborated feedback addressing students' conceptual understanding and mathematical reasoning. One way to tackle this could be to create tasks which request students to construct 'examples' that meet certain mathematical conditions (Olsher, Yerushalmy, & Chazan, 2016).

THE PLANNED STUDY

We plan to perform a study during autumn 2020. The focus will be on the design of DMS tasks and elaborated feedback of the ambitious type outlined above within a CAA system. The purpose is to provide feedback based on students' responses. The aim of the study is to investigate students' utilization and perception of various types of elaborated feedback provided in a CAA system. The study is planned to be conducted in a first year engineering mathematics course involving approximately 200 students. To better understand students' way of using (or not using) the provided feedback, we plan to perform a survey that will be followed up by focus group interviews. The main focus will be on comparing and contrasting what impact various types of elaborated digitized feedback might have on students' learning strategies.

REFERENCES

Fahlgren, M. (2015). *Designing for the integration of dynamic software environments in the teaching of mathematics*. Dissertation, Karlstad, Sweden: Karlstad University.

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How Might the Use of Technology in Formative Assessment Support Changes in Mathematics Teaching? *For the learning of mathematics, 36*(3), 11-18.

Rønning, F. (2017). Influence of computer-aided assessment on ways of working with mathematics. *Teaching Mathematics and its Applications, 36*(2), 94-107.

Sangwin, C. (2015). Computer aided assessment of mathematics using STACK. In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 695-713). Switzerland: Springer.

Silent video tasks and the importance of teacher collaboration for task development

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In a design-based research project aiming to develop silent video tasks, results from a first implementation phase indicated that silent video tasks might be a helpful tool for teachers practicing formative assessment. Consequently, the second implementation phase was planned with upper secondary school teachers who had previous experience with formative assessment methods. This poster displays important aspects of collaboration with teachers in task design.

Keywords: task design, formative assessment, use of technology, silent video tasks, upper secondary school.

Silent videos are short (less than 2 minutes long) animated video clips that show mathematics dynamically without text or sound. With the aim to assess students' understanding of the video's mathematical topic, the teacher selects and shows a silent video to students as a whole group before splitting them into groups of two that prepare and record their voice-over for the video. The video topic is usually connected to a topic covered in the previous week(s). For example, the video topic could be chosen to be the area of a triangle, linear functions, or the unit circle. Students' responses get listened and reacted to in a whole group discussion lead by the teacher addressing issues such as word use, clarity, meaning, and understanding. This discussion's goal is to reach some common understanding of the mathematics shown in the video. After the first implementation phase of this design-based research project, where I (the presenting author) worked with four randomly selected Icelandic upper secondary school mathematics teachers and their students, results indicated that silent video tasks could be a useful tool for teachers using formative assessment. Comparing them to the following six characteristics of technology-based formative assessment strategies, defined in the FaSMEd project (Wright, Clark, & Tiplady, 2018), silent video tasks seemed to fulfil all but the first criteria: i) *provide immediate feedback*: this was not fulfilled as feedback was given 1-3 day(s) later; ii) *encourage discussion*: students communicated both when preparing and listening to their responses to the silent video task; iii) *provide a meaningful way to represent problems and misunderstandings*: on the basis of students' responses, misunderstandings could be discussed in teacher-lead group discussion; iv) *give opportunities to use preferred strategies in new ways*: when preparing recordings, students–who normally sat silently in mathematics class–talked about mathematics among themselves v) *help raising issues that were previously not transparent for teachers*: students' responses uncovered misunderstandings that teachers did not expect and had not been present in other conventional tasks; and vi) *provide different outcomes feedback*: it was possible to give feedback to the whole class via discussion but it depended on the teacher and their habits or beliefs, whether they did so or not (Kristinsdóttir, Hreinsdóttir, & Lavicza, 2020). To develop the silent video tasks further in a second phase, I worked with three teachers and their students in two upper secondary schools. The schools were purposefully selected to have teachers with previous experience of using formative assessment and technology in their mathematics classrooms. Even though I had indications that silent video tasks might be useful for formative assessment, I wanted to see how teachers with experience of formative assessment would use them. Would they use students' responses and the results from the group discussion to make decisions about the next steps in instruction as suggested by Wiliam (2011, p. 43). The research question was: How can silent video tasks be used for formative assessment in the mathematics classroom? During preparatory interviews in the second implementation phase, before implementing a silent video task, the three teachers underlined the importance of immediate feedback. They suggested to bring the whole group discussion forward such that it would be included in the lesson immediately after receiving students' responses – as opposed to playing them in a follow-up lesson 1-3 day(s) later. Also, they suggested that instead of only selecting some sample responses from students, all responses would be listened and reacted to. Similar to findings by Hoppe, De Groot, and Hever (2009), teachers were active and critical not only as adopters of this new tool in the classroom but also as co-designers in the development process. Further preliminary results regarding the participating teachers' influence on the development of the task instructional sequence will be presented on this poster.

REFERENCES

Hoppe, H. U., De Groot, R., & Hever, R. (2009). Implementing technology-facilitated collaboration and awareness in the classroom. In B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of Knowledge through Classroom Interaction*. (pp. 130–142). Routledge. doi.org/10.4324/9780203879276

Kristinsdóttir, B., Hreinsdóttir, F., & Lavicza, Z. (2020). Using silent video tasks for formative assessment. In B. Barzel, R. Bebernik, L. Göbel, M. Pohl, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *Proceedings of the 14th International Conference on Technology in Mathematics Teaching—ICTMT 14* (pp. 189–196). University of Duisburg-Essen. doi.org/10.17185/duepublico/70763

Wiliam, D. (2011). *Embedded Formative Assessment*. Solution Tree Press.

Wright, D., Clark, J., & Tiplady, L. (2018). Designing for formative assessment: A toolkit for teachers. In D.R. Thompson, M. Burton, A. Cusi, & D. Wright (Eds.), *Classroom assessment in mathematics: Perspectives from around the globe* (pp. 207–228). Cham, Switzerland: Springer.

Theme 4

Theoretical Perspectives and Methodologies/Approaches to Conduct Research in Mathematics Education in the Digital Age

Papers

To learn about differential equations by modelling

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Expressive modelling is in harmony with the Danish mathematics education narrative where students must work intellectual independently and perform creative mathematical reasoning. This paper reports from a case study of students' modelling differential equations systems using exploratory, tailored interactive ICT tools. The study aimed to throw light on the question, if and how the students' use of exploratory tools contributed to their concept formation in terms of emergent models. The students in the study's two cases took distinct starting points for the modelling process. However, their reports in both cases showed signs of emergent models of the differential equations models.

Keywords: Differential equations modelling, exploratory ICT tools, emergent models of differential equations, epidemic models.

INTRODUCTION

This paper presents a case study of students' modelling with differential equation (DE) systems using a variety of ICT tools. The aim of the case study was to give an insight of the interplay between the DE model, the problem modelled, and the tool used for modelling. It focused on how exploratory ICT tools could not only mediate the *exploration of a model*, but also support *expressive modelling* in the meaning of students' reinvention of the DE model. The study took a mathematics education point of view on modelling, with emphasis on modelling for concept formation in terms of *emergent modelling.* Its basis included results about the interplay between students' modelling and the use of ICT (Andresen, 2006).

THEORETICAL FRAMEWORK

Modelling for concept formation: 'Emergent Models'

Hans Freudenthal's view on mathematics and the principles of Realistic Mathematics Education (RME) are basic to the articulation of concept formation in terms of *emergent models* (Gravemeijer & Stephan 2002): Horizontal and vertical mathematizing are characterised in RME, in (Gravemeijer & Stephan 2002), by the passing of four levels of activity (situational, referential, general and formal). The progressive mathematizing is driven by reflections where a new mathematical reality is created at each level. This four-layer model was basis for the design heuristics of emergent (mental) models (Gravemeijer & Stephan 2002) and, according to Cobb (2002, p193) the model might '*facilitate (...) the analysis of mathematical learning in instructional situations*'. Further:'*The explication of a mapping between a situation and a model might then be viewed as a description of the way that the situation became structured during modeling activity.*' (Cobb 2002 p 193). In this study, the four layer model serves to facilitate the analysis of students' concept formation by offering a structure: the textual analyses display signs of students' activity stratified with regard to the levels. The progressive mathematisation, then, is detected as variation between the levels and interpreted as (elements of) the students' concept formation in the form of an emergent model of the mathematical content. Accordingly, the learning outcome is viewed as emergent models of essential mathematical concepts, following Cobb (2002). In this study of modelling with DE the essential concepts encompass solution, slope field, equilibrium solution and null-clines (examples below). Besides, students' learning about DE models as such was also a goal in the two cases.

The distinction between explorative and expressive modelling

This study draws on a distinction between expressive and explorative modeling. *Expressive modeling* is understood as an activity aiming to capture a problem by a mathematical model, be it ready-made or under creation. It encompasses, hence, expression in mathematical terms of quantities and relations in connection with some sort of problem solving. Expressive modeling is in harmony with Niss' description of a Danish mathematics education narrative where students have to work intellectual independently (Niss 2020 p 320). The creative mathematical reasoning (CMR) by Lithner (2008) is pivotal in this narrative. Expressive modeling is also in harmony with the design heuristics of emergent models. *Explorative modeling* is understood as an activity aiming to explore a mathematical model, be it ready-made or under creation. Explorative modeling is, for example, the prevailing form in school mathematics modeling tasks that present a model and ask for the results of giving a certain input. (Berget & Bolstad 2019, in Norwegian). In the case of DE models students can explore a model, i.e. the epidemic SIR models (see 1.1 in Table 1), by the use of tailored, interactive ICT tools. Explorations may encompass students' producing graphs or running simulations based on various input and adjustments of parameters. Such tailored ICT tools were in the study labelled '*exploratory tools*'. Sequences of expressive modeling will normally encompass processes of explorative modeling. The modelling sequence in Blomhøj and Jensen (2003) is overall expressive: It encompasses six sub-processes, each of them requiring creative non-routine activities. However, those sub-processes will include exploration and try out of (parts of) the (mathematical) models under construction. Andresen concluded (2007) that sequences of explorative work may serve to support the students' concept formation and at the same time prepare them for expressive modelling. In this study, the direction (explorative vs expressive) of modeling is detected in smaller and larger subsections of the overall modeling proces described in the texts, by condensation and interpretation of the meaning of the subsection.

Interplay between tool and modelling

In this study the students used exploratory (interactive ICT) tools designed for DE models, i.e. the epidemic SIR model [1]. According to the theory of instrumental genesis an artefact, i.e. the exploratory tool, becomes useful, and then denoted an instrument, only after the user's formation of mental utilisation scheme(s) (Drijvers & Gravemeijer 2005). The instrumental genesis proceed through activities in a two-sided relationship as a process in which the tool impacts the learner's thinking simoultaneously with the learner's getting familiar with the tool. In this study students explored the explorative tool and, simultaneously, explored the SIR model by use of tools. Thereby, they would generate an instrument from the explorative tool. For the students, this process of instrumental genesis was tightly intertwined with the process of modelling the real-world problem with DE systems. Modelling a real world problem would request expressive modelling and imply students' use of tools for that purpose. The students, therefore, needed to generate one or more instruments to fulfil the demands of expressive modelling of the real-world problem. Similarly, the students' getting familiar with the exploratory tool by exploring it, implied that they explored the SIR model. The requested generation of an instrument was intertwined both with explorative and expressive modelling of the epidemic. This entwinement is the core of the interplay between tool and modeling in the study.

METHODS

Materials for analysis

This study analyses 2 cases of modelling epidemics with DE systems. The cases occurred in the course 'Modelling in and for mathematics teaching and learning' which is a 15 ECTS compulsory course in our masters' programme in mathematics education for teachers, in the following called students. Our programme requests at least 60 ECTS in mathematics and at least two years of professional practice as mathematics teachers. The course was based, among others, on the textbook (Blanchard, P., Devaney, R.L. and Hall, G. R., 2002), and included lectures on DE, mathematical modelling, and on mathematics education content; mathematics tasks; and two projects. In the first of the two projects the students worked in groups (2-3 persons) with the aim "*To formulate, complete and present a project that encompasses a simple differential equation model.*" Students' learning goals of this project were: 1) to learn about DE by doing a modelling project, and 2) to get personal experiences with learning mathematics from doing a modelling project. Each group was free to choose what problem and what DE model they wanted to study, and what tools they wanted to use for the study. The students were not familiar with mathematical modelling; neither did they know the DE models in advance. At the course's oral, individual examination, the students were interviewed about their projects (10 minutes) besides 10 minutes talk about DE. In the final course evaluation, each of the two project reports counted 25%, and the oral examination counted 50%.

The two cases are based on group reports from the modelling project, picked out of 17 reports prepared and submitted by groups of students and evaluated by the author, in 2014 – 2019. In the two reports the students used exploratory tools tailored for the epidemic models SIR, SEIR and SEIRS. The reports were picked out from the sample because they had the DE models and the exploratory tool in common and were prepared by gender – mixed groups. They were graded 'B' (best 10 - 15%) which reflected the general level of reports in the course. Further, they were prepared in two different semesters, and the modelling processes had distinct overall directions (explorative and expressive, respectively+).

Research question:

How can exploratory tools contribute to students' concept formation.

Analysis

Each case implied a qualitative textual analysis (Kvale 2001; Ju & Kwon 2007) of one report (15 and 30 pages, in Norwegian). The overall direction of the modelling process and all the applied tools were identified. For each of the mathematical concept(s) related to DE, signs in the text were identified and interpreted (not disjunctively coded) regarding and the level of activity in accordance with (Gravemeijer & Stephan 2002): 1) situational level with descriptions in natural language and own wordings, 2) referential level where a 'model of' was created and inquired. A 'model of' was identified by the students' use of situation related terms and half-way formalised explanations, for example, that 'the amount of sick persons will grow exponentially over time', 3) general level with creation and handling of a 'model for'. A 'model for' was identified by the students' use of general expressions and terms with no visible relation to the situation, for example, that 'We find that the graph of I(t) hits the maximum value if the parameter has a value of 0.259' and 4) formal level with general reasoning and considerations, which were very rare in the reports. In each case, then, the concept formation in terms of progressive mathematization was condensed. The 'direction' of modelling was interpreted based on meaning condensation of naturally delineated subsections of the text. The delineation of subsections was not a division into disjointed classes; smaller subsections of explorative modelling could be embedded in an expressive modelling section and vice versa. The subsection's direction was labelled a) expressive modelling, i.e. the process was driven by the problem or b) explorative modelling, i.e. the model was the starting point. Finally, the two cases were juxtaposed; the combinations of tools, concept formation and directions of modelling were discussed with the aim to throw light on the research question.

DATA AND FINDINGS

Case 1

In case 1 the students reported on modelling an epidemic flu in a boarding school. Data was found in a table in the textbook (Blanchard et al. 2002). The SIR model was chosen from the outset. The students used exploratory tools tailored for SIR (and, later on, for the SEIR and SEIRS epidemic models). In the excerpts (Table 1) from the first part of the report the students use 'Solver for the SIR epidemic model' by Warren Weckesser as their tool. In general, and in the sub-processes, the students took as their starting point the DE model rather than the flu epidemic. The models were explored successively by applying them at the original set of data and juxtaposing the results. Both in the case of SIR and, later on, SEIR and SEIRS, the exploratory tool was used to explore the model. The subsections where the model's parameters were fitted with data by use of the tool, though, were expressive according to the description above. From an overall perspective, the successive exploration of various models, fitting their parameters with data and, finally, juxtaposing the results, could be interpreted as an expressive modelling process. The frequent variation between the levels of activity indicate progressive mathematising which lead to emergent models, e.g. of DE systems and of the SIR model's mechanisms and parameters.

Case 1, Excerpts. Translated from Norwegian by the author	Level of activity	Tool, direction	Concepts, notions
1.1 This leads to the following model: Equation 1: $dS/dt{=}{-}\alpha SI$ Equation 2: $dI/dt{=}\alpha SI{-}\beta I$ Equation 3: $dR/dt{=}\beta I$ The system is autonomous, since the changes of S, I and R over time t depend only on the dependent variables, and not on the independent variable t	3): model for	No tool Explor.	SIR model, DE system, autonomous system
1.2 The parameters α and β in equations 1,2 and 3 in the SIR model from section 2.1 are a mathematical description of how the population (pupils) will move between the three groups S, I and R. The parameters are fixed number values which vary according to which disease spread to be modelled. The parameter α describes how quickly the population gets infected, while β describes how quickly the infected get healthy	3) → 2): model for → model of	No tool Explor.	Parameters of the SIR model
1.3 To determine β Warren Weckesser's model was used. After 6 days, the peak for I (t) appears to be 282763≈0.37 when β = 0.44	3): model for	Tool. Explor	SIR model: recovery rate β;
1.4 We also read the same result from figures 2 and 3. This value, which is 0.025, says that 2.5% did not get the disease during the course of the epidemic. In our report on the outbreak of illness at a boarding school in England, this means that out of 763 people, 19 will not be registered with symptoms.	3) → 1): model for → situation	Tool.	Parameters in SIR

1.5 The curve shows an increase in the number of infected in group I until the vectors change direction and shows a rapid decrease in the number of infected individuals in I.	2); model of	Tool. Explor.	Direction field, mechanisms of SIR

Table 1: Case 1

Case 2

The students in case 2 studied the development of epidemics, based on WHO data about Ebola in Congo. Firstly, though, they used GeoGebra for a preliminary inquiry of mechanisms in the SIR model, and for an exploration of the SIR model (using the same data as in case 1). They turned to model the Ebola epidemic in Congo by SIR using successively GeoGebra, the ‘Slope and Direction Field’ tool and the Ness SIR tool. The students modified SIR to SEIR and then inquired the mechanisms in the new model (SEIR). They pinpointed characteristics not only of the models but also of the epidemics, such as the population’s geographical distribution, the individual and collective interaction, and realistic values of the models’ parameters. In case 2the students explored the models by applications of them on data and, vice versa, they also used the models to inquire data about epidemics. The excerpts (Table 2) from case 2 illustrate a frequent variation between the levels of activity which indicate students’ concept formation in terms of emergent models, i.e. of the SIR model and its characteristics.

Case 2, Excerpts. Translated from Norwegian by the author	Level of activity:	Tool, Direction	Concepts, notions
2.1 β = 0.0714 was probably not the best value to choose. This implies that it takes an average of 14 weeks for a person to recover. If we instead think that on average it takes a week to recover from the flu, then $\beta = 1$.	1) → 2); situation → model of	GeoGebr. Expres.	Recovery rate β
2.2 At lower α there is no solution as the two curves are not intersected. Therefore, we see that to get $Imax$=282/763 with $\beta = 1$ must α=3.64. R0=α/β=3.64/1=3.64 that is about the same value we got last time	3); model for	GeoGebr. Explor.	Ranges of α; infection rate α; recovery rate β; reproduct. factor R_0
2.3 We will look at how the model can be expanded and adapted to work more in line with our data and initiatives to stop the Ebola outbreak.	4) Formal	SIR tool Expres.	SIR

2.4 We added 300 days here to compare with the SIR curve, but do not see if the total number of infected is affected by this. Therefore, we also look at 500 days. It appears that S (t) and R (t) flatten out at the same values as for SIR.	2) → 1); model of → situation	SEIR tool Explor.	SEIR and SIR

Table 2: Case 2

CONCLUSION

The exploratory tools were used for exploration in both cases. In case 1 the students' modelling process took outset from the DE models, either expressed in formal mathematics (1.1; 1.2) or expressed by the explorative tool (1.3; 1.4; 1.5).Since the students took the SIR model as their starting point, the exploratory tool was closely fitted with the model and strongly supported their mathematising. In case 2 the students' modelling process took outset from the epidemics (2.3;2.4). They used the tools for expressive modelling (2.1) which encompassed subsections of explorations (2.2) by use of the same tools. Table 2 includes examples (2.1; 2.2) of expressive and explorative use of GeoGebra which is not designed as an exploratory tool as well as examples (2.3; 2.4) of expressive and explorative use of exploratory tools. This illustrates that the students in case 2 managed to use either types of tool in either modelling situation. Both types of tools supported their mathematising.

The textual analysis in both cases showed formation of central concepts in DE, exemplified in Table 1 and 2 regarding the SIR model and its parameters. The exploratory tools served to support students' concept formation in modelling processes, either as tools for explorative or expressive modelling subsections of the process.

NOTE

1. Tools used by the students:
   - SIR tool Weckesser, Warren (2007) Solver for the SIR Epidemic Model;
   - SEIRS tool Nesse, Hans (2015), Global Health-SIR model (with vaccine);
   - SEIR tool: Nesse, Hans (2015), Global Health-SEIR-model; all three found at: http://math.colgate.edu/~wweckesser/software/
   - Slope and Direction Fields, found at: https://homepages.bluffton.edu/~nesterd/java/slopefields.html;
   - GeoGebra; found at: https://www.geogebra.org/classic

LITERATURE

Andresen, M. (2006). *Taking advantage of computer use for increased flexibility of mathematical conceptions* (Ph.D. dissertation). Copenhagen. Danish University of Education. ISBN: 877684069-7

Andresen, M. (2007). Understandings of 'modelling'. In Demetra Pitta-Pantazi and George Philippou (eds.): *Proceedings of the Fifth Congress of the European Society* *for Research in Mathematics Education. 22-26 February 2007, Larnaca, Cyprus* (2042-2049).

Berget, Ingeborg & Bolstad, Oda. (2019). Perspektiv på matematisk modellering i Kunnskapsløftet og Fagfornyinga. *Nordisk tidsskrift for utdanning og praksis*. 13. 83-97. 10.23865/up.v13.1882.

Blanchard, P., Devaney, R.L. & Hall, G. R. (2002). *Differential equations*. Brooks/Cole, USA. ISBN 0-534-38514-1

Blomhøj, M. & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and its Appl. 22* (3) 123-139

Cobb, P. (2002). Modeling, symbolizing and tool use in statistical data analysis. In: Gravemeijer, K., Lehrer, R., van Oers, B. & Verschaffel, L. (eds.) *Symbolizing, modelling and tool use in mathematics education* (171-195). Kluwer Academic Publishers, Dordrecht.

Drijvers, P. & Gravemeijer, K.(2005). Computer Algebra as an Instrument. In Guin, D., Ruthven, K. & Trouche, L. (eds.) *The Didactical Challenge of Symbolic Calculators* (163-196). Springer.

Gravemeijer, K. & Stephan, M. (2002) Emergent models as an instructional design heuristic. In: Gravemeijer, K., Lehrer, R., van Oers, B. & Verschaffel, L. (eds.) *Symbolizing, modelling and tool use in mathematics education (*145-169). Kluwer Academic Publishers, Dordrecht.

Ju, Mi-K., & Kwon, O. N. (2007). Ways of talking and ways of positioning: Students' beliefs in an inquiry-oriented differential equations class. *Journal of Mathematical Behavior 26*, 267-280

Kvale, S. (2001). *Interview. En introduktion til det kvalitative forskningsinterview.* Hans Reitzels Forlag. Copenhagen, Denmark. (In Danish)

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies of Mathematics 67,* 255-276

Niss, M. (2020). The impact of Dutch mathematics education on Danish mathematics education. In: Heuvel-Panhuizen, M. van den (ed.). *International reflections on the Netherlands didactics of mathematics.* (317-324). ICME-13 Monographs, Springer

Students' experiences with dynamic geometry software and its mediation on mathematical communication competency

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This paper presents preliminary findings of a design experiment conducted in an everyday classroom addressing the interplay between mathematical communication competency and the use of GeoGebra. Results indicate that both the students' instrumentation behavioural profiles and their communication profiles are tightly interconnected and both are influenced by the students' mathematical knowledge.

