

## WORKING GROUP 9

### Tools and technology in mathematical didactics

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# TOOLS AND TECHNOLOGIES IN MATHEMATICAL DIDACTICS

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## Introduction

The integration of tools and technologies is an important and actual theme today. It has been a topic of CERME Working Groups since its first edition. In the proceedings of CERME3, Jones and Lagrange point out that ‘the potential of computer-based tools, in particular, to enhance the teaching and learning of school subjects, including mathematics, is reflected in the increase in Government spending over recent years on providing such technology for schools’ (Jones and Lagrange, 2003). They indicate, however, that classroom implementation is not advancing in line with spending: ‘for upper secondary school pupils (aged 14-16), the proportion never, or hardly ever, using computers in their mathematics lessons is as much as 82%’ (Jones and Lagrange, *ibid*). Such considerations underline the importance of the work of the thematic group on tools and technologies in mathematical didactics. Although the word ‘tool’ is quite general, the Group’s focus is on computer tools. Of course, this does not exclude considering tool use in general.

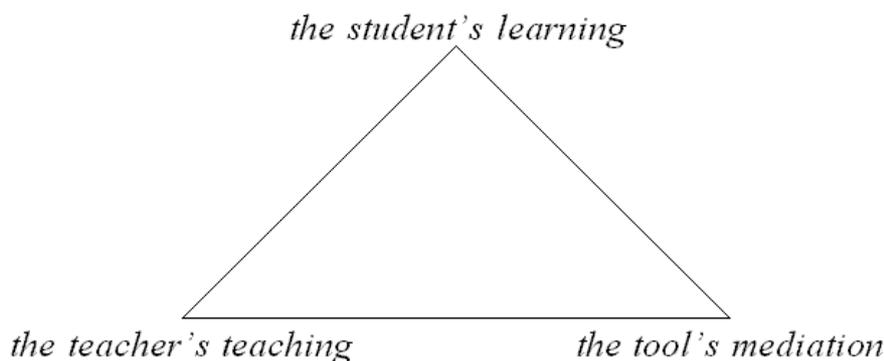


Figure 1 The triangle of themes

For WG9 at CERME4 three main themes are proposed, which form the three vertices of the triangle which frames our work (see Fig. 1): the relation between the use of technology and learning, the role of the teacher in technology-rich mathematics

education, and the characteristics of technological tools which can foster the learning of mathematics. Let us briefly address each of these dimensions.

### 1. The relation between the use of technology and learning

In fact, this theme can be considered as the ‘proof of the pudding’: what is the relation between the use of technology and students’ learning? The idea that technology allows simply ‘to leave the work to the device and to concentrate on conceptual development’ seems to be inadequate. Rather, a complex interplay between the work in a technological environment and the development of mathematical understanding and skills is noticed. How can this relation be investigated? How does paper-and-pencil work fit in? What is the relation between private and social aspects of the use of technology in learning?

### 2. The role of the teacher in technology-rich mathematics education

As students often work individually or in pairs while using technology, it may seem that the importance of the teacher in a technology-rich learning environment is decreasing. However, the teacher needs to point out the main issues, to invite reflection and classroom discussion, and to ‘orchestrate’ and guide the development of techniques, while being aware of the constraints and affordances of the available technology. The question is therefore how teachers can deal with the new pedagogical context of technology-rich learning. What support do teachers need while integrating technological tools in their teaching? What pedagogical resources are available or should be developed? What would virtual communities of practice look like? How can the professional development of teachers be established?

### 3. The characteristics of technological tools which can foster the learning of mathematics

Technological tools for mathematics education show much variety. Some tools, such as applets, can be considered local ‘dedicated’ software environments. Others, such as graphing calculators, computer algebra systems and dynamic geometry systems, provide more general pedagogy-free environments. What are the characteristics of each of these tools? What is the impact of the distinction between micro worlds and expressive tools (Hoyles and Noss, 2003)? What kind of technological tools do we need in our teaching, and what future developments can we foresee?

As the triangle suggests, these three themes are far from independent. Rather, they influence each other in a reciprocal way. The triangle model will be used throughout this final report as an organizing structure to position the different contributions. In the following sections we address each of the vertices. The paper ends with some overarching conclusions and reflections.

### The relation between the use of technology and learning

The papers, presentations and discussions concerning the relation between the use of technological tools and learning lead to three main observations. First of all, we notice a considerable variety of topics and approaches within the working group

contributions. Within this variety, we observe that the research domain of the use of technology in mathematics education is more and more aware of the *potentials of the tools* and in particularly their influence on the students' knowledge construction and learning process. New technologies provide efficient tools for widening the mathematical landscape that can be discovered by the students (Dana-Picard). Also, they may enhance the development of flexible conceptions (Andresen), and foster the genesis of connections within complex scientific ideas through the use of quasi-concrete objects in dynamic settings (Jones). Bescherer et al show how the use of a virtual seminar made distributed knowledge become shared knowledge. Albano indicates how technology essentially changes the didactical triangle. Both Delice and Pitalis point out that teaching the same topic in quite different technological environments gives rise to different learning processes and conceptions. With respect to this point, Ughi shows the need for variation in tools used for learning purposes: the use of even quite simple tools may result in deep but tool-related insights.

A second observation on the issue of tool use and learning concerns the *theoretical frameworks*. Within a general constructionist approach (Jones), a great variety of—sometimes shared—theoretical frameworks is used: the notion of collaborative work (Bescherer et al), the dialectics process-object and tool-object (Andresen, Delice), activity theory (Delice), the theory of didactical situations (Albano), the notions of cognitive tool (Fuglestad) and cognitive acceleration (Pittalis), the importance of mediations in every learning process (Reggiani), the question of computerized transposition, and the notion of windows on mathematical meanings. The instrumental approach, however, turns out to be the most central theoretical framework in the contributions