Keywords: communication, competencies, instrumental genesis, GeoGebra.

INTRODUCTION

Both mathematical competencies and use of digital technologies have influenced the Danish school system (Niss & Højgaard, 2019). The mathematical competency framework (KOM) consists of eight competencies, all embedded in the Danish mathematics curriculum for primary and secondary school (Ministry of Education, 2019). But, research on the interplay between competencies and the use of digital technology is very limited, specifically with respect to mathematical communication. Some research indicates that using technology may foster communication in class (Drijvers, Ball, Barzel, Heid, Cao, & Maschietto, 2016). Jungwirth (2006) describes that mathematical communication changes when using technology into "computer-related talk", i.e. *empractical talk*, which tends to be "less connex, less coherent, but more linked to the context than conversational talk" (p. 378). In this case, mathematics is used more separately from the digital tool. Yet, it is not clear, how mathematical communication and technology use are related. Research in this area is highly relevant because technology use in class necessarily entails communication as talking is present in everyday teaching and students' mathematical communication competency may mediate their experience with digital technology and vice versa. We therefore aim at exploring how mathematical communication competency and technology use in mathematics mutually mediate on each other. To achieve this aim, a dedicated task is designed for an empirical case study conducted in a lower secondary classroom. Research questions are: *How is the students' (aged 14-16) use of a digital tool (i.e. GeoGebra) related to their oral mathematical communication and what conditions hinder or foster using the tool as well as activating mathematical communication competency?*

RESEARCH FRAMEWORK

KOM defines mathematical competency as "someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations" (Niss & Højgaard, 2019, p. 14). This is applied to the mathematical communication competency and students' use of technology. Mathematical communication competency contains two aspects: First, the student as a *sender* of information must be able to express oneself mathematically; second, the student as a *receiver* of mathematical expressions must be able to interpret and study other's expressions as mathematical ones (Niss & Højgaard, 2019). Since *mathematical objects* are mental object, they are only accessible by *mathematical representations*. Representations can be multifunctional, not specific for mathematics (e.g. *natural language* and geometric figures) or mono-functional, specific for mathematics (e.g. symbolic languages and graphs) (Duval, 2017). In mathematical communication, at least one mathematical representation is used. In oral mathematical communication a term can expressed, graphs described and so forth. Mathematical communication can take various forms: written, oral, visual and gestural. A competent student can engage in mathematical communication with different people and in different contexts. Three dimensions characterise a student's competency in a given situation. *Degree of coverage* concerns student's ability to bring both aspects of the competency into action. With respect to communication, it is to *send* and *receive*. *Radius of actions* refers to how many task contexts the student is capable to activate when communicating mathematically (e.g. linear functions only, or linear functions applied to velocity). *Technical level* refers to how complex the student communicates with respect to conceptualisation and technicalities (Niss & Højgaard, 2019).

Using Guin and Trouche (1998), instrumental genesis describes the process of mastering transforming an artefact into an instrument for a student. Such a process involves instrumentation, which describes how the tool affects students' actions and their use of the tool, for instance, strategies or techniques when solving a task. Guin and Trouche have identified five student instrumentation profiles when using CAS, which entail their mathematical knowledge as well as their behaviour, addressing the relation between CAS and the graphical environment. The profiles vary and they are situational, meaning that they describe how students act and handle a task in a certain situation. Students with a *random work method* show difficulties with both CAS and paper-pencil, many errors and no use of verification strategies. These students use trial-error and copy-paste techniques. The *mechanical work method* characterises students who do limited calculations and only simple manipulations. Reasoning bases on machine results and mathematical argumentation is shortened and needs investigations to understand. The *resourceful work method* shows various kinds of investigation strategies: calculator, paper-pencil techniques and theory knowledge. Students with a *rational work method* are good at reviewing available information and tools and they do not prefer to utilise the calculator. Instead, they tend to use paper-pencil techniques for argumentation or proving. Finally, students with a *theoretical work method* use mathematical knowledge systematically, verify machine results and do semiotic interpretations to understand.

METHODOLOGY AND METHOD

Our case study was conducted in a (regular) 9th grade mathematics classroom (students aged 14-16) in Denmark in December 2019. The participating class utilises GeoGebra in everyday mathematics teaching and in testing situations.

Presentation of the task

We designed our task using the principle of *zooming-in/zooming-out*, allowing us to focus on geometry and functions, as well as on using technology and communicating alternately. The task environment is created for communication by forcing the students to share their experience, distribute responsibilities, explore a conjecture and test it. By embedding different representations in GeoGebra, we expected to activate further communication (Guin & Trouche, 1998; Niss & Højgaard, 2019). The task is based on a task described in Johnson and McClintock (2018), originally focusing on quantitative reasoning and the understanding of functions as co-variations where students "conceive of functions as specialised relationships between quantities" (p. 303). We revised this task to make it fit our purpose of research. First, we built on concepts that the students are already familiar with (e.g. rectangles and triangles). Second, we rearranged the setting. The task consists of two parts: 1) filling a rectangle, 2) filling a triangle. The students got a paper for notetaking and links to the pre-set of GeoGebra Templates as in Fig. 1.

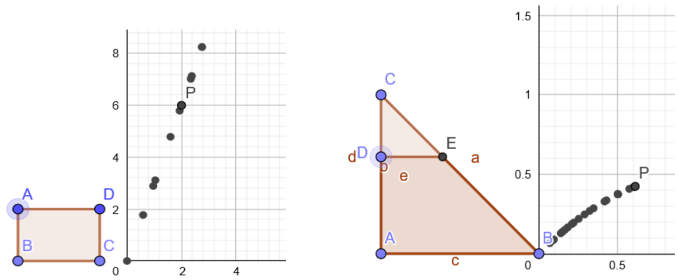

Figure 1. Left side: Task 1 – Filling rectangles. Right side: Task 2 – Filling triangles.

Both tasks have eight subtasks. Subtasks for the individual students alternate with tasks for student pairs. The questions concern the relationship between the figures to the left and point *P* placed in the coordinate system. In both tasks, the students have to present the algebraic expression of the function *p* and construct the graph in the coordinate system in GeoGebra. The function *p* states the relationship between the height of a figure (task 1: rectangle, task 2: triangle/trapezoid) and its area. In task 1, *p* is a linear function $y=3x$. In task 2, *p is* a quadratic function $y=-0.5x^2+x$.

Data include three subtasks. In subtask 1, the students identify the figures on the left and measure height/width and state the area. In subtask 5, the students individually fill a table, stating the area of the rectangle to a given height (e.g. Fig. 2). In subtask 6, the students have to construct the functional relation between *AB* and the area of the rectangle by writing the algebraic expression in the input field. Further, they have to name the type of function and identify the dependent and independent variables.

Data collection and aggregation

Data are collected in a class of 23 students who worked for four hours in pairs. They used one computer each because we want to understand the individual process of instrumental genesis (Guin & Trouche, 1998). The teacher chose two focus pairs. Both pairs are high-achievers in mathematics, and one of the pairs (the one in transcript 3) is proficient with technology, too. The purpose of this case study is to reveal theoretical insight. To achieve this, contrasting cases sharing a specific commonality are useful to consider. With respect to our topic, we have chosen proficiency with technology as the contrasting dimension, and high achievement in mathematics as the commonality. The latter choice was done because difficulties high achievers show may exist among other students, but not vice versa.

Data comprise video recordings of the two focus pairs working together, and four individual screencasts from all students in the focus pairs. Screencast included audio screens and webcam recording, and the students' worksheets. Video data are transcribed *true verbatim*, after translation *intelligent verbatim* but including the students' interactions with GeoGebra. Data are analysed based on the theoretical concepts provided in the framework.

TRANSCRIPTS OF STUDENTS' WORK AND THEIR DATA ANALYSES

Transcript 1: Clara and Dea work with subtask 1, talking about the rectangle.

Clara and Dea just opened the task (Fig. 1, figure to the left).

1	Clara	Talk about what you see in GeoGebra. Isn't is a rectangle?
2	Dea	Yes it is just a rectangle. Identify height, width and area of the figure.
3	Clara	You just have to use the functions there [*she is looking through all the buttons to find the right one*].
4	Dea	The height, we can see it.
5	Clara	principally, we can calculate it. [*Still turning the mouse around to find the right button to hit*]. 12.28 [*the area is stated in the rectangle*].
6	Dea	The height must be 4. Can't you measure, you can see how long the line segments are? [*Clara looks at buttons, Dea in the input field*].
7	Dea	We can just look in the side. Line segment *BC* is 3.
8	Clara	That must be the width, right?
9	Dea	Yes.
10	Clara	Then the other must be…
11	Dea	We can see that…
12	Clara	Line segment *AB*. (Dea waits *4.09* and *points on the screen with her pen*.]

Transcript 2: Clara and Dea try to construct the function *p*

In Task 1, Subtask 5, they individually fill the table in Fig. 2. In transcript 2, Clara and Dea work on subtask 6 trying to find an algebraic expression for the function *p*.

Højden af AB	1cm	3cm	5cm	7cm	10 cm
Areal af rektangel, ABCD	3	9	15	21	30

Figure 2. Clara's table stating the relation between the height, AB, and the area of ABCD.

14	Dea	We can see that it increases with 3 cm^2 for every cm AB increases with 1. That is the relationship. Height of *AB* is 1, then the area of the rectangle is 3. [*After a short while, a teacher comes in*.]
15	Teacher	Yes, what is the value of *x* when looking at the figure? If you look at point *P*, what is the x-coordinate then the same as? And the *y*-coordinate for *P*?
16	Clara	The area.
17	Teacher	What characteristics does a function have? How do you make a function?
18	Dea	An *x* and a *y*.
19	Clara	But what if it changes all the time [*pointing at the point P*].

Dea begins to write in the input field, x=, then x=3, instead of an equation. She realises that it is wrong and writes x=6. This is also wrong, she quits. Finally, she writes x=ab3.

Clara writes f(x)=, then she googles 'function equation', turns back to GeoGebra and writes f(x)=1x+3, looks at the graph done by GeoGebra and deletes it.

Transcript 3: Adam and Ben construct the function p.

In Transcript 3, Adam and Ben do not have the same results in subtask 5. As Adam and Ben construct the graph in subtask 6, the conversation starts:

20	Adam	What kind of function is it?
21	Ben	It is this one [*he points at the table from subtask 5*]. I think we should use yours. It is a little easier.
22	Adam	Then we write down here [*he points with the mouse at the input field*]
23	Ben	Every time it moves …
24	Adam	Isn't just like *y=3x* [*He writes the function in the input field and moves a bit around in GeoGebra with the navigation, looking for P – to see if P is on the line*]. So now we actually have made that function, right?
25	Ben	Yes. It is a linear function.
26	Adam	Function equation. It is a table equation. Linear function?
27	Ben	Linear.

After, they identify the dependent and the independent variable in the function.

ANALYSIS OF MATHEMATICAL COMMUNICATION COMPETENCY

In this paper, we focus on mathematical representations of mathematical objects (Niss & Højgaard, 2019; Duval, 2017) by identifying the use of representations, and how the students' relate them (Duval, 2017) and whether and how they understand functions as co-variation (Johnson & McClintock, 2018).

Communication in Transcript 1 concerns rectangle as *object*, and students' ability to identify width, height and area. *Representations* involved are rectangle, numbers and mathematical terms. Both students are active as senders and receivers by building on each other's expressions. The talk however is empractical: Focus is on the use of GeoGebra as they discuss how to find height and area (Jungwirth, 2006).

The *object* changes to functions in the following transcripts. Transcript 2 shows communication consisting of the following *representations*: table, mathematical terms, algebraic expression in GeoGebra, graphs in the coordinate system and rectangle. Clara and Dea show difficulties when translating between representations: Dea writes $x=3$ and $x=6$ when trying to construct the function p. Translation is also difficult for Clara who writes $y=1x+3$ (instead of $y=3x$). They are able to express the relationship between the height and the area in the table in natural language (line 14) (Duval, 2017), but they do not identify variables. Therefore, they cannot regard functions as specialised relationships between variables; hence, they do not address co-variation (line 19). Missing to translate the representations correctly makes their communication less clear because the relationships between the representations are not stated. They rather use terms and state relationships within the rectangle (line 14), not identifying the variables within the function and sharing information and techniques nor build on each other's statements. Drawing together, their communication competency appears to be *empractical*, since they *receive and do* (teacher sends, the students receive and do in GeoGebra, line 15-18).

In Transcript 3, Adam and Ben utilise table, natural language using mathematical terms, algebraic expression in GeoGebra, point P and the graph. Adam and Ben translate from one representation to another (line 20-25) (Duval, 2017). They show well developed understanding of functions as co-variations, knowing dependent and independent variables, hence, regarding functions as a specialised relationship. Adam and Ben interpret and state the different representations, taking receiving roles and they use representations as senders. Adam asks questions when sending, which clarifies his interpretation and understanding of Ben's and his own work (line 20, 24 & 26). Only at the beginning of the conversation, their talk has empractical parts (line 22 and 24). While their communication is influenced by the tool, mathematics is not separated, but rather explicitly related to the tool by stressing relations between the representations. Adam and Ben are active and both send and receive simultaneously, which makes the communication *participatory*.

Adam and Ben's degree of coverage (i.e. receiving and sending information), is broader than Clara and Dea's because Adam and Ben participate actively by sending-receiving to each other based on the tool (Transcript 2) and Clara and Dea are receiving from another one-sending to the tool, where communication between each other is only dominated by handling the tool. This leads to two profiles: a *participatory 'sending-receiving' communication* and an *empractical 'receiving-doing' communication*.

Transcripts 1 and 2 emphasise that students' communication competency profiles differ in situation and context (Niss & Højgaard, 2019).

Analysis of students' Instrumentation behaviour

To analyse the instrumentation behavioural profiles, we identify how the students understand the mathematical knowledge involved (e.g. semantic interpretation or comparisons), use information tools (e.g. use of theory, paper-pencil, calculator, GeoGebra, Google, peers, or the teacher) and strategies (e.g. trial-error) (Guin & Trouche, 1998).

Clara and Dea use GeoGebra each to get information to solve the task in Transcript 1. They also use the button that finds the area in figures. They use GeoGebra as their information tool, for instance, the input field. Clara keeps trying all buttons to find the right one, indicating a trial-error strategy (e.g. line 6). Their use of GeoGebra indicates a *mechanical work method* based on their dependence on the machine, and no use of paper-pencil. The situation changes in Transcript 2, when the task concerns functions. Instead of using each other as information tools, they use the teacher and Google (line 15, 17 & 21). They do not translate from table to algebraic expression (Duval, 2017). Dea tries to construct the function by writing $x=3$ and $x=6$ in the input field. In the graphical window, she validates her results as being incorrect, but she does not provide mathematical knowledge to correct them. Here we find a trial-error method, also shown in Clara's use of GeoGebra. Clara also uses Google to do the copy-paste technique. Clara also validates the results without being able to adjust it to the correct function equation. In Transcript 2, Clara and Dea mostly have either a *random work method* or a *mechanical work method* because they do not take mathematical knowledge into account. Their strategies, *trial-error* and *copy-paste* indicate a *random work method*. However, validation based on machine results suggests a *mechanical work method*. This reveals a combination of the two profiles: *a random-mechanical work method.* Adam and Ben show two other profiles, *theoretical work method* and *resourceful work method*, because they utilise semantic interpretation as a tool to understand (line 21-24), when they translate from table to algebraic expression and to graphic, the latter by GeoGebra. Their information tools are theory (knowledge about translations between representations), the students themselves, the input field and GeoGebra. They use graphs in GeoGebra to validate their results (line 24-27) (Guin & Trouche, 1998).

CONCLUDING DISCUSSION

This paper indicates that *participatory* communication and *theoretical-resourceful work* method are closely linked as well as *empractical* communication and *mechanical-random work* method. In addition, the students' understanding of functions affects both the communication profiles and the work method in GeoGebra. Adam and Ben understand functions as co-variation and they can easily translate representations using their level of theoretical knowledge. They use a theoretical-resourceful work method fostering the activation of communication competency. Communication is participatory due to their understanding of functions. In contrast, Clara and Dea's communication is more empractical due to their lack of mathematical understanding, the latter thus constraints activating communication competency. Key for all profiles is their dependence on the ability to understand and translate representations (Guin & Trouche, 1998; Niss & Højgaard, 2019), where weakness in translating between representations (Duval, 2017) hinders both, communication competency as well as instrumental genesis of GeoGebra. Available information tools instead foster the students' use of GeoGebra, and thus their communication (Guin & Trouche, 1998). If students are too contingent on one tool and lack mathematical knowledge, communication cannot flexibly refer to mathematics linked to the tool, thus, resulting in empractical talk (Jungwirth, 2006) as in the case of Clara and Dea. This paper indicates that the relation between the tool and students' mathematical knowledge is the important aspect rather than just the relation between the tool and the students' instrumentation processes.

REFERENCES

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of Technology in Lower Secondary Mathematics Education – A Concise Topical Survey.* Switzerland: Springer.

Duval, R. (2017). *Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representation.* Cham, Switzerland: Springer Nature.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of computers for Mathematical Learning, 3*(3), 195-227.

Johnson, H. L. & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics*, *97*, 299-316.

Jungwirth, H. (2006). Everyday computer-based maths teaching: The predominance of practical activities. In: J. Novotná, H., Moraová, M. Krátká & N. Stehlíková (eds). *Proceedings30th conference of the International Gorup for the Psychology of Mathematics Education, 3* (pp. 377-384). Prague: PME

Ministry of Education (2019). *Matematik – Fælles Mål.* Copenhagen: Undervisningsministeriet. https://www.emu.dk/grundskole/matematik/faelles-mal [20.09.2019]

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9-28.

Programming as a mathematical instrument: the implementation of an analytic framework

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This paper relates to an ongoing project using design-based research as a methodological approach in which students with no prior experiences of using programming as a mathematical tool are observed trying to solve mathematical problems with the help of programming. The Instrumental Approach is used as conceptual framework in which the concept of instrumental genesis describes the process where the programming environment as an artefact together with student-developed mental schemes forms an instrument in order to solve mathematical problems. The development of schemes is of special interest in this paper where Vergnaud's components of a scheme provide a framework for analysing transcripts of talk between student pairs and the programming code that they generate.

Keywords: mathematics education, computer programming, instrumental genesis.

INTRODUCTION

During the past decade, there has been a renewed recognition of programming as an important digital competence to be developed as part of the general education of all students, and of its particular relationship to mathematical competence. This has been recognized in changes to curricula in many countries: in France, Finland, and Sweden, for example, programming is included in mathematics curricula. In Sweden, where the study described in this paper took place, programming was in 2018 included in mathematics from year 4 in lower secondary school. In upper secondary school, programming is to be used as a tool for mathematical problem solving.

Papert (1980) argues that using programming in school and in mathematics could have positive effects on children's learning and could help students to develop new cognitive skills. According to Hoyles and Noss (2015), the use of programming in mathematics education is seen by most students as an engaging activity where they can independently "build, learn from feedback and debug" (p. 7). Programming is also a means for developing creativity and ability in problem solving (Romero, Lepage, & Lille, 2017) and offers a natural opportunity for students to be exposed to mathematical concepts closely related to programming, e.g. iterations (Noss, 1986). But although the introduction of programming has the potential of offering new possibilities for learning, Drijvers and Gravemeijer (2005) argue that the integration of new technologies in mathematics education can be complicated and that it would be naïve to believe that "we can separate techniques from conceptual understanding and that leaving the first to the technological tool would enable us to concentrate on the latter" (p. 164). Instead they argue that machine techniques and conceptual understanding must be interwound and be developed simultaneously. Drijvers and Gravemeijer

(2005) consider this interwinding as a fundamental part of *The Instrumental Approach* which will be described in the following section.

THE INSTRUMENTAL APPROACH

The Instrumental Approach originates from the field of cognitive ergonomics (Rabardel, 2002) and considers the process where a subject, involved in a goal-driven activity, uses an artefact (a material or abstract object) to act towards a given objective. During the process when the subject appropriates the artefact to her/his needs and integrates the artefact with her/his activity the subject develops mental utilization schemes associated both with use of the artefact and with the objective of which the artefact should act towards (Rabardel, 2002). These schemes can be usage schemes – directed towards the artefact itself - or instrumented action schemes - directed towards the object of the activity. The artefact together with the associated mental schemes constitutes an instrument for the subject, where the instrument is regarded as a psychological construct. The process through which the instrument is formed is called the *instrumental genesis* and is, in this ongoing research project, followed with special interest when studying how students (the subjects) use a programming environment (the artefact) when solving a mathematical problem (the objective).

The development of schemes

In order to study the instrumental geneses of students, the development of mental schemes is therefore of special interest. Vergnaud (1998) argues that mental schemes can be divided into four different components; *goal and anticipations, rules of action, operational invariants, and possibilities of inferences* and this paper will focus on students' use of different rules of action and operational invariants. The rules of action are considered by Vergnaud (1998) as the generative part of the scheme, directed by operational invariants (Buteau, Gueudet, Muller, Mgombelo, & Sacristán, 2019). Every action is built upon some information or concepts and Vergnaud (1998) thus regards concepts-in-action as a vital part of the operational invariants. The second part of the operational invariants consists of theorems-in-action, regarded as "proposition[s] which [are] held to be true" (p. 168) by the subject when s/he acts. Vergnaud (1998) argues that there is a relationship between concepts-in-action and theorems-in-action since "concepts are ingredients of theorems" (p. 174).

Buteau et al. (2019) used Vergnaud's (1998) four components of a scheme as an analytic frame in order to analyse how university students engage in mathematical inquiries using programming as a mathematical tool. They argue that their use of the framework has "deepen[ed their] understanding of what is at stake in terms of students' learning in this particular context" (p. 17) and has served as a means to illustrate students' instrumental geneses. In accordance with the work of Buteau et al. (2019), Vergnaud's (1998) components of a scheme will serve as an analytic framework for this study and the research question that the study is addressing is: *What are the instrumental geneses of upper secondary school students' use of programming environments in trying to solve mathematical problems pre-designed to lend* *themselves to programming?* The question of methodology which this specific paper addresses is: *How can Vergnaud's (1998) components of a scheme be operationalized for this study?* Since the instrumental geneses in this study relate to students' use of programming environments as a mathematical problem-solving tool, the schemes developed during the intervention relate to the problem-solving process as a whole and not to specific parts of it. But due to limited space within this paper, only a specific section of the problem-solving process will be described.

METHOD

The findings presented within the paper are part of a research project using design-based research as the overarching research method. In 2019, 27 eleventh grade students in a Swedish upper secondary school participated in the teaching intervention of the first design cycle of a lesson in which students are intended to solve mathematical tasks using a non-standard problem-solving strategy (e.g. an exhaustive trial) involving use of the programming environment. The students were, at the time of the intervention, taking the same introductory course in programming and thus had basic knowledge of coding but no experience of using the programming environment as a mathematical tool during their ongoing course in mathematics. Due to the students' prior study of coding, it was assumed that they had already developed basic usage schemes related to the use of the artefact (programming environment). The focus of this study is thus on the development of instrumented action schemes directed towards mathematical problem solving.

Data collection and data analysis

During the intervention students worked in pairs and three of the pairs were followed more closely through the use of screen-capturing software which also recorded the conversation between students in each pair. This data was of special interest when studying the development of schemes since it allowed the researcher to identify stable behaviours important when analysing students' instrumental geneses (Buteau et al., 2019). The researcher (who acted as the teacher) also wore a microphone to record conversations between students and the researcher. All voice recordings collected during the teaching intervention of the first design cycle were transcribed using NVivo. During the analysis, data from the recordings were grouped into themes relating to different parts of the problem-solving process. Then these themes were coded in an iterative process using Vergnaud's (1998) components of a scheme. Both verbatim abstracts and code generated by students have served as evidence when analysing the development of students' schemes.

RESULTS AND ANALYSIS

In this section, examples will be given of how verbatim abstracts from conversations between students during the intervention have been used together with the generated programming code to analyse the development of students mental schemes using Vergnaud's (1998) components of a scheme as the analytical framework. In this paper, the use of rules of action and operational invariants will be of special interest. The mathematical problem will not be described in detail in this paper but concerns finding the ages of three sisters. The problem can be mathematised algebraically in terms of relationships involving the ages of the sisters but not in a manner which permits the students to solve the problem using algebraic methods already known to them. Therefore, it was anticipated that the students might use the programming environment in order to conduct an exhaustive trial, a process which will be analysed in the following sub-section.

Developments of schemes relating to the implementation of an exhaustive trial

During the intervention, the researcher asks a pair of students called Sophie and Richard to describe their problem-solving strategy and how this had been implemented using the programming environment. Sophie explains how the pair have used nested loops in order to conduct an exhaustive trial:

Sophie: Yes, we have a nested *for*-loop so the first one... Eh... Or starting with *a* is zero and every time it goes around then *a* increases by one. But before that happens, *b* is set to *a* plus eleven and then comes the next *for*-loop which then tests every age between *a* and *b*, which is then the mid sister. And if this formula we came up with is true then the loops should stop. […]

The outer loop (Fig. 1) is thus used by the pair to systematically increase the value of the variable *a* which concerns the age of the youngest sister. Within this loop the value of *b*, the age of the oldest sister, can be calculated using a given relationship between *a* and *b*. The inner loop is used to vary the variable concerning the age of the mid sister and thus runs for integer values between *a* and *b*. Within the inner loop an IF statement is used to test a mathematical condition within the task involving the ages of the sisters.

```
int a;
for(a = 0; a < 1000; a++){
    int b = a + 11;
    for(int f = a; f < b; f++){
        if ((b+1)*(f+1)*(a+1)-(b+1)*(f+1)*a == 432){
            break;
        }
    }
}
System.out.print(a);
```

Figure 1: Screen shot visualizing the nested loops generated by Sophie and Richard

Based on the verbatim abstract above, several rules of action used by the pair could be identified: (a) *formulating the problem situation as amenable to solution through exhaustive trial;* (b) *making use of programming to implement a solution strategy based on exhaustive trial*; (c) *creating iterations through defining conditions for loop(s); and* (d) *making use of the conditional operator IF to (i) evaluate given conditions in order to (ii) perform different actions based on the validity of given conditions*. It could be argued that these rules of action are justified through several concepts-in-action involving the ideas of (a) *conducting an exhaustive trial*; (b) *systematically combining variables*; (c) *establishing a loop relating to a variable*; (d) *nesting loops (and statements within them) in order to achieve an appropriate sequence* *of variable-related actions*; and e) *using conditions within loops and conditional operators in order to extract a solution within a given range*. These concepts-in-action form two theorems-in-action which guide the actions of the pair: (a) *Systematically combining variables serves as a means of achieving an exhaustive trial when more than one variable is in play* and (b) *Establishing nested loops relating to the key variables in play is a means of systematically combining these variables*.

The ideas of students Emilia and Fredrik, on the other hand, are less developed. In particular, they struggle to articulate – either orally or in code – how to systematically combine variables. In the nested loops shown in Fig. 2, the outer WHILE loop is used to check if the variable *guldmynt_f* (dependent on the sisters' ages, and recalculated with each traversal of the loop) is less than or equal to 432 (a crucial value in the problem). The inner FOR loop includes the same condition and initialises a control variable *i* incremented on each traversal, which is then (mis)used within the loop, apparently with the intent of systematically increasing the variable *b*, the age of one of the sisters.

```
16          while(guldmynt_f<=432){
17              for (int i = 0; guldmynt_f <=432 ; i++) {
18                  b = b+i;
19
20                  guldmynt_f = (a+12)*(a+1)*(f+1)-(a+12)*(a)*(f+1)-432;
21
22              }
23          }
```

Figure 2: Screen shot visualizing the nested loops generated by Emilia and Fredrik

Fredrik then realizes that the other age variables *a* and *f* are never assigned new values within the loops.

Fredrik: We should increase everyone? I think.

Emilia: No, but we just need... We just need to increase the age of Cinderella (variable *a*) because the others increase automatically because we have written *a* plus twelve there and a little something else as well.

Fredrik: Yes but... We still need to increase *f*.

Fredrik deletes the calculation of *b* in line 18 and inserts $f = f + i$ instead. Later, Fredrik returns discussion to the control variable *i,* which he relates to testing values associated with an (unspecified) year and age:

Fredrik: But we need to find out what year it is... Because now they are zero years old... That's why we have to test values all the time.

Emilia's and Fredrik's way of coding indicates the use of concepts-in-action based on the ideas of (a) *conducting an exhaustive trial;* (b) *establishing a loop relating to a variable*; and (c) *nesting loops (and statements within them) in order to achieve an appropriate sequence of variable-related actions*. But their failure to code loops which will systematically increase the values of variables indicates that the pair, unlike Sophie and Richard, have underdeveloped rules of action relating to how to use nested loops to combine variables in order to conduct the exhaustive trial. As a consequence, there is also a lack of theorems-in-action guiding the pair's actions.

A third pair of students, Christian and David, initially have a clear view about how to use the programming environment as a mathematical tool.

Christian: Actually, you could set it up mathematically… Or... and then just use *brute force* by testing lots of different combinations with the computer.

Although not made explicit by the verbatim extract above, the program structure (Fig. 3) later reveals the pair's intention of conducting an exhaustive trial in order to solve the given task. Unlike the other pairs, Christian and David also realize that they can take advantage of the fact that the ages of the sisters must take integer values, and so use this to establish an additional condition.

Christian: But yes, what we can do is just have a eh... WHILE TRUE and then we have... And then we have... And then we check for that one so then you have like eh... If this… and then you have like *and*-signs for... Eh.... And check if something is INT (integers).

```
8         while(true){
9             As+=1;
10            Be = As + 11;
11            Fu = 432.0 / Be;
12            if(As<Fu && Fu<Be && Fu-Math.round(Fu)==0) {
                  System.out.println("As: "+As+", Fu: "+Fu+", Be: "+Be);
14            }
15        }
```

Figure 3: Screen shot visualizing the single loop generated by Christian and David

This additional condition allows the pair to create a program (Fig. 3), which only uses a single loop involving three variables corresponding to the age of each sister. The variable controlling the loop is *As* (age of the youngest sister), initialised as 0, and, at the start of each iteration, increased by 1. Thus, the loop systematically examines what happens as *As* increases from 1. Within the loop, the *Be* (age of the oldest sister) is then specified as $Be = As + 11$ and the third variable *Fu* (the age of the mid sister) is calculated as $Fu = 432 / Be$ (although it should be noted that the underlying definitions of the ages used in these two calculations are not compatible with each other). The IF-statement in line 12 checks three conditions which need to be met in order for the combination of ages to be a solution to the problem. The first two conditions check if *Fu* is the mid sister and the third condition checks if the value of *Fu* holds an integer value. If *Fu* is an integer, the difference between *Fu* and the rounded value of *Fu* equals zero. If all the three conditions are met the program should print the ages of the sisters.

The verbatim extract together with the code generated by Christian and David expose how the pair has used several different rules of action during the development of the program: (a) *formulating the problem situation as amenable to solution through exhaustive trial;* (b) *making use of programming to implement a solution strategy based on exhaustive trial*; (c) *creating an iteration through defining conditions for a loop; and* (d) *making use of the conditional operator IF to (i) evaluate given conditions* *in order to (ii) perform different actions based on the validity of given conditions*. These rules of action are justified through several concepts-in-actions involving ideas of (a) *conducting an exhaustive trial;* (b) *computing linked variables when more than one is in play;* and (c) *using conditions within loops and conditional operators in order to extract solutions within a given range*. These concepts-in-action are related to a theorem-in-action used by Christian and David stating that: *Computing linked variables when more than one is in play serves as a means of reducing the number of variables to be systematically varied in an exhaustive trial*.

The examples given in this section are small extracts from students' problem-solving process when trying to solve a mathematical problem using a programming environment as a mathematical tool. Yet, they illustrate different approaches used by students in order to conduct an exhaustive trial and also difficulties relating to conceptual and computational understanding. The way Sophie and Richard try to conduct an exhaustive trial differ from the method used by Christian and David. This is illustrated by the components comprising their developed schemes, although some generic components relating to the problem-solving strategy are common. Emilia's and Fredrik's scheme could be regarded as deficient since it lacks several essential components relating to the use of nested loops, an action often perceived as conceptually difficult for novice programmers (Mladenović, Boljat, & Žanko, 2018).

DISCUSSION

Following Buteau et al. (2019), we have explored approaches to operationalizing Vergnaud's (1998) components of a scheme when studying students' instrumental geneses. Using conversations between students (and between students and the teacher) together with their generated code has made it possible for the researchers to extract different components of schemes explicitly stated by students (or shown in their program structure). This in turn has presented a possibility for the researchers to search for similarities and differences within different schemes as well as analyzing which components are missing from deficient schemes. This is illustrated by the example where Emilia and Fredrik, just like the other two pairs, had begun to develop an instrumented action scheme directed towards mathematical problem solving, involving the use of exhaustive trial to solve the mathematical problem. But the lack of well-functioning rules of action relating to the use of nested loops, in order to systematically combine variables, hindered this pair to implement their problem-solving strategy. This deficiency within their instrumented action scheme illustrates that Emilia and Fredrik may have had under-developed pre-existing usage schemes directed towards the artefact itself relating to the use of (nested) loops.

We also argue that defining specific components of the scheme based on conversations between students should not be seen as a straightforward process. In the analytic and iterative process there has to be a balance between defining components, on the one hand, generic enough to be able to search for commonalities across cases, but on the other hand still specific enough not to lose the key characteristics of each case.

Since most of the data involves conversations between students during an ongoing problem-solving process involving a particular situation, it cannot strictly be argued that the findings provide evidence of "the invariant organization of behavior for a certain class of situations" (Vergnaud, 1998, p. 167). But, at the least, the data shows how, in solving the task, students generate proto-schemes or schemes-in-progress which together with the artefact start the formation of an instrument. Indeed, this is no more than the notion of a dynamic, constructive process of instrumental genesis.

REFERENCES

Buteau, C., Gueudet, G., Muller, E., Mgombelo, J., & Sacristán, A. I. (2019). University students turning computer programming into an instrument for 'authentic' mathematical work. *International Journal of Mathematical Education in Science and Technology*, 1-22. doi:10.1080/0020739X.2019.1648892

Drijvers, P., & Gravemeijer, K. (2005). Computer algebra as an instrument: Examples of algebraic schemes. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 163-196). Boston, MA: Springer.

Hoyles, C., & Noss, R. (2015). A computational lens on design research. *ZDM, 47*(6), 1039-1045. doi:10.1007/s11858-015-0731-2

Mladenović, M., Boljat, I., & Žanko, Ž. (2018). Comparing loops misconceptions in block-based and text-based programming languages at the K-12 level. *Education Information Technologies, 23*(4), 1483-1500. doi:10.1007/s10639-017-9673-3

Noss, R. (1986). Constructing a conceptual framework for elementary algebra through Logo programming. *Educational Studies in Mathematics, 17*(4), 335-357. doi:10.1007/BF00311324

Papert, S. (1980). *Mindstorms: children, computers, and powerful ideas*. New York, NY: Basic Books, Inc.

Rabardel, P. (2002). *People and technology - a cognitive approach to contemporary instruments* Retrieved from https://hal.archives-ouvertes.fr/hal-01020705

Romero, M., Lepage, A., & Lille, B. (2017). Computational thinking development through creative programming in higher education. *International Journal of Educational Technology in Higher Education, 14*(42), 1-15. doi:10.1186/s41239-017-0080-z

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior, 17*(2), 167-181. doi:10.1016/S0364-0213(99)80057-3

Instrumental orchestration of using programming for mathematics investigations

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We investigate how the instrumental orchestration can contribute to our understanding of the teaching to university students of using programming technology for mathematical investigation projects. Our case study highlights a dual role the instructors play, as policy maker and as teacher, to orchestrate students' instrumental geneses, and the integration of projects as a key element of the exploitation mode.

Keywords: instrumental orchestration, schemes, programming, university education.

INTRODUCTION AND CONTEXT

In the field of mathematics education, programming for learning has a legacy of half a century that started with the designing of the LOGO programming language for learning (Papert, 1972). Studies working in this area have largely focussed on learning, whereas pedagogical design was mainly tangentially analysed (e.g., Noss & Hoyles, 1992). However, with the recent increased integration of programming in schools and curricula, we see a crucial need for studies about teaching and teacher education concerning programming such as that of Benton et al. (2018).

In this paper, we address this need under the fourth theme of the conference, namely *Theoretical perspectives and methodologies/approaches for researching mathematics education*, by presenting a preliminary study concerning the theoretical contribution of the instrumental orchestration (Trouche, 2004) to analyse the teaching to university students of using programming for mathematical investigation projects.

Our study is part of a five-year naturalistic research that takes place in the context of a sequence of three university mathematics courses, called *Mathematics Integrated with Computers and Applications* (MICA) I-II-III taught at Brock University since 2001. In these project-based courses, math majors and future math teachers learn to design, program, and use interactive environments to investigate mathematics concepts, conjectures, and applications (Buteau et al., 2015). The research aims at understanding how students learn to use programming for 'authentic' mathematical investigations, if and how their use is sustained over time, and how instructors support that learning.

The question guiding the study presented in this paper is: *What do we learn about the teaching of using programming for authentic mathematical investigations by using the theoretical frame of the instrumental orchestration, considering programming as an artefact?* Building on previous work on students' instrumental genesis of using programming (Buteau et al., 2019a) and on constructionist facets of a related teaching

(Buteau et al., 2019b), this study focuses on teaching aspects that aim at steering students' instrumental geneses. Next, we present the instrumental approach, and how we use it when the artefact is programming. We then present our methods, and illustrate the use of instrumental orchestration by analysing the case of the MICA II teaching. Finally, we discuss insights gained from using the instrumental approach.

INSTRUMENTAL APPROACH: SCHEMES, GENESIS, ORCHESTRATION

The instrumental genesis approach (Rabardel, 1995) provides a lens to describe how a student, in an activity with a math goal, learns to use an *artefact* (e.g. programming) and learns mathematics at the same time, through the development of schemes. It introduces a distinction between an artefact, which is produced by humans, for a goal-directed human activity, and an *instrument*, developed by a subject along his/her activity with this artefact for a given goal through a process called *instrumental genesis*.

The instrument is composed by a part of the artefact and a scheme of its use (Vergnaud 1998), either as a *usage scheme*–"oriented towards the management of the artefact"– or as an *instrumented action scheme*– "oriented to the carrying out of a specific task" (Trouche, 2004, p.287). In math education, the instrumental approach was first used to study learning processes of secondary school students using calculators (Guin et al. 2005). These studies used a detailed definition of schemes based on Vergnaud's work. Namely, a *scheme*, defined as a stable organization of the subject's activity for a given goal, comprises four components: i) the *goal* of the activity; ii) *rules-of-action* (RoA), generating the behaviour according to the features of the situation; iii) *operational invariants*: concepts-in-action and theorems-in-action (TiA), which are propositions considered as true and governing the RoAs; and iv) possibilities of *inferences*.

Trouche (2004) added that students' instrumental genesis may need to be guided by a teacher and proposed the concept of *instrumental orchestration*. He explains that the instrumental orchestration refers to the teacher's intentional organization, arrangement and didactic use of various artefacts in the class (including digital ones), with the purpose of steering the students' instrumental genesis. Drijvers et al. (2010) added the idea of didactical performance to explain the different adjustments, which are made in response to the events of the class. These authors (2010, p. 215) summarize the resulting three elements of the instrumental orchestration: i) *Didactical configuration*: "an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it"; ii) *Exploitation mode*: "the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions [it] includes decisions on the way a task is introduced and worked through, on the possible roles of the artefacts to be played"; and iii) *Didactical performance*: "involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode." We next describe how we use the instrumental approach when considering programming as the artefact.

Programming for Mathematics Investigations: Students' Instrumental Genesis

The general goal of the students' activity is to investigate a complex situation, combining mathematical knowledge and programming. By using programming for this goal, students develop an instrument, associating some aspects of programming (artefact) and schemes of use for specific sub-goals such as those described in the development process model (see Fig.1). For example, the *scheme of articulating a mathematics process in the programming language*, as a sub-scheme of the *scheme of designing and programming an object* (Step 3 in Fig.1). Thus a student's instrumental genesis in this context means that a student develops a complex web of schemes which ramifications include, among others, those in Fig.1 (Buteau et al. 2019a).

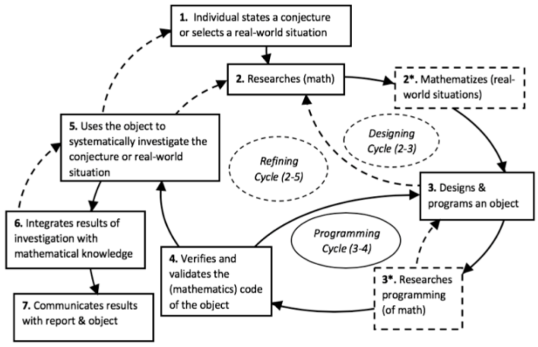

Figure 1. Development process (DP) model of a student engaging in programming for a pure or applied mathematical investigation (Buteau et al., 2019a).

METHOD

We investigate the teaching of using programming for authentic mathematical investigations using the case of the instrumental orchestration of one MICA II instructor, Bill, during Winter 2019. Bill is a mathematician, a remarkable teacher with long teaching experience (over 30 years) who played a key role in the development and teaching of MICA courses–he is not researcher in this project. Data included all course material and semi-structured task-based interviews with Bill for each of the 4 assigned and 1 original math investigation projects. Data also included a 2000 departmental program description document when the MICA courses were adopted. For this paper, we use only Bill's first project assignment for which he chose 4 short investigation problems involving Monte Carlo integration (we discuss Bill's choice further below): P1-Buffon Needle problem; P2-area between two curves; P3-hypervolume of the unit hyper-sphere in R^4; and either P4- Buffon-Laplace problem or P5-the infinite limit of the probability that two randomly selected integers smaller than n are relatively prime. Bill's interview and project guidelines were analyzed by identifying potential students' schemes that might have, implicitly or explicitly, intentionally been promoted by Bill. In this study, we did not directly observe Bill's teaching. Only a part of his teaching was accessible through the course material he produced. This is a limitation of our naturalistic research, but we also collected student data from Bill's class allowing to relate shortly their instrumental genesis with Bill's orchestration. Next, we outline each orchestration component of a MICA instructor using our understanding of the teaching of MICA courses (e.g. Buteau et al., 2019b), and illustrating them using Bill's teaching.

INSTRUMENTAL ORCHESTRATION OF PROGRAMING: A CASE STUDY

Didactical Configuration

The choice of programming technology use in MICA courses and the teaching setting configuration was established in 2000 by the mathematics department (including Bill) at Brock University, and has since remained. The teaching format involves 2 hrs of lecture (in a regular lecture room) and 2 hrs of computer lab (1 computer per student), weekly. In regards to the artefact, there is an agreement among the instructors that MICA I-II courses mainly use vb.net language with Visual Studio. There are now two MICA III courses; one for math and science majors moving on to C++ programming language with GNU IDE, and the new MICA III course for future mathematics teachers using vb.net, Scratch and Python with Jupyter Notebook. Specifically, in the case of Bill's teaching in 2019 of MICA II, he decided to also introduce Excel technology as part of his second assignment. Bill justifies: "I also think… that every math major has to be able to use Excel, because this is one of the standard tools in the outside world." (B.A2.143)

Exploitation Mode

The main didactical intention grounding how programming technology would be integrated into a sequence of three MICA courses was also established in 2000 by the mathematics department: students would learn to exploit programming for mathematical work. The 2000 departmental document stipulates:

> [Students] will confront problems from pure and applied mathematics that require experimental and heuristic approaches. In dealing with such problems, students will be expected to develop their own strategies and make their own choices about the best combination of mathematics and computing required in finding solutions.

Also, the core of each MICA course was to be pure and applied programming-based mathematics investigation projects that account for 70-80% of a student's final MICA course grade. This is a key element of the exploitation mode (here again, the 'instructor' is viewed as a 'policy maker faculty' rather than a 'teacher'): Through these projects, the department appears to thus intend that students engage in the process described in Figure 1 (Buteau et al., 2019a). During lectures, the instructor introduces students to mathematics that is needed for the assigned individual mathematics investigation projects which are worked on during the labs. We interpret such projects as aiming at developing and/or re-enforcing various students' schemes, such as the *scheme of articulating a math process in the programming language*. Since the instructor chooses the topic and direction of these mathematics investigations, and communicates it through detailed guidelines, we interpret such projects to aim at students developing their web of schemes associated mainly to steps 3 to 7 of the DP model (Fig.1). Each MICA course also involves a final original project, in lieu of a final exam where students work individually or in pair, and choose a topic of their own and the direction of the mathematics investigation. Such final projects can be viewed as an intention for students to develop further or mobilize their complete web of schemes including those associated to steps 1 and 2. For the individual MICA instructor (as a 'teacher'), the ways s/he decides to exploit the didactical configuration in order to meet the didactical intentions envisioned and decided by the department, include decisions about the mathematics content and related investigation projects. It also includes decisions about the ways the content is developed in lectures and synchronized with the investigation project work in the labs. In terms of the choice of mathematics content in MICA II course, the instructors over the years have selected various topics and areas of mathematics relevant to a computational approach for investigations, often according to their own, evolving mathematics interests and research (Buteau et al., 2019b). For example, Bill comments on the computational relevance of P5 investigation and his personal interest in this area:

> [P5] generated a lot of great discussion… I love the idea… I like analytic number theory, I love the idea that, um, there are patterns… I jump up and down about that with the [students]... I can sell this assignment to my students. (B.A1.174)

We associate Bill's choice of 'relevant topic' to Step 1 (Fig.1), and interpret it as an implicit guidance to students (to develop a scheme of) identifying when a programming approach is an added value for the work, such as for math that cannot be done by hand. Using various resources (including their own research), the MICA II instructor *designs* programming-based mathematics investigation projects and *develops guidelines aligned with both* the planned lecture content and planned guidance in lab as student work through the projects. For example, Bill *designs* MICA II project assignments by 'playing on the computer with some math' and decides on parts of an assignment (and guidelines) by thinking on the potential difficulties that MICA II students may confront e.g. when programming the mathematics (Steps 3-5 in DP model). Bill mentions:

> I actually tried many many many things before we got the formula that you have here and I would try something and I'd say "That's too hard, that comes too fast, this has to go, this has to be sequenced differently." (B.A1.43)

We also interpret Bill's expectations from students to be able to mobilize usage schemes of programming in vb.net (Step 3) developed in MICA I. He says e.g. that P1 is "to keep them calm" as "there are no new programming tricks..it's all review". In P1 he gives students a code to build from. By providing the students with a "well written piece of code", Bill says that it "helps them review …proper coding practice"; e.g. "how to change from math coordinates to graph coordinates… separately and clearly;" etc. Bill requires students to submit, for P3, a print out of their code rather than the program; he says "I'm telling them I'm going to actually read the code on the page, it sends that signal" (B.A1.154). We interpret it as Bill's intention to students' developing their scheme of coding with rule-of-action 'I write codes according to standards'. As for the *project guidelines,* they outline the topic to be investigated, within the mathematics context developed (i.e. *synchronized) in the lectures*, together with some details of the investigation design (such as input and output) sometimes complemented with some partial code (as for P1 mentioned above). We associate these respectively to Steps 1, 2, and 3 in the DP model. For example, Bill says:

> the idea that we can take a real-world situation, and we can distill from that the mathematics, and then take that mathematics and write a simulation based on that mathematics… I'm thinking about that all the time, that, that sequence. (A1.96-98)

The *guidelines* sometimes also detail how to use the program for the mathematical investigation (e.g. by suggesting range of parameter values) and/or emphasize the need to interpret output within their mathematics knowledge (e.g. by requiring to justify their conclusion from the investigation). We associate these respectively to Steps 5 and 6 in the DP model. For example, Bill's guidelines for P3 read:

> The output should show the mean and standard deviation of the samples. Estimate the hypervolume accurate to one decimal place and use your observations to explain why you are confident that your first decimal place is correct.

This suggests to the students that they must apply their statistics knowledge in order to appropriately use their program and justify their answer (Step 5-6 cycle). In fact, Bill mentions that he revised the guidelines due to his dissatisfaction from past students' poor interpretation of their program output (i.e., more guidance needed for Step 6).Bill deliberately includes a more challenging question (selection between P4 or P5) as part of assignment 1 where *he plans close to no extra guidance* beyond the statement of the problem (i.e. Step 1): "But there has to be a question on every assignment that, is something to think about… This one is solo. Um, they don't get very much help from me" (B.A1.162-4). We interpret Bill's intention that students mobilize and develop, without his help, their whole web of schemes for this particular investigation task.

Didactical Performance

The ways a MICA instructor teaches in lectures and labs involves ad hoc decisions aligned with how they have planned to support students' learning to use programming technology for pure and applied mathematical work mainly through their individual mathematical investigation projects. Based on interactions with students, individually or collectively, and on observations of interactions among students, the instructor takes decision as to how to respond. This response may take form as *individual help* addressing an identified student's difficulty *to develop/mobilize a certain scheme*, or as a class intervention aiming at steering the *collective development a certain scheme*. Bill recalls many individual interventions during the labs. E.g. we interpret Bill's expectation from students to mobilize their scheme of debugging (programming cycle in DP model) when needed. He explicitly mentions it to students: "it will be unusual for either me or the TA to debug your code, that's not our job" (B.A1.207-8). He recalls an intervention with a student, aligned with his expectations, as he sits down beside the student: "Explain the principles and the ideas… If you're desperate, we might look through your code." (B.A1.216) We interpret Bill's response as a reminder to the students of this schemes' effective rules-of-action: '*step back from the code', 'think* *through the big picture of the code design'*, and *'think of the different parts of the code'*. Aligning with his planned *'close to no extra guidance'* for P5 that we associated with students' mobilizing their whole web of schemes for this task, Bill mentions helping individual students by responding with guiding questions: "Investigate: what does it mean?...How big should n be? And my answer is 'I don't know'." (B.A1.164; 170). As for a collective response in lab, Bill e.g. comments addressing the students' difficulty of explaining their output from the program in P3, which we interpret as steering the collective mobilization and development, for this task, of schemes associated to step 6:

> I'm very interactive in the lab … so when we get to working on this question I'll be talking about variability… it's an opportunity to work on the board with them… none of this sits by itself. (B.A1.140)

DISCUSSION

The research question guiding this paper concerned what we learn about the teaching of using programming for authentic math investigations by using the frame of the instrumental orchestration. Drawing on our case study of MICA II teaching, we discuss here elements of answer to this question, and indicate directions for future research.

The identified **didactical configuration, main didactical intention** and **the project element as part of the exploitation mode** turned out to be the same for all MICA courses adopted by the department and also followed by MICA instructors (i.e., can be viewed as operational invariants of the collective 'MICA instructor'). This configuration and exploitation mode element stress a 'student-centered' approach, whereby the core of the courses is on individual student projects that aligns with a constructionist approach (Papert, 1980). The 20 years of sustained MICA implementation could suggest that this didactical configuration and exploitation mode element support well the teaching of programming-based math investigations.

The instructor aligns with the collective **exploitation mode**; namely through his/her choices of 'content' through *project guidelines* and *planned guidance in lab and lectures* according to his/her intention of steering the collective students' instrumental genesis of their complex web of schemes associated with the programming-based mathematical investigation activity. This gives insights on how the institutional decisions support well the individual instructors. The exploitation mode of MICA II teaching also highlights that, unlike most technology-rich mathematics courses, the choice of integrating programming comes *before* to the choice of mathematics content, and has led to describe the math content at the *individual* level, rather than the usual *collective* level (as a 'policy maker level'). The **didactical performance** of MICA II teaching pointed to the significance of the lab setting as a key element of the didactical configuration to facilitate the MICA II instructor to steer both individual and collective students' development of schemes. Unlike the didactical configuration, these other two components of MICA instrumental orchestration seem to be evolving. For example, Bill's refining of the project guidelines to explicitly steer the students' mobilization or development of the scheme to mathematically interpret the program output.

In terms of future research, we note that the project guidelines, as a collection, appear to steer students to develop or mobilize their whole complex web of schemes associated with the math investigation activity. Studying aspects of investigation project tasks, as part of the whole task collection, that affect which and how different schemes are guided in the project guidelines (and in lectures and labs), will lead to essential recommendations for practice.

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REFERENCES

Benton, L., Saunders, P., Kalas, I., Hoyles, C., & Noss, R. (2018). Designing for learning mathematics through programming: A case study of pupils engaging with place value. *International journal of child-computer interaction*, *16*, 68-76.

Buteau, C., Gueudet, G., Muller, E., Mgombelo, J., & Sacristán, A. (2019a). University Students Turning Computer Programming into an Instrument for 'Authentic' Mathematical Work. *IJMEST.*

Buteau, C., Muller, E., & Ralph, B. (2015). Integration of programming in the undergraduate math program at Brock University. In *Proceedings of Math+Coding Symposium*, London, ON.

Buteau C., Sacristán A. I. & Muller E. (2019b). Roles and Demands in Constructionist Teaching of Computational Thinking in University Mathematics. *Constructivist Foundations 14*(3), 294–309.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213-234.

Hoyles, C. & Noss, R. (1992). *Learning Mathematics & Logo*, Cambridge: MIT Press.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York, NY: Basic Books.

Rabardel, P. (1995). *Les hommes et les technologies; approche cognitive des instruments contemporains.* Paris, France: Armand Colin.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *IJCML*, *9*(3), 281–307.

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Dordrecht: Kluwer.

Developing an analytical tool of the processes of justificational mediation

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Within the Instrumental Approach (IA) the newly developed notion of justificational mediation (JM) describes mediations that aim at establishing truth of mathematical statements in the context of CAS-assisted proofs in textbooks. Here we study JM with the intent to broaden the notion to the context of informal justification processes of early secondary students interacting with GeoGebra. Seeing JM as a process that has the objective of changing the status of a claim, we use Toulmin's model and combine it with the IA to unravel the structure of the process through an analytical tool. The study is part of a broader project on the interplay between reasoning competency and GeoGebra with lower secondary students.

Keywords: digital environment, Instrumental Approach, justificational mediation, reasoning competency, Toulmin's model.

REASONING COMPETENCY AND JUSTIFICATIONAL MEDIATION

During the last decades, the use of digital technologies in mathematics education has increased, as well as the body of research in this area (e.g., Hoyles & Lagrange, 2010). In Denmark, this development has coincided with the promotion of *mathematical competencies*, seen in the KOM-framework as "…someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations." (Niss & Højgaard, 2019, p. 6). In the wake of this development, a need has arisen for understanding the interplay of students' enactment and development of the specific mathematical competencies and their use of digital technology (Geraniou & Jankvist, 2019). What might "readiness to act appropriately" mean in the context of digital technology? How can such readiness be identified and nurtured? These are examples of broad questions that gave rise to this study.

We follow Geraniou and Jankvist (2019) who took some first steps in weaving together the KOM framework with the Instrumental Approach (IA), which is also widely used in the European research community. The IA suggests that the use of tools involves *pragmatic mediation,* concerning the subject's actions on objects and *epistemic mediation*, concerning how the subject gains knowledge of objects' properties through the tool (Rabardel & Bourmaud, 2003). However, Jankvist and Misfeldt (2019) suggest that a third form of mediation, *justificational mediation* (JM), may be useful in the context of CAS in proofs and proving activities. JM concerns how the status (e.g. probable, likely, true or false) of statements for a student is modified through the use of a digital environment (Jankvist & Misfeldt, 2018; 2019). However, the authors have advanced the notion of JM within the context of CAS-assisted proofs in textbooks in upper secondary school, which touches on the more formal part of the *reasoning competency*. Still the authors ponder whether students think about justification, insight and performing mathematical labor as different things and how (Jankvist & Misfeldt, 2019). So, other situations relating to the less formal side of the reasoning competency spectrum should be considered and studied separately within this frame.

Within the KOM-framework's reasoning competency, we study students' mathematical informal argumentations that take place within the digital environment GeoGebra, focussing on the processes through which an uttered statement changes status: it may either be rejected or believed to be true to a greater degree than in its initial form. The ways in which students justify their claims within an environment like GeoGebra can assume forms that are closely related to the environment itself, as well as to the underlying mathematical theory within which the objects are placed. Hence we ask: *how can we analyze JM and what insight into it can we gain*?

Seeing JM as a process of argumentation, our analytical tool is derived from Toulmin's model, and, because JM occurs in a digital environment, we make use of constructs from the IA. We now explain how the theoretical frame is set up.

THEORETICAL FRAMEWORK: CONSTRUCTING A TOOL OF ANALYSIS

Although the original intention of Toulmin's model was to analyze finalized argumentations (Toulmin, 2003), there are numerous examples in mathematics education where it is used to analyze students' processes of argumentation (e.g. Pedemonte, 2008; Simpson, 2015), also in the context of digital environments (eg. Hollebrands, Conner & Smith, 2010). These studies, however, do not usually situate the model within the research field of educational use of digital technologies in mathematics, and hence do not draw on the theories used in this field. In this study, we suggest an analytic tool that does exactly that.

With respect to the IA, we consider GeoGebra as an *instrument*. Such a notion arises from the use of an *artefact* and the development of *scheme.* In this context the artefact is GeoGebra itself, but in other cases it could be a specific tool within it (such as dragging, or a slider). *Schemes of utilization* are developed by a solver to accomplish a specific task (Rabardel, 2002). *Scheme* is understood according to Vergnaud's construct: "the invariant organization of activity for a certain class of situations" (Vergnaud, 2009, p. 88), that relates an "invisible part" to a student's visible actions. Schemes are made up of various aspects, including a *generative aspect:* rules to generate activity; namely the sequences of actions; information gathering; and controls and an *epistemic aspect:* operational invariant; namely concepts-in-action; and theorems-in-action, with the function to pick up and select the relevant information and infer from it goals and rules.

In the following, we will introduce elements from Toulmin's model and explain how we interpret them within the IA and with respect to JM.

JM through Toulmin's model in the context of GeoGebra

In Toulmin's model, the *claim* is a statement of the speaker, uttered with a certain indication of likelihood (*qualifier*); the claim is justified through other elements of the argument (data, warrant, backing). The first utterance of the claim indicates the start of the JM process, in which the *aim* is to change the qualifier. In younger students' informal argumentation, the aim is seldom to construct rigorous mathematical proofs but rather to convince themselves of the existence of mathematical relations and facts (Jeanotte & Kieran, 2017). Hence, a change in the status of the qualifier will often be from likely to more likely, and less often from likely to true. We recognize such a change of status of a claim by students' restatement of the claim accompanied by a new qualifier. The change of the status is reached through the generation of *data* that for the solver constitutes evidence and facts supporting the claim, and through the *warrant* that consists of inference rules that allow the solver to connect the generated data to the claim (Toulmin, 2003). The warrant is often implicit, in which case, it must be inferred from the utterances and gestures of the students. We can infer the warrants and analyze the generation of data through the notion of scheme introduced above. The *generation of data* is the product of the generative aspect of the schemes used (e.g., dragging, creating objects on the screen and interacting with them, utterances and other hand-gestures) that are carried out by students. Warrants are the epistemic aspect of the schemes used. One last element remains; *backing*. This element requires some careful consideration, which we elaborate in the next section.

Toulmin describes the *backing* of a warrant as "…other assurances, without which the warrants themselves would possess neither authority nor currency" (Toulmin, 2003, p. 96). However, Simpson (2015) identifies three different uses of backing in mathematics education research. In the context of JM, we consider the backing to be an explanation of why the warrant is relevant (Simpson, 2015). Central is, that the aim of JM is to change the status of the claim, so the backing must explain why the warrant is relevant for generating data that allows the change in the status of the claim. Thus, the backing becomes fundamental to the JM process. Currently, we have reached the following formulation of *backing* in JM processes:

If the claim is true, I can generate data, within the specific instrument, that is consistent with the claim.

This seems closely related to Vergnaud's (2009) notion of *theorem-in-action*, a sentence that the solver believes to be true, but that may in fact be false. Though it can be, it is not a mathematical theorem, and it can bridge domains of different natures. In our case it bridges the phenomenological domain of GeoGebra with the theoretical domain of algebra (also see Baccaglini-Frank, 2019). We recognize, that there might be variations of such a formulation, but we are currently studying this form.

METHODOLOGY

The task we analyze in this paper comes from a broader project, in which a series of tasks were designed by the first author and assigned to students in three classrooms of grade 7 students (in all 61 students). All students had prior experience using GeoGebra's geometric tools, as well as constructing points and sliders in the algebra view, but they had never used the slider to vary points, which is central in this task. The students worked in pairs for two 90-minute sessions while being video recorded. All together 17 pairs was recorded. The video recordings captured the screens and the students, both from the computer's camera and from a second handheld camera controlled by the first author, who was present during all the sessions.

The example below, is of a pair students, Lilly and Mia, who were described by their teacher as a particularly "talkative" pair, who usually participated with confidence to math class, even though they were not considered to be "the best" students. The tasks was posed and solved in Danish. The task as well as the excerpt have been translated to english for this paper.

We selected this example because of its short length and the fact that it contains many aspects of the process of JM. Indeed, in these 75 seconds the students changed the status of an initial claim from likely to more likely. This episode, therefore, constitutes a unit of analysis.

AN EXAMPLE AND ANALYSIS OF JM

We use the following transcript to illustrate the analytical tool and how it is applied in an analysis of students' justification processes. The two students are working on a task, where they are asked to predict how two given points $A = (1,s)$ and $B = (s,1)$ will move in the coordinate plane in GeoGebra. If the two points are constructed in the algebraic view, a slider for the interval [-5,5] will appear for the variable *s,* the slider can either be dragged or animated, and its movement induces the points to move in the coordinate plane as *s* varies. To ensure that the students predict, rather than construct and animate/drag the slider, the GeoGebra interface in this specific task is limited to the graphics view, showing a coordinate plane along with the cursor, the point tool, and the pen tool. An orange textbox also appears with the coordinates of the given points.

Lilly and Mia make a conjecture about a *line* through *AB* and discuss it, despite the task does not mention any lines. Lilly holds the mouse throughout the excerpt.

1. Lilly: [Reads out the task] Show in the coordinate system how you think point *A* and *B* move as *s* changes value.
2. Mia: I have the feeling they are making such a slanted line like this (Fig. 1a).
3. Lilly: Yes.
4. Mia: That is what I imagine.

5. Lilly: If I make a point now called *A* right? [Places point with point tool in (2, 2.98)] So this, this is *A*.

6. Lilly: And then we can say that ehm, *A* is equal to one comma *s*, right? [moves the curser to point at the coordinate sets in the orange text box (Fig. 1b)]

7. Mia: What should *s* be?

8. Lilly: One here, and then *s* could be… [Moves *A* towards (1,0)]

9. Mia: Four.

10. Lilly: Four, so it will be here then [Moves *A* to (1,4) (Fig. 1b) along x=1]

11. Mia: Yes.

12. Lilly: Then we do *B*.

13. Mia: [Points to approx. (4,1) with her index finger (Fig. 1c)] Yes, that is what I said, then it becomes such a slanted line.

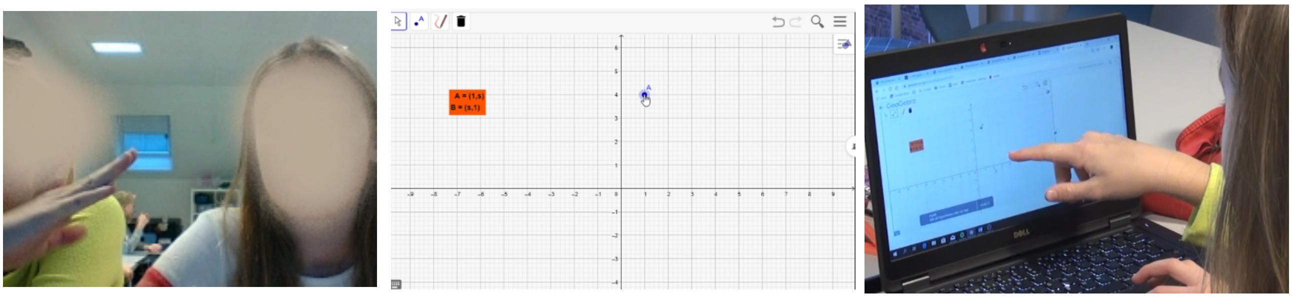

Figure 1. a, b, and c:
a) Mia's gesture, b) screenshot of Lilly's placement of point A, c) Mia points to screen approximately at (4,1)

Analysis of the example

In the analysis, we identify the structural elements and relate them to JM. Figure 2 on the next page visually illustrates Lilly's and Mia's JM process.

A process of JM starts in Lines 2-4 when the following claim (C_1) is stated and gestured: "they [*A* and *B*] are making such a slanted line like this" along with the qualifier "feeling" which indicates likelihood, not certainty. Lilly seems to base her claim on the initial data consisting of the algebraic expressions $A = (1,s)$ and $B = (s,1)$; moreover, she describes the line in her claim through a gesture (Fig. 1a), identifying certain geometrical features of such a line, possibly its "slant". Now the students go on to generate data for the claim using the instrument with the aim of changing the status of the claim, as we are about to show.

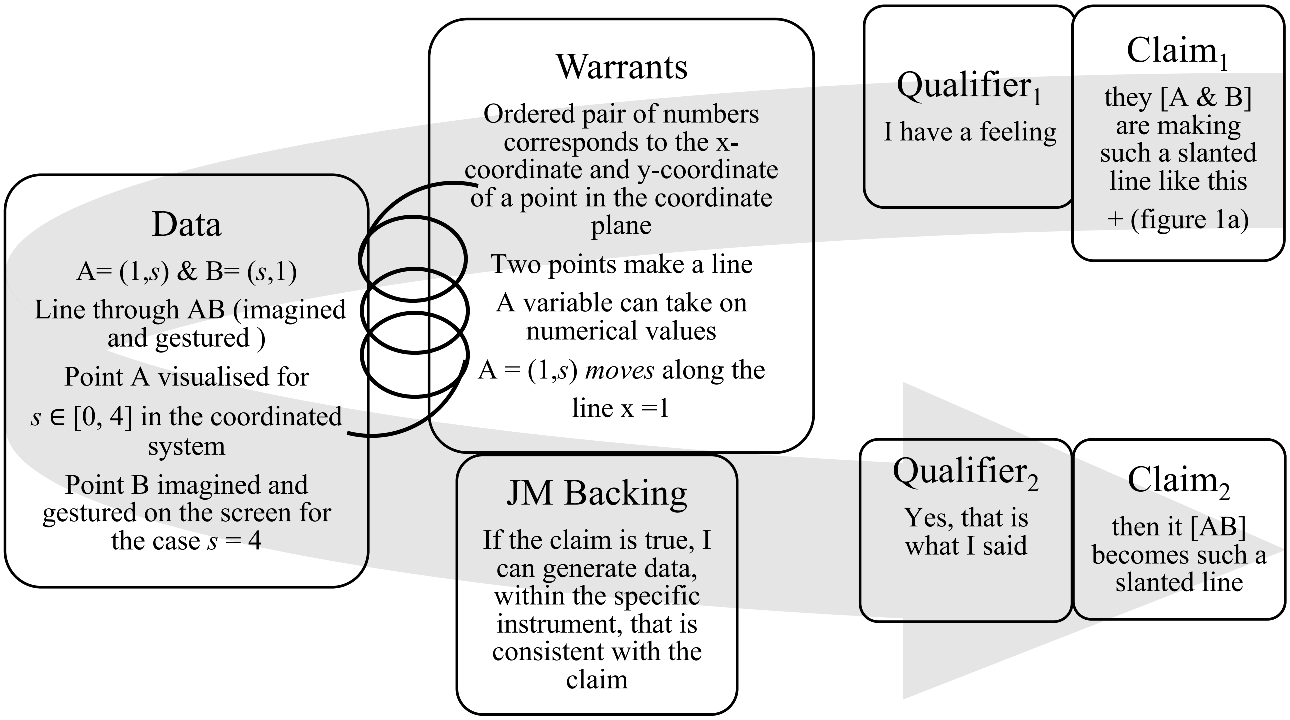

Figure 2: Illustration of Mia's and Lilly's JM process

Throughout lines 5-11 new data is generated by using the instrument. In line 5 and 6, the students create a point and establish the relationship between the algebraic notation of A and the created point. Throughout lines 7-11 data on this relationship is expressed by moving the point on $(1,s)$ from $(1,0)$ to $(1,4)$. On this basis we infer the warrants, schemes, used by the students to connect the data to the claim: an ordered pair of real numbers corresponds to the x-coordinate and y-coordinate of a point in the coordinate plane; two points makes a line; a variable can take on any real number; and $A = (1,s)$ moves along the line $x = 1$. We note that the third warrant depends on the instrument, as the *movement* of points only exists tacit within the instrument. This is an example of how warrants can contain both theoretical elements and phenomenological elements, linking the algebraic domain to the GeoGebra environment, as we discussed earlier. We infer the backing to be what we conjectured: If the claim is true, I can generate data, dependent on the specific, instrument that is consistent with the claim.

In lines 12 and 13, the students generate data regarding point B that is imagined and gestured on account of the same warrant and backing as lines 5-11. In addition, the restatement of the claim in line 13 indicates a change in its status of the claim: the utterance "Yes, that is what I said" suggests that the qualifier has changed from likely to more likely. Overall, to reach the change in status the students drew on their conceptual knowledge, as well as their knowledge about how variables are expressed within the tool. The restatement of the claim and change in its status also concludes a unit of analysis for the process of JM.

CONCLUDING DISCUSSION OF OUR ANALYSIS

In this study we seek to gain insight into informal argumentation processes as part of what we consider the students the reasoning competency and how this interplay with their use of GeoGebra. We do this with a focus on a particular form of mediation - justificational mediation (Jankvist & Misfeldt, 2019), arising from research that combines the KOM-framework and the IA (Geraniou & Jankvist, 2019). Here we designed an analytical tool inspired by Toulmin's model and grounded within the IA. In the following we will discuss and reflect upon the insights we have gained of this endeavour.

The use of Toulmin's argumentation model has allowed us to identify and amplify the importance of the qualifier as indication of change of status of a claim. This has served as a structure for identifying a unit of analysis of what can be considered a processes of JM. This supports that such a mediation is governed by the aim of changing this status of a claim; it has also allowed us to connect the generative aspects and epistemic aspects of schemes (Vergnaud, 2009) to the structure of an argument.

However, there are also limitations with this approach that relate to Toulmin's argumentation model. We do not yet find that this tool appropriately captures the crux of the matter, which is the interplay between theoretical and phenomenological components in students' informal argumentations. Aspects of this interplay can be seen through the notions of scheme and theorem-in-action, that we have adapted to the warrants and backing of the model. This adaptation feels like a long "stretch" with respect to what Toulmin's model has been previously used for in mathematics education (Simpson, 2015). Moreover, we have transformed Toulmin's model into a structure with two claims (or rather a first claim and then its restatement) and two qualifiers, to highlight the process of change in status of the claim and how it occurs. These stretches seem to be leading rather far from the initial model, and we wonder how appropriate it might be to still refer to Toulmin's model at all, also considering a posteriori how we have sort of "substituted" elements from the IA to parts of the model. Moreover, we have not yet been able to explicitly interweave the KOM-framework with the theoretical lenses used. To sum up, has referring to the IA and to Toulmin's argumentation model together supported us in understanding JM? To some extent yes, as it has provided some insight into students' instrumented activity involved in changing the status of a claim; however, it does not yet completely satisfy us.

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REFERENCES

Baccaglini-Frank, A. (2019). Dragging, instrumented abduction and evidence, in processes of conjecture generation in a dynamic geometry environment. *ZDM, 51*, 779–791. doi: 10.1007/s11858-019-01046-8

Geraniou E., Jankvist, U.T. (2019). Towards a definition of “mathematical digital competency”. *Educ Stud Math 102*, 29–45. doi:10.1007/s10649-019-09893-8

Hanna G. (2014). Mathematical Proof, Argumentation, and Reasoning. In: S. Lerman (eds) *Encyclopedia of Mathematics Education*. Springer, Dordrecht.

Hollebrands, K., Conner, A., & Smith, R.C. (2010). The nature of arguments provided by college geometry students with access to technology solving problems. *Journal for Research in Mathematics Education, 41*(4), 324–350.

Hoyles, C., & Lagrange, J.-B. (2010). Introduction. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics Education and Technology-Rethinking the Terrain: The 17th ICMI Study* (pp. 1-11). Boston, MA: Springer

Misfeldt, M. & Jankvist, U.T., (2018). Instrumental genesis and proof: understanding the use of computer algebra systems in proofs in textbook. In D.L. Ball et al. (Eds.), *Uses of Technology in Primary and Secondary Mathematics Education, ICME-13 Monographs*, (pp. 375–385). doi:10.1007/978-3-319-76575-4_22

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educ Stud Math 96*, 1–16.doi:10.1007/s10649-017-9761-8

Jankvist, U.T & Misfeldt, M.,. (2019). CAS assisted proofs in upper secondary school mathematics textbooks. *REDIMAT – Journal of Research in Mathematics Education, 8*(3), 232–266.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educ Stud Math (2019) 102,* 9–26. doi: 10.1007/s10649-019-09903-9

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM, 40*, 385–400.

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. Université Paris 8. https://hal.archives-ouvertes.fr/hal-01020705#

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691.

Simpson, A. (2015). The anatomy of a mathematical proof: Implications for analyses with Toulmin’s scheme. *Educ Stud Math, 90*(1), 1-17. doi:10.1007/s10649-015-9616-0

Toulmin, S. E. (2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. (First published in 1958.)

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94.

The semiotic potential of Zaplify: a touchscreen technology for teaching multiplication

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The Theory of Semiotic Mediation suggests that artefacts might evoke mathematical meanings if they have semiotic potential. Multiplication models that have traditionally been used as artefacts tend to have a static structure, which might limit students' physical interaction with the artefact and might hinder the representation of multiplication as a dynamic process. This paper presents an analysis of semiotic potential of a dynamic array model that is embedded in a touchscreen application. The analysis shows that this artefact might provide students with both visual and haptic experiences of multiplication, and it has the semiotic potential to evoke multiple meanings of multiplication.

Keywords: Theory of semiotic mediation, semiotic potential, digital artefacts, multitouch technology, multiplication.

INTRODUCTION

Multiplication is often introduced in second grade as repeated addition which can be problematic when children encounter multiplicative relations involved in more advanced topics (Squire, Davies, & Bryant, 2004). Indeed, there are important ontological differences between repeated addition and other models of multiplication (Confrey, 1994; Schwartz, 1988; Vergnaud, 1988).

Many researchers have conducted intervention studies to develop multiplicative thinking through models different from repeated addition. Some of these studies have incorporated the use of manipulatives (e.g., Tzur et al., 2013); others have used conceptual tools such as T-tables (e.g., Vergnaud, 1988). In this paper, drawing on the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), I examine the semiotic potential of Zaplify, a multitouch digital artefact designed to develop multiplicative thinking through a dynamic array model.

THE THEORY OF SEMIOTIC MEDIATION

The Theory of Semiotic Mediation (TSM) draws on Vygotsky's theoretical construct of semiotic mediation, and focuses on the relationship between artefacts, tasks conducted with the artefacts, and mathematical meanings. Bartolini Bussi and Mariotti (2008) define an artefact as any object that is made by human beings. Even though all artefacts share this material aspect, they are separated into three types, according to the phenomenological experience of their users (Wartowsky, 1979, as cited in Bartolini Bussi & Mariotti, 2008). A primary artefact is used to navigate one's environment. A secondary artefact is used to preserve and to transmit skills, which is necessary for the use of primary artefacts. Finally, tertiary artefacts do not have a practical goal in the sense of primary artefact yet have an autonomous world with certain rules. For example, if I move the handle of a door, I can open the door physically. Here, the handle is the primary artefact which helps me open the door. If I hold my hand in the air as if grabbing the handle and rotating my fist downwards, this gesture helps me remember how to open the door. Therefore, it constitutes a secondary artefact. The rotation of the handle might be modelled as an angle, which is a mathematical concept. In this case, this model constitutes a tertiary artefact.

Even though this categorization of artefacts might present them as mutually exclusive entities, the same artefact might be used both as a primary and as a secondary artefact (Maschietto & Bartolini Bussi, 2009). For example, for a novice learner, an abacus might be interpreted as a tool to record the counting activity. Whereas for an expert, such as a teacher, the abacus might represent the place-value. Mariotti and Bartolini Bussi explain this phenomenological aspect of the artefact use in Rabardel's' instrumental genesis.

According to Verillon and Rabardel (1995), an artifact is a man-made material object, either concrete or symbolic. An artefact becomes an instrument for a subject when s/he integrates the artefact with his/her activity. Therefore, an instrument is a psychological construct which has two components: the artefact and the subject's utilization schemes [1]. For example, a glass is an artefact which is initially designed for containing liquids. If a cook uses the glass to crash some walnuts, this artefact becomes an instrument for the cook. The cook's crashing scheme involves placing the walnuts on the cutting plate and pressing the bottom of the glass on the walnuts.

In a social context, these utilization schemes allow individuals to use an artefact to achieve a given task. While using the artefact, individuals conduct certain operations and create signs both to achieve the given task and to create shared meanings with others to collaborate in the task. This semiotic activity is essential for the process of internalization, which is defined as the individual elaboration of the previous socially lived experiences. Artefact use evolves into signs through this internalization process.

Mariotti and Bartolini Bussi (2008) distinguish artefact signs from mathematical signs in semiotic activities. The former plays a role in expressing the relationship between the task and the artefact and the latter expresses the relationship between the artefact and mathematical knowledge. In other words, the artefact signs are associated with the operations conducted to achieve the task and mathematical signs are aligned with the existing mathematical culture. Thus, the artefact mediates two meanings and this double relationship constitutes the semiotic potential of the artefact. The evolution of artefact signs into mathematical signs is the aim of mathematics education and this is achieved by the semiotic mediation of the artefacts and the cultural mediation of the teacher. At this point, Mariotti (2012) considers the analysis of an artefact's semiotic potential as an a priori phase in designing a successful teaching sequence because the specific utilisation schemes can be predicted from examining the tasks in relation to the artefact. In this paper, the analysis of Zaplify's semiotic potential aims to unearth various meanings of multiplication it could mediate.

MULTIPLICATION

The literature indicates two approaches to define multiplication: the static and the dynamic. While the former approach emphasizes the relationships between the quantities, the latter focuses on the underlying actions in multiplicative situations.

Schwartz (1988) situates the meaning of multiplication within modelling activity. He claims that the identification of quantities and referents is the basis for such an activity. Within this context, Schwartz defines multiplication as a referent changing operation that maps a quantity in one space to another quantity in a different space and suggests a graph model to represent multiplication, in which axes represent the distinct referents of multiplication. Thus, multiplicative situations require the identification of three referents and three relationships between them.

Vergnaud (1988) also emphasizes the context of multiplicative situations and claims that different aspects of a concept can be illustrated with varying situations and a situation cannot be analysed via only one concept. Therefore, he suggests a multiplicative conceptual field that involves a set of situations, schemes, concepts and theorems and formulations and symbols. Like Schwartz, he proposes that identifying the relationships between three variables is essential for multiplication. However, he points out four quantities of multiplication and the functional relationship (one-to-many correspondence) between them. In order to represent these relationships, Vergnaud suggests T-tables to model multiplicative situations.

Confrey (1994) takes a dynamic approach, describing multiplication in the context of the construction of numeric quantity. She proposes a splitting model for multiplication, which is described as “an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram” (p. 292). Even though counting the results of splitting action might correspond to the model of repeated addition, Confrey warns that “the cognitive act of recognizing a situation as multiplicative and displaying it appropriately occurs prior to this counting action” (p. 311). Confrey therefore approaches multiplication in terms of process rather than the structure of the product.

Davydov (as cited in Boulet, 1998) takes a mixed approach by pointing out both a multiplicative action and the relationships between the quantities. He defines multiplication as an arithmetic procedure that reflects an operation of transfer from a smaller unit of count to a larger unit of count. According to this, in 3x4, 3 is the multiplicand and represents the quantity of the smaller units; 4 is the multiplier and represents the quantity of the larger units. He also identified two types of relations between the quantities modelled in Figure 1: (1) many-to-one correspondence between the smaller units and the larger unit, (2) inclusion relations in composition of the product on two levels.

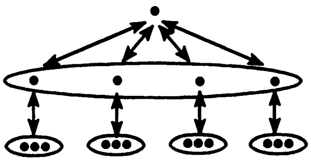

Figure 1: Multiplication as transfer of units (taken from Boulet, 1998).

In this paper, I focus on a dynamic array model, which is found in the Zaplify world of the multitouch application *TouchTimes* (Jackiw & Sinclair, 2019). I analyse this tool for two reasons. First, arrays are useful to model the multiplicative relationships (Maffia & Mariotti, 2018). Second, unlike the static models, a dynamic model might allow users to experience multiplication also as an action that brings simultaneous change in the units.

ZAPLIFY

Zaplify is an iPad application designed to enhance multiplicative thinking. It starts with an empty screen. When the tablet is placed horizontally on a surface, four of the fingerprints appear just above the lower horizontal edge of the tablet, while the other three fingerprints appear on the left vertical edge of the tablet (Figure 2a). Then a diagonal and seven fingerprints appear on the screen at the same time (Figure 2b). While the fingerprints automatically disappear in a few seconds, the diagonal line stays on the screen until the user touches on the screen (Figure 2c).

Figure 2: (a) Fingerprints, (b) Both fingerprints and diagonal line, (c) Diagonal line.

These fingerprints and the diagonal line are introduced automatically by the app to guide users to place their fingers both horizontally and vertically in the designated areas which are separated by the diagonal. When a user places and holds any finger on the screen, a “lightening rod” (referred to as “lines” henceforth) that passes through the point of touch appears either horizontally or vertically, according to the position of the touch. Touches in the upper triangular area produce horizontal lines, while the touches in the lower triangle produce vertical lines. Screen contact can be made with one finger at a time or with multiple fingers simultaneously. Multiple fingers that maintain continuous contact can create either only horizontal lines, only vertical lines or both, according to the position of the fingers (Figure 3a-c).

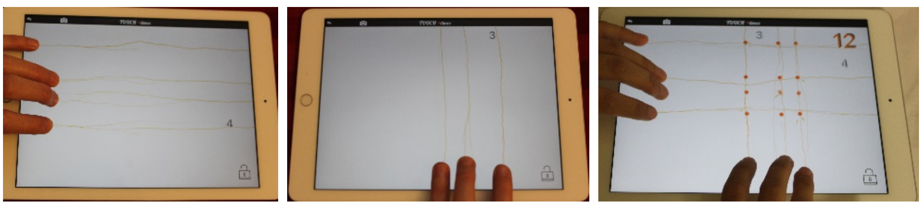

Figure 3: (a) Horizontal lines, (b) vertical lines (c) both horizontal and vertical lines.

Whenever a horizontal line intersects with a vertical line, an orange disc gradually appears on the screen. The numerical value of the total number of intersections, which is the product of two factors, appears at the upper right corner of the screen (Figure 3c). No numeral appears at the upper right corner without any intersection (Figure 3a,b).

SEMIOTIC POTENTIAL OF ZAPLIFY

An analysis of possible utilization schemes that are associated with specific tasks reveals the semiotic potential of an artefact (Mariotti, 2012). Therefore, I identify the semiotic potential of Zaplify by examining the possible utilization schemes that might emerge during two Zaplify tasks that are designed to mediate several features of multiplication. I illustrate the semiotic potential of Zaplify with some examples from ongoing research.

Task 1: Unitizing

In this task, students are asked the following questions respectively: 1) How can you make one dot?; 2) How can you make a pair of dots?. Students must create perpendicular lines to create a single dot. So, they should place one of their fingers below the screen and the other finger on the left of the screen. Placing fingers on different sides of the diagonal would create one horizontal and one vertical line. This spatial separation of the fingers together with the distinct orientation of the lines might evoke Schwartz's (1988) distinction of two separate referents in multiplication.

Creating two perpendicular lines, students would obtain a dot at the intersection point. This intersection is the necessary action for the dot to appear. Therefore, even though the children do not create the dot directly, this functionality might trigger signs that would point out the relationship between the intersection point of the lines and the dot. Thus, it might evoke the relationship between the lines and the point, which is aligned with the relationship between the factors and the product.

The drawings in Figure 4, made by a second grader, demonstrate how this task creates signs related to the factors and the product of a multiplication. The student made the first drawing after her first interaction with Zaplify. She was asked to play freely and explore Zaplify with a partner and explain it with drawings. She made the second drawing after the unitizing task, which was her second interaction with Zaplify.

 

Figure 4: Drawings of a second grader after different tasks with Zaplify.

As seen in the drawings, the multiplicative relationship between the number of lines, fingers and the dot was mediated after the unitizing task. Students written accounts also point out the evolution of signs towards a multiplicative meaning. For example, in the first drawing, the student wrote "We touch the balls and let's say if you place two fingers, we obtain four points". In the second drawing, she wrote "We made one straight (line). We also made one perpendicular (line). A ball was created."

In order to make two dots, students must place two fingers on the left, creating two horizontal lines and one finger below the screen creating a vertical line, or vice versa. Placing two fingers would not create any dots until the students place their finger below. Placing one finger would intersect the other two lines at two points and create two dots simultaneously. These simultaneous intersections at multiple points on a line might evoke the idea of unitizing through many-to-one correspondence. In other words, a smaller unit (represented by each horizontal line) would be transferred into the larger unit (represented by the vertical line) and creates the unit of unit (represented by the dots), according to the Davydovian approach.

Task 2: Skip counting

Prior to this task, a teacher models a skip counting sequence by making a pair of dots on a horizontal line and increasing the number of horizontal lines one by one. Then the teacher poses the skip counting task by asking students to "find a different way to skip-count by twos, by making vertical lines without changing the number of horizontal lines". Students might focus on how changing the number of lines affects the composition on *all* perpendicular lines. This is a more transformational approach to multiplication than repeated addition and focuses on how each line "spreads" across every perpendicular line.

As in the previous task, students should place their fingers on two different sides of the diagonal. This spatial separation of fingers again evokes two distinct units of multiplication. As the students place a new finger below the diagonal, a new vertical line and multiple dots will appear, respectively. The number of the dots will be determined by the number of horizontal lines because each horizontal line intersects with many vertical lines. Each vertical-making finger will create the same number of dots consecutively. Thus, this action evokes the simultaneous spread of each smaller unit on the larger units, one-to-many correspondences between the units. At the same time, this simultaneous spread might evoke two-level inclusion relations in which each smaller unit is included among them and in the larger unit at the same time.

CONCLUSION AND DISCUSSION

The analysis of the semiotic potential of Zaplify shows that this artefact provides both visual and haptic experiences, and triggers richer utilization schemes to evoke various aspects of multiplication. Placing fingers separately and creating dots at the intersection points of distinct lines are aligned with the three variables of multiplication (Schwartz, 1988). Moreover, unitizing and spreading actions evoke multiplication both as a dynamic process (Confrey, 1994) and as the web of functional relationships between the variables (Vergnaud, 1988).

Drawing an array on a paper or making composite units by connecting unifix cubes might seem similar to Zaplify in terms of their semiotic potential. However, these utilization schemes fail to represent the simultaneous change in multiplication. For example, drawing is a continuous action. Therefore, intersections between the perpendicular lines and the dots would be temporally separate. Whereas in Zaplify, both the intersections and dots would be created simultaneously and might evoke Confrey's description for multiplication which is creating simultaneously multiple versions of an original.

The experience of simultaneous change might strengthen students' understanding of the multiplicative relationships between the factors and the product. Increasing a factor by one changes the product with respect to the other factor. However, students may apply additive reasoning in these situations and may conclude that the product would increase by one (Squire, Davies, & Bryant, 2004). Experiencing simultaneous changes upon manipulating each factor in Zaplify, learners would sense both many-to-one and one-to many correspondences between the factors in multiple ways.

Aligning two numbers vertically or horizontally in T-tables might suggest many-to-one correspondences between the quantities in a more simultaneous manner compare to drawing arrays or connecting cubes. However, writing two entries in a T-table may not evoke a multiplicative relationship between the quantities. For example, in a T-table, aligning 1 and 2 horizontally might evoke an additive meaning where 2 is obtained by adding 1. Whereas in Zaplify, the change in the units is accomplished with both visual and haptic experiences that are not compatible with additive thinking: the user must press three fingers to see two dots on the screen.

In summary, the analysis of the semiotic potential of Zaplify unearthed important nuances in potential utilization schemes which can mediate richer meanings for multiplication compared to the existing artefacts which are also designed to develop multiplicative thinking. The analysis of the semiotic potential of artefacts may help teachers distinguish the semiotic potentials of different artefacts and unfold their potential to a great extent in the classroom. The next step would be to study to what extent Zaplify mediates these meanings in mathematics classrooms.

NOTES

1. Verillon and Rabardel defined the utilization schemes in the Piagetian tradition as "the structured set of the generalizable characteristics of artifact utilization activities." (1995, p. 86).

REFERENCES

Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics*, *18*(3), 12–19.

Bussi, M. G. B., & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L.D. English, *Handbook of international research in mathematics education* (pp. 746–783). New York, NY: Taylor and Francis.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291–330). Ithaca: State University of New York Press.

Jackiw, N. & Sinclair, N. (2019). TouchTimes [iPad application software]. Burnaby, BC: Tangible Mathematics Group.

Maffia, A., & Mariotti, M. A. (2018). Intuitive and formal models of whole number multiplication: Relations and emerging structures. *For the Learning of Matheamtics*, *3*(38).

Mariotti, M. A. (2012). Ict As Opportunities For Teaching-Learning In A Mathematics Classroom: The Semiotic Potential Of Artefacts. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (pp. 25–40). Taipei.

Maschietto, M., & Bussi, M. G. B. (2009). Working with artefacts : Gestures , drawings and speech in the construction of the mathematical meaning of the visual pyramid. *Educ Stud Math, 70*, 143–157.

Tzur, R., Johnson, H. L., Mcclintock, E., Kenney, R. H., Yan, P., Si, L., … Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA, 7*(3), 85–101.

Schwartz, J. L. (1988). Intensive quantities and referent transforming arithmetic operations. *In J. Hiebert, & M. Behr (Eds.), Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale: Lawrence Erlbaum.

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology*, *74*(4), 515-532.

Vergnaud, G. (1988). Multiplicative structures. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp.141-161). Hillsdale: Lawrence Erlbaum.

Verillon, P. & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77-101.

Hybrid environments of learning: potential for student collaboration and teacher efficiency

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It is presented an exploratory study around the impact on transforming school teaching and learning practices by means of designing and implementing hybrid learning environments, and, in this case, it was set up one on the subject of functions. This topic is studied, in the first year of a bachelor's degree in finances, during two weeks of face-to-face activity in the classroom; but in this exploratory study, the topic was addressed using a hybrid environment of learning where, in the first week, the students worked alone in a virtual environment previously designed, and teacher intervention happened until the second week, in the classroom. The results of this exploratory study showed refinement or validation of student conceptions through collaboration, and efficiency in the teacher's practice.

Keywords: hybrid environments of learning, learning collaboratively, autonomous use of online materials, change of classroom practices.

INTRODUCTION

According to Heffernan et al. (2012, p.101), if school practices must change in order to keep pace with the development of new technologies and to meet students' expectations regarding their use, then the efforts on teacher's education and in-service teacher development must be altered, there must be a greater number of interactive educational technologies developed in the cloud and implemented in the classroom. In the same way, if technologies are intended to have a powerful and lasting impact on the way that teachers teach and how much students learn, then technology developments and how they are used matter (Idem, p.102).

This paper presents some of the results of an exploratory study of the impact on transforming teaching practices and student learning, by using a hybrid learning environment, in this case to deal with the topic of functions in a first year of a bachelor's degree in finances. To carry out the exploration, two sequences of activities were designed, all of them were uploaded in a digital platform type *moodle* (http://pascal.ajusco.upn.mx/moodle). The first sequence focused on the general definition of a function, its characteristics and examples; the second sequence dealt with linear and quadratic functions, its definition, characteristics and examples. Three videos related to the content already specified were also elaborated, one appropriate to the first sequence, and two more for the second sequence. The main tool for virtual interaction between students consisted in promoting their participation at online forums. There, students were distributed in five groups (with 10 students in each) to upload, individually, their own examples of functions, so they should share and comment on them. It was planned that the students worked autonomously on all the designed activities in the digital platform (uploading own examples, commenting the other ones, reviewing elaborated videos on the topic, answering items of questions, etc.) during a week, and only during the following week both parts of the content (the general definition of function, and the linear and quadratic functions) were addressed conjointly by both students and teacher into the classroom.

The goal of the exploration we are presenting here, was to explore on the design and use of hybrid environment of learning, looking for student's support and engagement. More precisely, the interest of this exploratory study consisted of identifying the characteristics of the interaction patterns among the students, when they worked in the forums. We searched for productive collaboration among the students and for the learning of the topic of functions via a specific hybrid learning environment.

THEORETICAL FRAMEWORK

According to Heffernan et al. (2012, p. 92), it is clear that the materials and activities developed in a digital teaching platform can be used in a multiplicity of ways to change the teacher's routine, for example: in the planning of a lesson, because it can help the teacher to look towards a goal and a sequence; in the recovery of data from past lessons, to modify lesson plans based on current data on students' knowledge of the subject; in the evaluation of the lesson, since it can help the teacher to determine the success or retention of a lesson; and, finally, when students receive feedback from their classmates about their actions, which can later be capitalized by reviewing the topic in class or by solving questions associated with the feedback on the exam. Moreover, everything produced by the students in the virtual (or online) environment is registered in the digital platform, which is one of the main advantages present in the use and design of this digital medium (Dedé & Richards, 2012).

Besides, it is noteworthy that Sutherland and Balacheff (1999) announced the possibility, now already materialized, of implementing online courses, or using digital devices for teaching, freed from tutoring, accessible outside the school and managed by digital means like the Internet. Through these devices, such as videos or forums, students are left with the responsibility of unchaining their own ways of appropriating knowledge, and possible advances in learning a subject are made mainly through the exchange of opinions among peers. Actually, massive open online courses (MOOCs), for free preparation on college, and/or for professional development, have come to materialize these new educational trends (see Hoyos, 2016).

Pioneering implementation and conceptual tools to analyze massive online courses, G. Siemens (2005) and S. Downes (2010) introduced a new theory on connectivist learning. They introduced initial and simple terms as nodes, connection and network[1] (Siemens, 2005), which have been since then extensively used in empirical research (see Wang et al., 2018), seeking to establish a new alternative way of learning, the one denoted by the term connectivism. Moreover, discussing two distinct models for MOOCs' design (xMOOCs and cMOOCs), Wang et al. (2018, p. 45) argued that "the xMOOC model is now much familiar to learners and teachers in that they often use systems developed for more traditional online courses and use a predominance of video micro lessons and machine marked quizzes for student feedback. [These courses] have been criticized as not providing sufficient learner support and engagement."

While the design of cMOOCs is aimed at counteracting the deficiencies found in the earlier xMOOCs, for us, as researchers on the design of online mathematics education (i.e., not only interested on the design of MOOCs), this argumentation suggested another type of response and situation, precisely the one of exploring such criticism by means of using certain resources associated with the design of blended courses, or, more specifically, we decided to explore on the design and use of hybrid environment of learning, looking for student's support and engagement. In fact, this research proposal emerged from considering to exactly combine two very distinct ways of delivering a course: one part corresponding to a traditional online course (it function as above Wang et al. mentioned in relation to xMOOCs), specially designed for students to work autonomously on the study matter, and on the other hand, during a second part of the course, the topic would be addressed in a face to face classroom under teacher guidance. Finally, it's important to note that in general one of the research questions, in relation to the design and implementation of hybrid learning environments, is the search or observation of interaction patterns among participants. Therefore, the interest of this exploratory study consisted of identifying the characteristics of the interaction patterns among the students, when they worked in the forums. We searched for productive collaboration among the students and for the learning of the topic of functions via a specific hybrid learning environment.

As it was just mentioned above, connectivism is a theory that underlies the design and implementation of specific online and hybrid learning environments, and it was emerged linked mainly to the use of the Internet, as well as virtual education. However, many researchers still question what this theory explains, provides or suggests, for example, regarding the incorporation of technology in the classroom (see, for example, Kop & Hill, 2008). Whether it could do this regardless of previous theories or as an extension of some of them, or of theoretical models that have so far been applied to study the integration of technology in school (to see some of these models, see Zbieck & Hollebrands, 2008; Olive et al., 2010; Ruthven, 2014). Although, according to Downes (2010), what connectivism has to exhibit is to what extent is an emerging theory, and empirically proving in what sense is a new paradigm, that would specifically explain the case of network learning and collective distributed knowledge.

At last, it is also important to mention that for Kop & Hill (2008), the connectivist metaphor is particularly timely in the present time, given that Internet browsing and the means by which information is dispersed in this medium provide a point of reference to validate the claims of Siemens (2006) in relationship with the need to redesign education. From these authors' point of view, connectivism frames learning in terms of students connecting to nodes in a network, which in effect suggests that knowledge does not reside in a location but rather is a confluence of information that is originated from a multiplicity of individuals seeking to inquire regarding a common interest and providing feedback to each other. And, finally, it is worth to highlight that in the present exploratory study the connectivist metaphor has been concretized in students' networks, where they represented nodes connecting and interacting, virtually, by means of a series of digital forums we set up in a digital platform of teaching.

METHODOLOGY, ANALYSIS, AND RESULTS

In this exploratory study, a group of 61 students was enrolled to participate in the implementation of a hybrid learning environment in a college algebra course; 50 of them had active participation in the online forums that were opened to this effect. Three videos for presenting the topic of functions to the students were also elaborated and uploaded in the platform, definitions and examples were discussed there. As part of the activities, questionnaires were also prepared and uploaded. All the materials were hosted on a digital *moodle*-type platform, namely http://pascal.ajusco.upn.mx/moodle. It could be said that the main tool for student virtual interaction was their participation in digital forums. Five online forums (with 10 students each) were opened for students to upload (individually crafted) examples in each sequence of activities, following the written teacher instructions to share and comment on them. The general plan was that throughout the first week the students should work virtually, within the platform, autonomously, and during the following week the topic would be addressed in class, face-to-face, under teacher guidance.One member of the researchers' team was in charge of videos and quizzes' elaboration as well as of the topic's teaching at the classroom. She was formally interviewed by means of a semi-structured interview that was also registered, in order to retrieve what had happened in the classroom during the face-to-face week of teaching.

Besides, everything elaborated by the students in the digital platform was a subject of analysis and classification. Specifically, we used the SOLO taxonomy to classify student responses included in their activity in the forums, mainly those that showed their individually crafted examples of functions, in order to identify refinements and validation of conceptions on the topic, achieved or included in student's feedback or the exchanges among them. The SOLO taxonomy by Biggs and Collis (1982) was inspired by Piagetian descriptions of qualitative differences in handling the same tasks and was intended to help in the analysis of student's responses to open-ended questions in school (Marton, 2015, p.115). The different categories refer to different ways of handling the same task, and basically they consist in four different levels: in prestructural level, no crucial aspects of the task are mastered; in unistructural level, one crucial aspect of the task is mastered; in multistructural, several crucial aspects are mastered; finally, in relational level, several crucial aspects are related to each other. Moreover, it was useful for visualizing the classification of the students' elaborations, take a spreadsheet to identify productive exchanges, namely those related to refinement or validation of students' conceptions. Due to the space available in this paper, only a small part of the spreadsheet content is presented here (see Figure 1). Summarily, the SOLO taxonomy was used in this study as a tool for classifying all examples developed individually by the students on the topic of linear and quadratic functions. In addition, a spreadsheet was used to visualize student interrelationships derived from their activity in the forums.

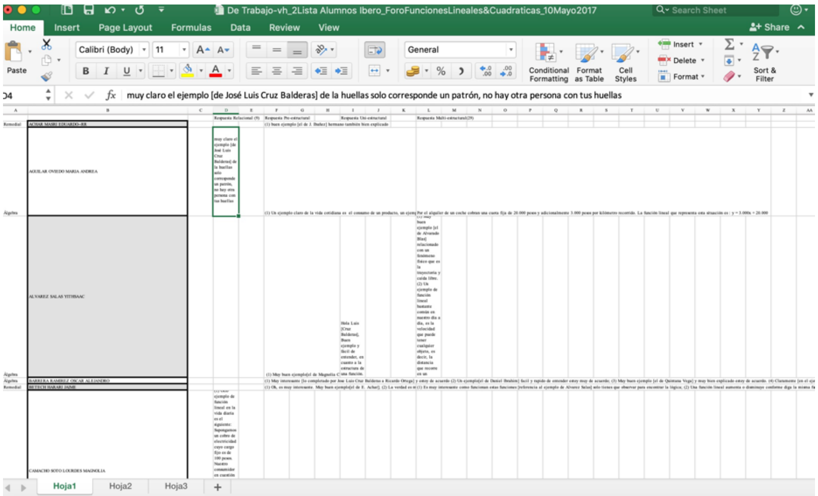

Figure 1. Compilation of student examples of functions, according to SOLO taxonomy

On the other hand, complementary data were obtained from the teacher interview, precisely on the role of the topic in the course's syllabus, and on the role of what the students had done online in the classroom, as it is shown in next segments.

> Teacher: In this case, I, as a mathematician, what I knew was that they [the students] needed to know any type of function [the teacher refers to the general characteristics of a function, that is, to its definition]; to be able to see its [functional] expression, to know how its graphic was going to be; and also see its graph and know what was the corresponding [functional] expression; and to know what were properties of those functions [she referred to the linear and quadratic functions], things that are immediate and automatic. The course's syllabus [on the topic of functions] goes a long way in that sense.
>
> ...
>
> Interviewer: How was everything that the students had done online used in the classroom?
>
> Teacher: In this respect what happened was that with the concept of linear function and quadratic functions, I would have worked two weeks into the classroom, if the virtual module of the course had not existed. However, what happened was that I decided that in a week, students would go on their own using the virtual module. What I did was that the following week, after they worked on this, I went back to what they had worked on, but instead of spending so much time [like I used to do] I spent only a week in face-to-face working. It is as if the two weeks that I would have dedicated to the subject, instead of doing it face-to-face, now I did one [week] in a virtual type for students to work with the concepts and to become familiar with them, and in the other [week], I did it face-to-face. What happened with the virtual [part of the course] the thing was that it allowed me to move faster, during the face-to-face week, knowing that they had reviewed the materials.

Schematic Analysis of Patterns of Interaction or Collaboration Among Students

From a relational level

Student MAA: Very clear the example [of student JL] of the fingerprints, [fingerprints of one person] only corresponds to one pattern, [for example] there is no other person with my fingerprints

Student LMC: Another example of a linear function in daily life is as follows. Let us suppose an electricity charge whose fixed amount is for 100 pesos. Our consumer in question has had a consumption of x watts amount and each watt price is 2 pesos. The function would be expressed in the following way: $f(x) = 2x + 100$. Thus 2 is our value in a, and 100 is the constant that we add. [Also] a very good example [the one of AS] about something that has an application in daily life, although it should be mentioned that this is only valid for uniform movements (where speed is constant). [In addition, I also] understand the example [the one given by CF] and it seems valid to me, but I consider that because having two antecedents for the same image (this does not meet the definition of function) we would need just one person wanting two things at the same time and not that two people are the same since x_1 could be equal to x_2 without this affecting the function as long as $f(x_1)$ is equal to $f(x_2)$. [Finally, also] I agree with another example [the one given by MAA], two phones can have the same price, therefore, the same image can have two antecedents, but a singular phone would not have two prices (obviously if we only talk from a provider) so an antecedent could not have two images.

From a multistructural level

Student YAS: Very good example [the one given by BA] related to a physical phenomenon, as it is the trajectory and free fall. [Another] example of a fairly common linear function in our day to day is the speed that any object can have, that is, the distance it travels in a given time. Speaking a little more specifically, assuming that a car on a flat road tends to travel 20 km in 5 minutes, with a linear function, the distance it will travel in 25 minutes could be determined. The algebraic expression, in this example, could be $f(x) = 4x$. Where x represents the [number of] minutes you want to calculate to see the distance traveled.

From an unistructural level

Student ALA: When throwing a ball, it first goes up and forward, then falls while continuing to advance, thus forming a path shaped like an inverted parabola.

From prestructural level

Student MAA: A clear example of everyday life is the consumption of a product, an example is the purchase of phones, there are different phones: price levels, with x = the phone and y = the price depending on which phone you would like, the price increases, but all phones have the same function: communicating. [Another] excellent example [is the one given by CS], it was very clear to me how we apply linear functions in daily life. [It is also] an excellent example [the one given by AS] because it helps you understand what a linear function is, very simple, with an example from daily life.

DISCUSSION AND CONCLUSIONS

Teaching and learning in a hybrid environment become productive when the implementation of digital materials previously designed serves to detonate conceptual questions that can be subsequently addressed in the classroom. By means of the exploratory study presented here we found different patterns of interaction among the students, derived from the autonomous activity of the students at the forums, as it's the interest of searching for the connectivist framework. We have shown that these patterns were established according to distinct levels of mastery of the topic in question. In other words, paraphrasing what the teacher said during the segments of the teacher interview we have shown above, we can see that "the students worked with the concepts and became familiar with them", at different levels of understanding we could add. It is to say that the virtual work of the students at the platform has suggested how the teacher could retake the topic in the classroom "to move faster", as she (the teacher) already said. Student interaction, or the collaboration among them, via the forums, became a real source of information of the distinct levels of student understanding of the topic in question, in this case on linear and quadratic functions at the first year of finance at college. It also allowed students to think beforehand about the topic in question, actively participating in the virtual modality, and trying to propose mathematical examples, which even if not mathematical correct were productive for triggering mathematical collaboration between peers. Moreover, student autonomous practice in the virtual environment really had an impact on classroom teaching, it enriched teacher information on student knowledge, prior to learning and teaching practices in the classroom. In fact, and according to Heffernan et al. (2012), the implementation of the hybrid environment of learning changed teacher routine and did make both learning the subject more meaningful and teaching more effective. In summary, this paper contributed to the study of the impact on transforming teaching and learning mathematics in the classroom, in particular exploring the opportunities offered by the design and implementation of hybrid learning environments that incorporate the use of multimedia in digital platforms for student collaboration and network learning, in the sense already indicated by Downes (2010).

NOTES

1. According to Siemens (2005), "a network requires at minimum two elements: nodes and connections. Nodes carry different names in other disciplines (vertices, elements, or entities). Regardless of name, a node is any element that can be connected to any other element. A connection is any type of link between nodes."

REFERENCES

Dedé, C. & Richards, J. (2012). *Digital Teaching Platforms: Customizing Classroom Learning for Each Student*. New York & London: Teachers College Press.

Downes, S. (2010). Learning networks and connective learning. In H. Yang and Ch. Yuen (eds.), *Collective Intelligence and E-Learning 2.0: Implications of Web-Based Communities and Networking*. USA: IGI Global.

Heffernan, N., Heffernan, C., Bennett, M. & Militello, M. (2012). Effective and meaningful use of educational technology: Three cases from the classroom. In C. Dede & J. Richards (eds.), *Digital Teaching Platforms: Customizing Classroom Learning for Each Student*. New York & London: Teachers College Press.

Hoyos, V. (2016). Distance technologies and the teaching and learning of mathematics in the era of MOOC. In M. Niess, S. Driskell And K.Hollebrands (eds.), *Handbook of Research on Transforming Mathematics Teacher Education in the Digital Age*. USA: IGI Global.

Kop, R. & Hill, A. (2008). Connectivism: Learning theory of the future or vestige of the past? *IRRODL, Vol. 9*, No. 3.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge

Olive, J., Makar, K., Hoyos, V., Kee Kor, L., Kosheleva, O. & Sträber R. (2010). Mathematical Knowledge and Practices Resulting from Access to Digital Technologies. In C. Hoyles and J.B. Lagrange (eds.), *Mathematics Education and Technology-Rethinking the Terrain (17 ICMI Study Book)*. New York, Dordrecht, Heilderberg, London: Springer, p.133-178.

Ruthven, K. (2014). Frameworks for analyzing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In Clark-Wilson et al. (Eds.), *The Mathematics Teacher in the Digital Era. Mathematics Education in the Digital Era 2*. Dordrecht: Springer.

Siemens, G. (2005). Connectivism: Learning as network-creation. Retrieved, on July 15, 2020, at *http://masters.donntu.org/2010/fknt/lozovoi/library/article4.htm*

Sutherland, R. & Balacheff, N. (1999). Didactical complexity of computational environments for the learning of mathematics. *International Journal of Computers for Mathematical Learning*, *Vol (4)*, pp. 1-26.

Wang, Z., Anderson, T., and Chen, L. (2018). How learners participate in connectivist learning: An analysis of interaction traces from a cMOOC. ***I**nternational **R**eview of **R**esearch in **O**pen and **D**istributed **L**earning, Vol (19)*, No.1.

Zbiek, R. & Hollebrands, K. (2008). A research-informed view of the process of incorporation of mathematics technology into classroom practice by in-service and prospective teachers. In *Research on Technology and the Teaching and Learning of Mathematics: Vol. 1. Research Syntheses*. USA: Information Age Publishing, pp.287-344.

The social development of knowledge in a new pedagogical setting: the same activity presented as three different interactive diagrams

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Teaching with technology-based interactive curriculum materials (TBICMs) should be considered more than a technological change; indeed, it is an attempt to create new paths for the construction of mathematical meaning. The paper investigates how students addressing the same activity presented as three different Interactive diagrams (IDs) participated in collaborative learning processes. The process allowed us to analyze the social development of knowledge in a new pedagogical setting. The results showed that the participants collaborated to generate an interactive text based on the given IDs. The text became an instrument which supported the development of knowledge.

Keywords: Technology-based interactive curriculum materials, collaborative learning, animation, interactive diagrams.

INTRODUCTION

There are profound differences between the traditional page in math curriculum materials that appears on paper and the new page that derives its principles of design and organization from the screen and the affordances of technology. In traditional curriculum materials, content is displayed in a static mode and students are invited to interpret it with limited possibilities of interaction, e.g., by pointing to a figure or tracing with a pencil. In contrast, recent technological advancements have enabled the production of TBICMs: a new type of materials that enables a broader interaction between the users and content. TBICMs comprise a set of IDs, namely, a relatively small unit of an interactive materials that can be used for different purposes: an exposition, a task, an exercise, etc. In such materials, students are invited to interpret the content by interacting with it, e.g., by playing a video clip, interacting with a graph, or changing the given examples.

Teaching with TBICMs should be considered more than a technological change; indeed, it is an attempt to create new paths for the construction of mathematical meaning. Computer technologies allow the design of a variety of IDs for the same task. The findings of our long-term research show that similar tasks with different IDs should be considered to be different learning settings (e.g., Naftaliev & Yerushalmy, 2011, 2017; Naftaliev, 2018). The interesting question is whether and how students who had been asked to address similar activities that included different IDs can share their work, participate in a group discussion and participate in social development of knowledge in a new pedagogical setting. This paper addresses the question.

FRAMEWORK FOR PEDAGOGICAL FUNCTIONALITY OF IDS

Naftaliev & Yerushalmy studied how, as the core of engaging tasks, IDs can be designed to form a pedagogical tool for various teaching intentions and students' needs and developed a semiotic framework for pedagogical functionality of IDs (e.g., Naftaliev & Yerushalmy, 2011, 2017; Naftaliev, 2018). The framework proposes three dimensions, or functions, for defining the pedagogical functionality of IDs, which address a variety of learning and teaching settings: a presentational function, which refers to the type of example in the ID and consists of three types of examples: specific, generic, and random; an orientational function, which refers to the mode of representations in the ID, i.e., metric, schematic, or metric/schematic; and an organizational function, which refers to the connection between all the components of the ID, i.e., elaborating, guiding, and illustrating).

Although examples in an ID are usually designed to be modified by the user, the example that initially appears in the ID determines the nature of its presentational function. Three types of examples are widely used in IDs: specific, random, and generic. Specific examples serve as a dynamic illustration that helps to analyze the situation without being able to change the information. Random examples are specific examples generated within given constraints. Generic examples are those in which the ID is structured to be representative; it presents a specific example as part of the given task, but it is not intended to present the specific data of the activity. Rather, it is aimed to help learners become aware of the representativeness of the example through a process of inquiry.

The tone in which the ID addresses the learner is subject to design decisions that regard the orientational function. Netz (1999) identified a connection between types of diagrams and the practices of ancient Greek mathematicians regarding their use. "The most significant question from a mathematical point of view is whether the diagram was meant to be metrical: whether quantitative relations inside the diagram were meant to correspond to such relations between the object depicted. The alternative is a much more schematic diagram, representing only the qualitative relations of the geometrical configuration. … they very often seem to be schematic in this respect" (ibid, p.18). Based on Fish and Scribner (1990), Mason (1995) drew attention to the importance of sketches rather than paintings as a metaphor for providing stimulus to students: "A painting has richness of detail, but its completeness of detail means that the observer has to work in order to see through the whole, to make contact with and examine details and yet retain a sense of connection to the whole; a sketch provides just enough to invoke Gestalt powers of closure and to initiate a process of construal" (Mason 1995, p. 386). The framework considers schematic vs. metric modes of the ID as an important factor in reader orientation. An example that appears in an ID can have an accurate metric appearance, namely, quantitative relations inside the diagram are meant to correspond to such relations between the objects depicted. The alternative is a schematic diagram, representing only the qualitative relations. The design of the ID has made it possible to address the given graphs as a sketch in a schematic mode, but, at the same time, the sketch can be interactively unfolded into a detailed metric diagram. For example, the graph that is represented in a schematic ID could also serve as a metric ID by providing the values of ordered pairs for any point on the plane, according to the users' choice.

The organizational function of IDs refers to the ways in which an ID can be organized, namely, illustrating, elaborating, and guiding. The three ways differ in their settings, as each is characterized by its own constraints and resources and is intended for a different aspect of inquiry. Illustrating IDs are simply operated, unsophisticated representations. For example, an animation with a limited degree of interventions by activation of controls in it (fig. 1). At any time, users can freeze the positions on the track, continue the run, or initialize the race. Elaborating IDs provide the means that students may need to engage in activities that lead to the formulation of a solution and to operate at a meta-cognitive level. The same animation that serves as an illustrating ID in the previous example can be part of an elaborating ID when set within other tools and representations (fig. 1). The elaborating ID provides four adjacent linked representations: a table of values that represents distance and time; a two-dimensional graph of distance over time; a one-dimensional graph which traces the objects' positions at each time unit; and an animation. The variety of linked representations and rich tools in this elaborating ID enables various options in viewing and interactions with the ID: as a schematic and/or as a metric diagram, as discrete information and/or as a continuous flow of information. Guiding IDs are used for guided inquiry; in addition to providing resources that promote inquiry, they set the boundaries and provide a framework for the process of working with the task. The guiding IDs are designed to call for action in a specific way that supports the construction of the principal ideas of the activity. They serve to balance constraints and open-ended explorations and to support autonomous inquiry. For example, the guiding ID in the fig. 1 was designed around a known conflict about a time-position graph describing a "motionless" situation over continuously running time. The ID consists of two representations of motion: an animation and a hot-linked position-time graph. The task is to establish a one-to-one correspondence between the graphs and the animation. The graph and the animation are only partially linked: motion occurs simultaneously on the animation and on the graph but there is no colour-match, so the identification process requires extracting data from the simulation and the graph in order to link them. The following constraints contribute to making the task an interesting challenge: the small number of animated representations, the partial linking between the representations, the absence of representations and controls that could turn the given schematic nature of the representations into an accurate metric diagram, and the exceptional example in a list of examples that are aimed at focusing on a motionless situation over time.

As computer technologies allow the design of a variety of IDs for the same concept, deciding which design to adopt in order to convey certain pedagogical functions is considered one of the urgent needs facing educators in the use of TBICMs.

THE SOCIAL DEVELOPMENT OF KNOWLEDGE IN A NEW PEDAGOGICAL SETTING

In collective practices, common goals are accomplished through the interrelated activities of individuals: individual activities are constitutive of collective practices; and at the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities (Saxe, 2015). Saxe defined three dimensions for analyzing the development of mathematical knowledge by students: the micro-, socio-, and ontogenesis. Each kind of development has its roots in activity as individuals use forms, like the IDs in our case, to serve varied functions to structure and accomplish emerging goals. Micro-genesis involves the short-term process whereby individuals structure forms into means to accomplish goals in activity. Socio-genesis involves the spread in use of forms as means for structuring and accomplishing goals in a community of individuals. Ontogenesis involves the interplay between the forms that students use and the functions that they serve over the course of student's development. A coordinated analysis of these three dimensions of activity was used in our research.

The fig.1 shows the research design. The interviews were sorted by students into groups of three. Each participant followed a four-step procedure that enabled us to examine and track the role of IDs in the student's knowledge development process. The activity, which asked the students to describe a motion situation, was first illustrated by a video clip and subsequently as an Illustrating, Elaborating or Guiding IDs, all based on the animation. At Stage A the students were given a preliminary task, presented by videoclip. The task designed to evaluate their knowledge and solution techniques. At Stage B the Students were given a task similar to the one they received in Stage A; the difference was that the task included the ID. The purpose of the interview was to learn how the students constructed their knowledge using the diagram. At Stage C, the three students who had been asked to address similar tasks that included different diagrams participated in a group discussion. The students could use all the diagrams they worked with in stages A and B. At Stage D the students were given the similarly printed task (fig. 2) and the question to be answered was "What can the students do alone, without the assistance of interactive texts?". The 14-year-old students volunteered to participate in after-school meetings. All interviews were video recorded. Each participant followed the four-step procedure that enabled examining and tracking the role of IDs in the students' knowledge development process concerning mathematical models of motion.

In this paper, I focused on the second step of the activity, asking the students (Elad, Helena and Or) who had already been asked to address similar activities that included different IDs to share their work and participate in a group discussion. This section includes an analysis of Elad, Helena and Or collaborative work.

At the first stage, in their individual work with the video activity, the learners put emphasis on getting the story right, which required attending to details such as the runners' body motion: "When they ran, they moved their body a little bit back and their feet a little bit forward and… this maybe gave them, I think, more acceleration. And in the end the one that was on the left won. They all made almost the same movements; just that there were some that started running and some that jumped out later and some that jumped a little sooner." The video clip kept the students too close to the situation and prevented them from thinking in the abstract.

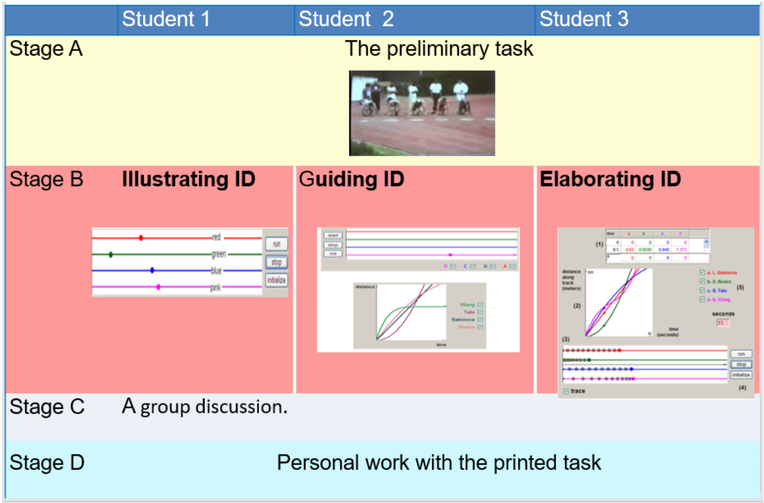

Figure 1. Four-step study procedure

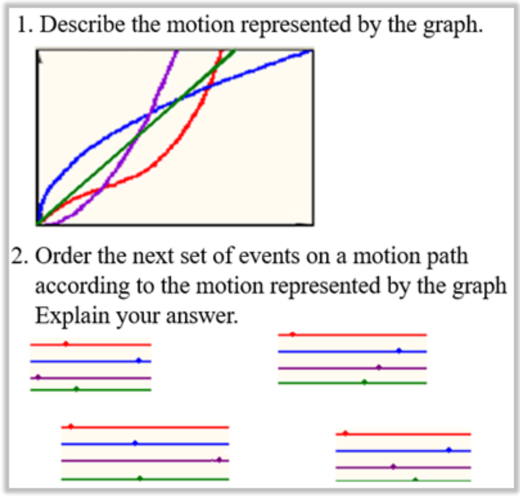

Figure 2. The printed task

At the second stage, Elad, the student who worked with the illustrating ID, started by activating the animation. Throughout the process, he stopped the animation several times. During each pause, Elad examined the runners' respective positions and described the changes in speed. Elad described each runners' changes in speed with reference to their relative positions at specific moments. He mistakenly interpreted continuous change of speed by comparing relative positions. For example, he argued that passing another runner must have meant speeding up, whereas, in reality, the runner maintained a constant speed. To cope with the challenge, Elad resorted to a failed attempt at drawing graphs by himself to complete the diagram.

Helena, working with the elaborating ID, started by activating the representation and tools in the ID. She learned about the wide variety of options and representations available in the ID, but there was no evidence showing developing knowledge concerning mathematical models of motion processes.

Or, the student who worked with the Guiding ID, began his work by identifying a visual and kinematic conflict: while all seven dots moved on the graphs, one of the dots in the animation stopped and remained still. To resolve this conflict, he focused on discrete events much like Elad, using discrete events to match the motions described in the animation and graph extracting discrete motion characteristics such as: average speed, time and distance. He successfully matched the dots yet failed to resolve the conflict.

At the third stage, in the group discussion, Or decided to open the conversation with the question which remained unsolved in his individual work (Fig. 3). He demonstrated the problem while activating the Guiding ID with which he worked. Elad and Helena were intrigued by the question and it became the goal of their collaborative work. They began by familiarizing themselves with the options of the ID to resolve the conflict. When they were unsuccessful resolving the conflict using the Guiding ID and realized their diagrams were different, Helena suggested using representations and tools from her Elaborating ID to accomplish the goal they defined for themselves. Each time she suggested adding only one option from the Elaborating ID. They used it firstly to develop meaning regarding the motion presented in the Elaborating ID. Then, they used the ideas which they developed to resolve the conflict using the Guiding ID. The following presents the process which took place in the last step of their work in which they successfully resolved the conflict.

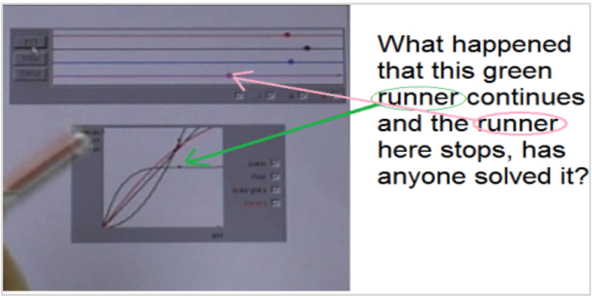

Figure 3. "…has anyone solved it?"

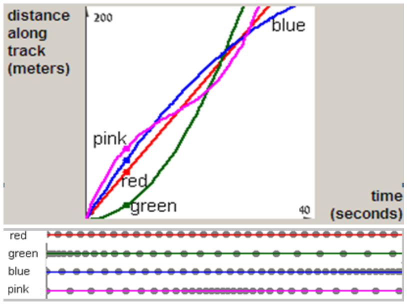

Figure 4. 2D and 1D (traces) graphs

Following a suggestion from Helena, the students activated the animation with traces, resulting in the generation of a 1D graph of the motion (Fig. 4). While running the animation and generating of a 1D graph, they read the race from the traced motion using the size of the spaces between the traces as a gauge for speed:

Helena: Press on traces. You see! Where they are stopping?

Or: Ahh… Yes, it describes every time point.

Elad: It describes the steps, the distance of the steps.

Helena: Here, you see the black starts [green] to advance more.

Elad: Pink starts with greater steps. If the traces describe the steps then here he starts to slow down as the time goes on and here it stays at the same speed.

Helena: And the black [green] is really fast.

Elad: But in the end he speeds a bit. The black [green] almost doesn't, he starts with slowness, as the time goes on, his steps only enlarge.

Helena: The red doesn't change… and the red… at the same speed

Elad: And the red, like I told you in the beginning, remember? That the red is always at the same distance, at the same speed, the same steps. And the blue at the beginning until the middle at the same speed, same steps and towards the end he starts to slow down.

Following the interpretation of the 1D graph as describing speed, the students checked whether this option was available in the Guiding ID. Once they verified it was not, they returned to work with the Elaborating ID. They began by interpreting the 2D graph based on the 1D graph in static mode with which they became familiar. In the end they were able to describe the speed by using only the 2D graph.

Knowledge development concerning Characteristics of Motion and the elements of IDs	In stage 2			In stage 3
	Elad with Illustrating ID	Helena with Elaborating ID	Or with Guiding ID	Helena, Or and Elad
Familiarize him/herself with the elements of the IDs	✓	✓	✓	✓
Discrete Characteristics (Animation): average speed, time for distance, distance	✓		✓	✓
Discrete Characteristics (Graph): average speed, time for distance, distance			✓	✓
Continues Characteristics describing the motion process (Animation and Graph): speed, time, and distance				✓

Table 2. Knowledge developments in the second and third stage

Helena: Wait, in his [Or's] diagram there is it [the traces]? Check.... It is interesting what happened with the pink in his [Or's] diagram [Guiding ID].

Or: I think that this [the Elaborating ID] is the best.

Helena: The red is running at the same speed. The black in the beginning runs really slow, and then he ups his speed more and more [they closed the 1D graph and continued work only with the 2D graph]. The blue runs really quick and then he starts to slow down. The pink runs fast, in the middle he slows down and then in the end again he runs fast.

Once they have succeeded in interpreting the 2D graph in the Elaborating ID, they were able to resolve the conflict they had about the motionless process presented by the Guiding ID:

Or: Yes. So, as the line is steeper, then his speed is... ehh... it is steep and… that's it, I see that in the end it turns into a straight line, plain, something like this. That means that he slowed the speed and even stopped in place.

Elad: If this shows distance, then it means that the distance here does not change.

The episode describes the students' exploration concerning the description of speed in the mathematical models in four stages: (1) analysis of a dynamic mode of 1D graph which was linked to the running animation; (2) analysis of a static mode of 1D graph;

(3) analysis of shapes of 2D graphs and (4) analysis of a motionless process represented by 2D graph. The Table 2 summarizes the changes which occurred in the students' knowledge at the end of the process. We can see that they were able to analyze not only discrete characteristics but also continuous characteristics describing the motion process.

The results showed that the development of knowledge occurred when the students engaged in a reflective activity concerning other members' reasoning and instruments involved in the collaborative process. As a result of the collaboration, students generated an interactive text: they posed a new question, decided what component from what ID to bring to discussion, decided on the sequence between the components, defined the role of each component, and created a representation of the data. The analysis clarified that choosing and combining representations from similar tasks, which were designed as different IDs, reflected students' personal choices to anchor their inquiry in the more familiar ones.

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REFERENCES

Fish, J., & Scrivener, S. (1990). Amplifying the mind's eye: sketching and visual cognition. *Leonardo, 23*(1), 117-126.

Mason, J. (1995). Exploring the Sketch Metaphor for Presenting Mathematics Using Boxer. In A. A. diSessa, C. Hoyles, R. Noss, & L. D. Edwards (Eds.), *Computers and exploratory learning*, 383-398. Berlin Heidelberg: Springer.

Naftaliev, E., & Yerushalmy, M. (2011). Solving algebra problems with interactive diagrams: Demonstration and construction of examples. *Journal of Mathematical Behavior, 30*(1), 48-61.

Naftaliev, E., & Yerushalmy, M. (2017). Design digital tasks: Interactive Diagrams as resource and constraint. In A. Leung, & A. Baccaglini-Frank (Eds), *The role and potential of using digital technologies in designing mathematics education tasks*, 153-173. Cham: Springer.

Naftaliev, E., (2018). Prospective Teachers' Interactions with Interactive Diagrams: Semiotic Tools, Challenges and Well-Trodden Paths. In Fan, L., Trouche, L., Qi, C., Rezat, S. & Visnovska, J. (Eds), *Research on Mathematics Textbooks and Teachers' Resources: Advances and Issues,* 297-314. Springer.

Netz, R. (1999). *The shaping of deduction in Greek mathematics***.** Cambridge University Press, UK.

Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *Journal of the Learning Sciences*, 11(2-3), 275-300.

Reasoning with digital technologies - counteracting students' techno-authoritarian proof schemes

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In this paper, we analyze and discuss an empirical example related to students' so-called techno-authoritarian proof schemes, i.e. when digital technologies come to play the authority of establishing mathematical truth. The example stems from a pilot study concerning the interplay between the uses of original historical sources, in this case Euclid's Elements, and digital technologies, here the dynamical software environment, GeoGebra. As part of the analysis, we make use of the constructs of justificational, epistemic and pragmatic mediations as analytical tools for planning the work with proofs and proving using digital technologies. We conclude that digital tools can indeed support students' work with original mathematical sources, and that these sources further can support an explicit and common awareness of difficulties and possibilities while working with proofs and proving within these technologies, not least in trying to counteract the activation of students' techno-authoritarian proof schemes.

Keywords: Digital technologies, GeoGebra, proof scheme, techno-authoritarian proof-schemes and original mathematical sources

THEORETICAL OUTSET

In a context of using digital technologies (DT) in proofs in mathematics textbooks, Misfeldt and Jankvist (2018) suggest that DT may play the role of an authority, leaving students with the impression that the DT itself establishes mathematical truth and justification. They do so by relying on Harel's and Sowder's (2007) notion of a *proof scheme*, i.e. "A person's (or a community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" (p. 809). While ascertaining is the process employed to remove a person's own doubts about the truth of an assertion, persuading is the process employed to remove other people's doubts. Harel and Sowder distinguish three types of different proof schemes. Our usually accepted *deductive proof scheme* within mathematics, i.e. that one is convinced of the truth of a mathematical result by means of logical deduction in the form of conventional proofs etc. Next, *empirical proof schemes* that come into play when using specific empirical examples to justify general statements. Finally, *external conviction proof schemes*, which are manifested as: an authoritarian proof scheme, e.g., that something is held to be true because some authority (e.g. the textbook or the teacher) says so. Misfeldt and Jankvist (2018) coin the term *techno-authoritarian proof schemes* as referring to students being convinced by DT of the truth of a mathematical statement (theorem, etc.). Yet, they only suggest that this was likely to be the case, based on their analyses of the blackboxing (Buchberger, 2002) use of Computer Algebra Systems (CAS) in Danish mathematics textbooks.

Furthermore, Misfeldt and Jankvist (2018) distinguish between three types of mediations; *justificational mediations, epistemic mediations* and *pragmatic mediations,* when CAS is used in proving activities. These three types of mediations build upon theoretical constructs from the area of using digital technologies (e.g. Rabardel & Bourmaud, 2003). Jankvist and Misfeldt (2019) explain how these three types of mediations are connected to proofs and proving:

Epistemic mediations are connected to proofs that explain (Hanna, 1990), as well as to deductive proof schemes (Harel & Sowder, 2007). *Justificational mediations* are related to proofs that only proves, i.e. without explaining. Furthermore, such mediations are connected to external conviction proof schemes. If statements are true because the CAS says so, CAS mediates a justificational process. *Pragmatic mediations* may be connected to one or more of the different proof schemes, including the empirical proof scheme, by providing necessary but laborious calculations and manipulations required for a certain argument. (p. 249)

Jankvist and Misfeldt (2019) used these three types of mediations to analyze Danish upper secondary mathematics textbooks addressing the didactical effects of CAS assisted proofs herein. Justificational mediations are to some extent related to techno-authoritarian proof schemes—or rather, they have a high risk to lead to such proof schemes. The risk is amplified, if students and teachers are not aware of this while using DT in their work with reasoning and proving. Jankvist and Misfeldt (2019) also refer to research literature (e.g. Dreyfus, 1999) emphasizing that students often do not know what a proof is and have not been told what counts as a mathematical argumentation. In this paper we present an empirical example from the first author's ongoing PhD study. This study addresses the interplay between a use of primary historical (original) sources and DT (for other related examples, see also Balsløv, 2018; Chorlay, 2015; Jankvist & Geraniou, 2019; Jankvist, Misfeldt & Aguilar, 2019; Olsen & Thomsen, 2017; Thomsen & Olsen, 2019), here in the form of working with Euclid's *Elements* and Dynamic Geometry Environments (DGS), more precisely GeoGebra (GG). We analyze this empirical example in the light of the three types of mediations related to techno-authoritarian proof schemes. The analysis to illustrates how working with the interplay between original sources and GG can be a potential way to address what a proof is and what counts as a mathematical argumentation—thereby trying to counteract the activation of techno-authoritarian proof schemes while working with GG.

CONTEXT OF THE EMPIRICAL CASE

The empirical case stems from the pilot study of the PhD. An expert teacher at a small school with only eleven 6th grade (12-13 years of age) students took place in the pilot study. The activities of the study were planned in collaboration with the teacher. Its outset was a previous teaching module, where both the teacher and the students had previously worked with one of Euclid's propositions and GG (as analyzed by Olsen and Thomsen, 2017). In general, the students were quite familiar with GG. The pilot study had a duration of two modules, 1x90 minutes and 1x180 minutes. The teaching sessions were a combination of classroom discussions, using a common projection screen, and group work of two-three students with a shared computer. At first, the teacher introduced the class to Euclid's five postulates from Book I of *The Elements* [1]. Next, they were to work with Propositions 6 and 7 of Book IV: (6) *To inscribe a square in a given circle*; (7) *To circumscribe a square about a given circle*. The students were provided with the construction part of Proposition 6 and were asked to follow this in order to inscribe a given circle using GG. Afterwards they got the proof of Proposition 6. The student groups were asked to discuss the meaning of the different steps in the proof using their construction in GG. For Proposition 7, the students were only provided with its title, i.e. to circumscribe a square about a given circle. Following their previous work on Proposition 6, they were asked to, first, construct the situation of Proposition 7 in GG and explain how they did this, and, second, to formulate a proof of why it was indeed a circumscribed square about a given circle. Throughout this work, the teacher had discussions with each of the student groups. They also had several common classroom discussions based upon selected groups sharing their work at the projection screen. Finally, the students had to answer questions about proofs and proving in general.

THE EMPIRICAL CASE

The empirical case is based on the students' work with Proposition 7. In general, the students found this work somewhat difficult. The group of students that we are concerned with here encounter several difficulties, and they were actually still in doubt about their proof when presenting it. They base their proof on their construction in GG. They rely on GG's functionality of producing two perpendicular lines. When presenting their proof in the classroom, the following conversation took place:

Student 2:	If *EG* is of equal length of *DH*, as the diameter of the circle, then *EG*, *AG*, *CE*…
Teacher:	*EG*?
Student 1:	[points to the figure] *AG*, *CE*…
Student 2:	And *CA* are the same length, if they all are equal to *DH*, then it must mean that they all are of equal lengths.
Teacher:	But are they?
Student 2:	Yes.
Teacher:	That is what you needed to show they are. If they are, then it is true. […] Do you see what I'm getting at?
Student 2:	Yeah, but that was the same for them. [the previous group that presented]
Teacher:	No, they mentioned the thing with the right angles.
Student 2:	But we were also going to do that.

Teacher: Okay. Let's continue then.

Student 2: We just haven't reached that part yet.

Teacher: Okay. Go ahead.

Student 2: Yeah well, but I just wrote something…

Teacher: Let's try. Let's try.

Student 2: [...] [pointing to the screen] In our construction, when we made them, then they intersected like this, perpendicularly. So, the corners must be the 90. I don't know. I just wrote something…

Teacher: [Directed to the class] It was a nice try.

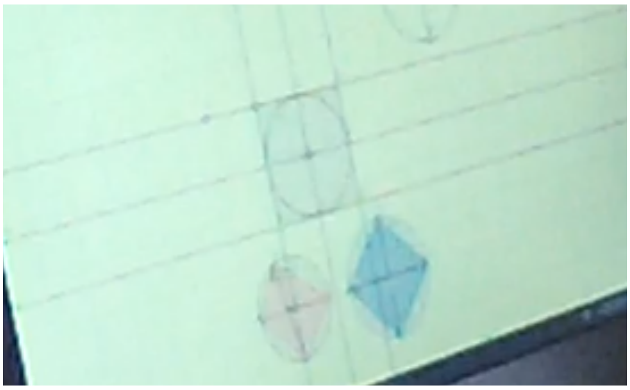
Student 2: I was going to ask you, you know. But you started… [the classroom presentations]

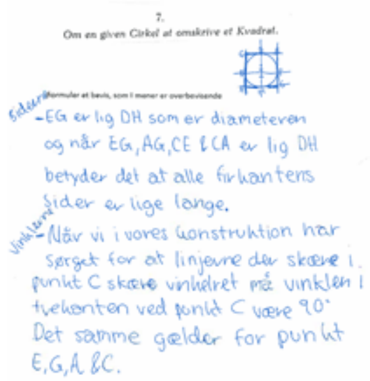
Teacher: It was a good attempt. Student 2 says that in the construction, they have constructed it so that [...] the two lines that meet in *C* were perpendicular by asking GeoGebra to make sure of this. Having done it like that they know that they are perpendicular. I don't know, if I think it is a 'bad' argument, because what you say is true. You did use GeoGebra. And in the way you constructed it, it was perpendicular. So, it must also be perpendicular now. You used GeoGebra to argue for this, the way you constructed it. It's alright.

It seems as the students are in doubt if their argumentation counts as a proof and that they find it somewhat confusing, when they end up arguing with GG as the authority. The teacher tries to make them see that they have constructed the proof by relying on a functionality in GG. After the common presentations, all groups have a little time to finish their work in GG and their written poofs. Figure 1 presents a screenshot of the film recording of the group's computer screen, while they are working with the task (the two constructions at the bottom of the screen are connected to the students' work with Euclid's Proposition 6, Book IV) and their final written proof.

Right side of Figure 1 reads:

"(The sides) -*EG* is equal to *DH*, which is the diameter, and when *EG*, *AG*, *CE* & *CA* are equal to *DH*, then it means that all the sides of the quadrilateral are of equal length. (The angels) -When we in our construction have made sure that the lines that intersect in point *C* intersect perpendicularly, then the angle in the triangle at point C must be 90 degrees. The same is true for points *E*, *G*, *A* & *C*."





7.

Om en given Cirkel at omskrive et Kvadrat.

Formuler et bevis, som I mener er overbevisende

Siderne

-EG er lig DH som er diameteren og når EG, AG, CE & CA er lig DH betyder det at alle firkantens sider er lige lange.

Vinklerne

-Når vi i vores konstruktion har sørget for at linjerne der skære i punkt C skære vinkelret må vinklen i trekanten ved punkt C være 90°

Det samme gælder for punkt E, G, A & C.

Figure 1. Left: The students' construction in GG. Right: The students' attempt at a proof for Proposition 7.

DISCUSSION

Euclid's proof for Proposition 7, following to some extent that of Proposition 6, involves, as the teacher insinuates, a long line of arguments before we are able to deduce that *DIB* is a right angle. Following this, it is argued that *DBC* is a right angle, and hence that *AGEC* is right-angled. Since *AGEC* was previously shown in Euclid's proof to also be equilateral, it is thus a square. Hence, a square has been circumscribed about a given circle.

The students, however, 'shortcut' the proof by applying GG's functionalities of making lines of equal lengths and making perpendicular lines. From a strictly deductive point of view, they simply do not 'clean up'—or at least they leave it to GG to do this. And it is in this sense, we believe, that we witness something where GG served as a justificational mediator, which—if it was not for the students' work with the previous proposition and the teacher's support and questions to their argumentation—could have lead to the students' activation of techno-authoritarian proof schemes. In one way, you can say that the 'authority' is embedded in the way GG is developed. This is a challenge in itself, when working with the interplay between original historical sources such as Euclid's *Elements*, and digital technologies (Jankvist & Geraniou, 2019; Olsen & Thomsen, 2017). From a discursive point of view (Sfard, 2008), this is also an example of viewing the original source and the digital technology as two different discourses (Olsen & Thomsen, 2017). The pilot study example indicates that if we want to support students' development of mathematical reasoning and proving, while working with the interplay between historical sources and digital technologies, we might have to take into account more specifically some of the embedded 'discourses' of GG, e.g. Euclid's geometry (Thomsen, in review). More precisely, it could be an idea to have the students investigate both the language and the method of building a deductive mathematical proof in, say, one of Euclid's propositions, and at the same time articulate which options the GG functionalities provide in relation to this. Such an approach might also support epistemic mediations and help the students realize what mathematics the GG 'buttons' have blackboxed (Lagrange, 2005), and in that way assist in counteracting activation of techno-mathematical proof schemes.

If more time had been available, the teacher could have organized a common discussion in the classroom putting Harel's and Sowder's (2007) concepts of ascertaining and persuading into play. The students could have discussed why the presenting group was not convinced about their own proof, and whether their classmates were convinced or not. This could have lead to a discussion of relating the fact that when reasoning and proving using GG, students probably will encounter similar challenges, where GG serves a pragmatic or a justificational mediation process. Even if the students do agree, and to some extent accept to rely on the functionalities of some of the GG buttons, they might do so in a more reflective way, not only in a techno-authoritarian manner. Working with the interplay between original sources and DT in this way have a potential to foster reflective discussions in the classroom about what count as mathematical proofs and argumentations. The students could also have used the dragging function in GG to investigate what happens to the length and the angles if some of the points are moved. In this way, GG can serve as a pragmatic or epistemic mediator depending on how the students reflect on the visualization of the examples.

On the other hand, if we look at the empirical example from the perspective of the three types of mediations, we observe that things may be more complex than merely an activation of techno-authoritarian proof schemes. Since the presented example is based on working with proving in the DGS of GG, it makes the starting point of using the three types of mediations as an analytical tool somewhat different than using them in addressing the didactical effects of CAS. The users of GG are in many ways offered a visual support when proving. Besides this, before making their own proofs, the students had worked with Euclid's Proposition 6, Book 4, which made them aware of the deductive way of proving. This might be the reason why the group was not convinced by their own proof before presenting it to the rest of the class.

In the empirical example, the students' use of GG actually can be seen as all three types of mediations. They started using GG as a pragmatic mediator, which turned into justificational mediations in an attempt to use it as an epistemic mediator. Allow us to explain. The students constructed a circumscribed square about a given circle by using the functionalities of GG and based their proof on that construction. Thereby it might be seen as they in one way used an empirical proof scheme, while they constructed a figure with equal lengths of the sides of the square and the diameters of the circle. This can be seen as a proof building on an empirical example of how to construct a circumscribed square about a given circle. Hence, the students' use of GG turned out to be as a justificational mediator. To some extent, the students ended up reasoning that the corner was right angled and the sides had equal length, because they had constructed it that way and GG said so. Their doubt concerning if this was a proof in some way show that they might have been aware of using GG as an epistemic mediator, i.e. they wanted to formulate a proof that explains (Hanna, 1990).

CONCLUSION

Based upon an empirical example, we have emphasized the importance of teachers being aware not to support students' activation of techno-authoritarian proof schemes when working with digital technologies. This might be a struggle in many classrooms when using GG, for example, because Euclidian geometry is embedded in the way the software is developed (Thomsen, in review). This implies that we should consider defining some didactical guidelines for working with proofs and proving in digital environments. Digital tools can support students' work with original mathematical sources, and these sources can further support an explicit and common awareness of difficulties and possibilities while working with proofs and proving within these technologies, not least in trying to counteract the activation of students' techno-authoritarian proof schemes.

NOTES

1. E.g. see https://mathcs.clarku.edu/~djoyce/java/elements/bookI/bookI.html #posts

REFERENCES

Balsløv, C. U. (2018). *The mutual benefits of using CAS and original sources in the teaching of mathematic*s. Master's Thesis. Copenhagen: Danish School of Education, Aarhus University.

Buchberger, B. (2002). Computer algebra: The end of mathematics? *ACM SIGSAM Bulletin, 36*(1), 3–9.

Chorlay, R. (2015). Making (more) sense of the derivative by combining historical sources and ICT. In E. Barbin, U. T. Jankvist, and T. H. Kjeldsen (Eds.): *History and Epistemology in Mathematics Education – Proceedings of the Seventh European Summer University*, pp. 485–498. Copenhagen: The Danish School of Education, Aarhus University.

Hanna, G. (1990). Some pedagogical aspects of proof. Interchange, 21(1), 6–13.

Harel, G. & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In Lester Jr, F.K. (Ed.). Second Handbook of Research on Mathematics Teaching and Learning (pp. 805–842). Charlotte, NC: Information Age Publishing.

Jankvist, U. T. & Geraniou, E. (2019). ICT as a way of making original sources more accessible to students. In E. Barbin, U. T. Jankvist, T. H. Kjeldsen, B. Smestad and C. Tzanakis: *Proceedings of the Eight European Summer University on the History and Epistemology in Mathematics Education*, Oslo, Norway, July 2018. Oslo: METU.

Jankvist, U.T. & Misfeldt, M. (2019). CAS assisted proofs in upper secondary school mathematics textbooks. REDIMAT – Journal of Research in Mathematics Education, 8(3), 232-266. doi: 10.4471/redimat.2019.3315

Jankvist, U. T., Misfeldt, M. & Aguilar, M. S. (2019). Tschirnhaus' transformation: mathematical proof, history and CAS. In E. Barbin, U. T. Jankvist, T. H. Kjeldsen, B. Smestad, and C. Tzanakis: *Proceedings of the Eight European Summer University on the History and Epistemology in Mathematics Education*, Oslo, Norway, July 2018. Oslo: METU.

Lagrange, J. (2005). Using symbolic calculators to study mathematics: The case of tasks and techniques. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument* (pp. 113–135). New York, NY: Springer.

Misfeldt, M. & Jankvist, U. T. (2018). Instrumental genesis and proof: Understanding the use of computer algebra systems in proofs in textbooks. In M. Tabach, H-S. Siller, L. Ball, P. Drijvers, S. Ladel & C. Vale (Eds.). *Uses of technology in primary and secondary mathematics education: Tools, topics and trends*, (pp. 375–385). Cham: Springer. ICME-13 Monographs https://doi.org/10.1007/978-3-319-76575-4

Olsen, I. M. & Thomsen, M. (2017). *Matematikhistorie og it i matematikundervisningen i grundskolen*. (Unpublished master's thesis). Copenhagen: Danish school of Education, Aarhus University.

Rabardel, P. & Bourmaud, G. (2003). From computer to instrument system: a developmental perspective. Interacting with Computers, 15(5), 665–691.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing.* New York: Cambridge University Press.

Thomsen, M. (in review). *Working with Euclid's geometry in GeoGebra – experiencing embedded discourses.* Submitted paper for the conference Norma 20

Thomsen, M., & Olsen, I. M. (2019). Original sources, ICT and mathemacy. *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (CERME11, February 6-10, 2019) (pp. 2060–2061). Utrecht the Nederlands: Freudenthal Group & Freudenthal Institute, Utrecht university and ERME.

Digital curriculum resources' connectivity: an attempt to conceptualization

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We attempt to conceptualize digital curriculum resources' (DCRs) connectivity feature by relying on the frameworks of boundary crossing/boundary objects and multimodality. We exemplify our framework on three example DCRs from Flanders and the USA and consider impact of connectivity on social relations and issues of control and empowerment.

Keywords: connectivity, boundary object, multimodality, social relations.

CONNECTIVITY: A MODE AS BOUNDARY OBJECT

Connectivity is a characteristic feature of digital curriculum resources [1] (DCRs) (Gueudet, Pepin, Restrepo, Sabra, & Trouche, 2018). Yet, its conceptual basis is underspecified. Therefore, in this contribution, we aim to conceptualize DCRs' connectivity, and do so by relying on the frameworks of boundary crossing and boundary objects (Akkerman & Bakker, 2011) and multimodality (Bezemer & Kress, 2008).

Akkerman and Bakker (2011) describe a boundary as an intersection of sociocultural practices, leading to discontinuity in action and interaction. It is in the dialogic nature of boundary crossing that learning potential resides. Boundary objects, then, are "artefacts doing the crossing by fulfilling a bridging function (Star, 1989)" (Akkerman & Bakker, 2011, p. 133). Here, we understand DCRs' connectivity as a boundary object: it is an artefact taking up a bridging function. Along this line of thought, connectivity is a decisive mode, distinguishing DCRs from their printed counterparts. Relating primarily to the framework of boundary crossing and boundary objects, we propose four modal resources that make up the mode of connectivity: the domains between/within which connections are made, the nature, the visibility, and the direction of the connection (See Table 1).

In their review, Akkerman and Bakker (2011) noticed that boundaries are encountered within and between the domains of work, school, and everyday life. Relating to DCRs, *domains* that can be thought of are school, everyday life, a virtual world, the content (e.g., mathematics), the publishing enterprise, and policy. People and objects play a central role at boundaries and boundary crossing, so both people and objects are central to a boundary's nature (Akkerman & Bakker, 2011). Hence, we understand the *nature of the connection* to be a second modal resource. In the context of this study, people can be thought of as the teacher, students, colleagues at school, curriculum designers, and family. Objects can be material and semiotic resources, content strands, and units within a resource. A third modal resource is the *visibility of a connection*. Akkerman and Bakker relate to Williams and Wake (2007) to describe that boundary objects, if unnoticed, might function as black boxes, and that the opening up of these boxes can make these objects as sites of learning potential. From this line of thought, it follows that the visibility of connections is an important feature, as invisible connections might hinder that a DCR's connectivity functions as a boundary object. Adler (2000) argues that resources need to be both visible and invisible: they need to be visible to be noticed, and then need to become invisible or transparent to let the mathematics shine through. As this process of transparency is not a feature from the resource per se, but from the use of the resource (Adler, 2000), this is an issue we don't explore in this contribution. Instead, the modal resource of visibility captures whether or not connections are made explicitly visible by the DCR's design. *Direction of the connection* is the fourth modal resource. Yerushalmy (2014) describes how digitalization has challenged the traditionally assumed role of authority that goes with the printed textbook. DCRs can have authority over its users, but they can also allow for more agency on the part of their users (e.g., Bezemer & Kress, 2008).

Domain(s) of connection	
Connections within and/or between the domains of school, everyday life, a virtual world, mathematics as a discipline, publishing enterprise, policy.	
Nature of the connection	
People	Teacher, students, colleagues (e.g., care teacher, principal), family members, curriculum authors.
Objects	*Material resources*: connections between resources (e.g., connection between print and digital resource, between two or more digital resources, between DCR and official curriculum). *Semiotic resources*: connections between different representations (e.g., link between a static visualisation of a relationship to a dynamic interactive visual). *Content domains/mathematical topics*: connections within and between different domains (O'Halloran, Beezer, & Farmer, 2018) (e.g., connection to previous learned theorems, connecting geometry content to fractions content). *Structure of material resources*: connections between different frames or units of the resource (Bezemer & Kress, 2015) (e.g., connections between/within chapters and lessons).
Visibility of connection	
Invisible	Connections that are not transparent to the user (Usiskin, 2013).
Visible	Connections that are accessible to the user (O'Halloran et al., 2018)
Direction of connection	

Resource ← other resources/ people	The connection is primarily focused toward the resource: other resources and/or people are connected to the resource (e.g., some DCRs allow teachers to add links and documents to a digital teacher's guide).
Resource → other resources/ people	The connection is primarily reaching out from the resource: the resource is connected to other people and/or resources (e.g., a DCR's learning trajectory can be designed in such a way that it becomes a tool of students' learning).
Resource ↔ other resources/ people	The connection goes in both directions, connecting people/other resources to the resource, while it simultaneously connects the resource to other people and resources.
No direction	The connection does not specify or impose a specific direction.

Table 1: Four modal resources that make up the DCRs' mode of connectivity

We will exemplify the proposed framework on three examples of DCRs, but first, we describe each of these three DCRs in brief.

THREE EXAMPLES OF DIGITAL CURRICULUM RESOURCES

Pepin, Gueudet, Yerushalmy, Trouche, and Chazan (2015) distinguish three types of DCRs. An integrative e-textbook is most closely related in nature to a printed curriculum resource. It typically includes a digital version of the printed resources, but also includes some additional adds-on, such as links to the digital world outside the DCR and the possibility to add own materials. An evolving e-textbook is a DCR that is constantly developing, typically by a core community of users. The interactive e-textbook is a toolkit based on a set of learning objects that students proceed through according to their own ability and interest.

The three DCRs included in this study are examples for the first and third type of DCR as distinguished by Pepin et al. (2015). Bingel, developed by VAN IN publisher in Flanders, Belgium, and the digital environment of Everyday Mathematics, developed by the University of Chicago, USA and published by McGraw-Hill are exemplary for integrative e-textbooks. Math-Mapper, developed by Scaling Up Digital Design Studies at North Carolina State University, USA, is an example of an interactive e-textbook. We provide a description of these three DCRs below. Given space restrictions, these descriptions are incomplete, but aim to help understanding how these resources fit in the abovementioned typology.

Bingel is a digital platform that includes a teacher version and a student version. The teacher version includes a planner, which helps to organize lessons according to a weekly schedule defined by the teacher. Bingel will then propose to order the lessons according to the sequence in the printed resource, but allows teachers to reorder the lessons. In a lesson mode, Bingel lists the specific lesson goals, goals of the official curriculum, needed materials and lesson guide in a similar way as in the printed version, but also allows to insert notes and links, and add or delete goals. It also provides lesson slides to be projected on an interactive whiteboard, such as digital versions of the student textbook pages, with the feature to record the notes made during class. Bingel has a feature that allows teachers to track student progress when solving exercises digitally. Progress can be monitored during the lesson, and outside of the lesson on a class, student or task level. Bingel also provides an overview of the goals achieved and the goals still to be achieved during a grade. It is adaptive to students in that it provides exercises for individual students based on their actual performance. Teachers can then see whether the difficulty level during a series of exercises went up, down, or stayed the same. Likewise, upon completion of a test and after the teacher submitted student scores, Bingel suggests a series of exercises per student based on their test score. It does not provide a rationale, however, about the grounds on which it selected specific exercises for specific students. Bingel also includes features to support teacher collaboration and additional student differentiation, but schools have to pay additionally for these tools. Also, Bingel allows the school principal to view a class's progress, and to compare the progress among classes. The student version includes a virtual world in which a class lives on an island. Students can choose an avatar, which they can modify depending on the money they earned. They earn money by means of solving exercises on their class Island. There is also a feature that allows students to take a picture of a page in their paper version curriculum resource, which then allows students to do more related exercises on their digital island.

The Everyday Mathematics digital environment has several features similar to Bingel. For instance, it also allows teachers to add notes and documents to the digital version of the teacher's guide. It differs, however, in at least two significant ways from Bingel. Everyday Mathematics visualizes the learning sequence so that teachers can check how the *Common Core State Standards* (CCSS) are addressed and built up over lesson activities, lessons, and units for a complete grade level. In contrast to the static learning sequence in the printed version, teachers can scroll over the content across an entire grade. Likewise, in the digital lesson mode, teachers can click on goals of activities, which brings them then to a spiral to which this goal belongs to, which supports teachers' understanding of the function of a particular activity in relation to the spiral. Another feature that distinguishes Everyday Mathematics from Bingel is the extensiveness of the virtual community where teachers can connect with other teachers and with the curriculum developers. The platform allows teachers to share materials and videos, and initiate and participate in discussions, in which curriculum developers can also join. The platform also provides professional development modules, most of which require an additional cost.

Math-Mapper is an example of an interactive e-textbook. It is designed for middle school mathematics and addresses four fields of topics, which are broken down into big ideas, clusters, and constructs. The clusters and constructs are structured according to a learning trajectory. All of this is highly visualized: students and teachers can zoom in on the map and see how big ideas are made up of clusters, and how clusters are made up of constructs. The order of the constructs indicates the order in which these constructs are supposed to be learnt, and zooming further in on the constructs visualizes a learning trajectory per construct. The learning trajectory is coupled to the CCSS and typical misconceptions. Math-Mapper is used both by students and teachers. It includes a diagnostic assessment tool, which allows students to visually situate their performance in relation to the learning trajectory. Likewise, it also allows the teacher to track performance of a class and students in relation to the learning trajectories in a highly visual manner. Math Mapper also contains a planning tool, which helps the teacher to plan lessons (and adjust planning) in accordance to the big ideas. Based on the planning set by the teacher, students can practice certain constructs and check their performance by means of taking an assessment. While practicing, students can follow the order of exercises as suggested by Math-Mapper or they can decide themselves what level in the learning trajectory they want to work with. The connection between the exercises and the learning trajectory is highly visible so that students easily can monitor their progress.

EXEMPLIFYING THE PROPOSED FRAMEWORK AND CONSIDERING IMPACT OF CONNECTIVITY ON SOCIAL RELATIONS

The specific configuration of modes and modal resources of curriculum resources has implications on the social relations at stake (Bezemer & Kress, 2008, 2015; Yerushalmy, 2014). In classroom teaching, the social relations are between the teacher, students, content, and artefact (Rezat & Sträßer, 2012). Here, we exemplify how the proposed framework (See Table 1) can help to analyse the mode of connectivity of the three example DCRs. We will also relate the specific configuration of the mode of connectivity to implications on social relations. More in particular, we will discuss how DCRs' connectivity a) configures the social relations between teacher, students, artefact and content, b) expands the web of social relations, and c) can in-source or re-source teachers.

Configuring typical social relations

As a connection (or boundary object), a learning sequence can bridge between multiple domains. For instance, it bridges between the domains of content, school, policy, and publishing enterprise when DCRs are being developed. Here, we focus on the social practice of classroom teaching, and focus on the connection between the domains of content and school. As to the nature of connection, the learning sequences embedded in the three selected DCRs connects content domains and/or mathematical topics to each other, but also to students, the teacher and the DCR. Though, as we describe below, the visibility and direction of this type of connection differs across the three DCRs, having a bearing on the social relations at stake.

Bingel adjusts the difficulty level of exercises based on students' individual performance. Its adaptivity is based on situating students' performance in relation of a thought sequence, but the sequence itself is not made visible to students and teacher. The direction of the connectivity goes from the students and teacher to Bingel. The relationships Bingel – students, and Bingel – teacher are affected.

In contrast to Bingel, the learning sequence in the digital environment of Everyday Mathematics is made visible to the teacher. Teachers can trace activities to their locations in the learning sequence and can see how a sequence spreads out over a number of activities or lessons. The connectivity goes primarily toward the resource: the learning sequence helps the teachers to understand how content in the resource is structured. In helping the teacher to understand how the curriculum resource is sequenced, the teacher is supported to make thoughtful modifications, which indicates that this connectivity also has the potential to go from the resource to the teacher. Particularly the relationship Everyday Mathematics – teacher is affected.

Math-Mapper's learning sequence is the most visible of the three DCRs. Contrary to Everyday Mathematics, the visible sequence also connects to the students. The sequence proposes a pathway, both for students and teachers, which can be adjusted from the students' or teacher's side. This indicates that the social relation between Math-Mapper and students, and Math-Mapper and teacher goes in both directions. Possibly, because the learning sequence becomes a tool for deliberative use by both students and teacher, the social relation between students and teachers, going in both directions is also stressed. The relationships Math-Mapper – students, Math-Mapper – teacher, and students – teacher are affected.

Expanding the web of social actors

A distinguishing feature of Bingel and Everyday Mathematics' digital environment is the possibility to connect to colleagues such as other grade teachers, care teachers, and the school principal; parents; and curriculum developers. These are people located in the domains of school, everyday life, and the publishing enterprise. Although relationships with these social actors can also exist in a non-digitalized environment, the connectivity of DCRs provides new opportunities. Bingel, for instance, has a feature that allows parents to track their child's results and performance. It has a similar feature so that school principals can track progress of classes. The Everyday Mathematics digital community platform allows teachers to exchange information such as video recorded lessons with one another, and it is also a platform where curriculum developers can interact directly with teachers. This type of connection, and the visibility of these connections offers unprecedented ways to expand the web of social actors in relation to teaching and learning mathematics.

In-sourcing versus re-sourcing

Our analysis of the connection *learning sequence*, discussed above, reveals that the direction of the connection differs among the three DCRs. In Math-Mapper, the connection is bidirectional between both resource and students, and resource and teacher. In Everyday Mathematics, it has the potential to go in both directions, but mainly goes from the teacher toward the resource. As for Bingel, the direction goes toward the resource as well. We see this pattern of direction toward the resource also in the *adaptivity* connection. Here we focus primarily within the school domain and zoom in on the teacher and the resource to describe the nature of adaptivity. In Bingel and Everyday Mathematics, teachers can make modifications and add materials and notes, but the main idea is that the teachers adhere to the resource's existing structure. In other words, teachers are sourced into the DCR. Math-Mapper, on the other hand, is designed to accommodate to a teacher's existing teaching practices and in such a way allows teachers to re-source their practice. The difference between in-sourcing and re-sourcing reflects Gueudet et al.'s (2018) idea that some DCRs have "a connectivity directed towards the publisher's productions", while other DCRs have "a more networking kind of connectivity" (p. 556).

INSTRUMENTS OF CONTROL AND EMPOWERMENT

Earlier, we wrote that learning resides in the dialogic nature of boundary crossing. Akkerman and Bakker (2011), relating to Oswick and Robertson (2009) also warn, however, that boundary objects can possibly reinforce power structures and occupational hierarchies. Looking back holistically on the three issues described in the previous section, we see this playing out to different extents in the connectivity of the three example DCRs. The invisibility and directionality in the learning sequence from teacher and students toward Bingel stresses the position of Bingel over the teacher. Providing parents and the principal opportunities to follow and compare progress may potentially empower these social actors over teachers. Characteristic of the integrative nature of Bingel, teachers primarily are sourced into the resource. This sketches a web of social relations in which other actors than students and teacher are empowered, potentially leading to more a submissive role for teachers.

Everyday Mathematics has an integrative nature as well, but by visualizing its learning sequence, this DCR supports teachers to take a more active role in the expanding web of social relations. Math-Mapper, due to the visibility of its learning sequence and connectivity toward students and teacher, functions as a tool to empower students as owners of their own learning and teachers and curriculum resources to be active and powerful partners in this process. These reflections suggest that there is value exploring how DCRs' connectivity impacts issues of control and empowerment.

NOTES

1. Digital curriculum resources (DCRs) refer to digital resources that are curricular in nature, in that they contain a scope and sequence and are designed to support instruction over time.

REFERENCES

Adler, J. (2000). Conceptualising Resources as a Theme for Teacher Education. *Journal of Mathematics Teacher Education, 3*(3), 205-224.

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132-169.

Bezemer, J., & Kress, G. (2008). Writing in Multimodal texts - A social semiotic account of designs for learning. *Written Communication, 25*(2), 166-195.

Bezemer, J., & Kress, G. (2015). *Multimodality, learning and communication: A social semiotic frame*: Routledge.

Gueudet, G., Pepin, B., Restrepo, A., Sabra, H., & Trouche, L. (2018). E-textbooks and connectivity: proposing an analytical framework. *International Journal of Science and Mathematics Education, 16*(3), 539-558.

O'Halloran, K., Beezer, R. A., & Farmer, D. W. (2018). A new generation of mathematics textbook research and development. *ZDM, 50*(5), 863-879.

Oswick, C., & Robertson, M. (2009). Boundary objects reconsidered: From bridges and anchors to barricades and mazes. *Journal of Change Management, 9*(2), 179-193.

Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. (2015). E-textbooks in/for teaching and learning mathematics: A disruptive and potentially transformative educational technology. In L. English & D. Kirschner (Eds.), *Handbook of International Research in Mathematics Education*. New York: Routledge.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation. *ZDM, 44*(5), 641-651.

Star, S. L. (1989). The structure of ill-structured solutions: Boundary objects and heterogeneous distributed problem solving. In *Distributed artificial intelligence* (pp. 37-54): Elsevier.

Usiskin, Z. (2013). Studying textbooks in an information age—a United States perspective. *ZDM, 45*(5), 713-723.

Williams, J., & Wake, G. (2007). Black boxes in workplace mathematics. *Educational Studies in Mathematics, 64*(3), 317-343.

Yerushalmy, M. (2014). Challenging the authoritorian role of textbooks. In K. Jones, C. Bokhove, G. Howson, & L. Fan (Eds.), *Proceedings of International Conference on Mathematics Textbook Research and Development* (pp. 13-20). Southampton, UK.

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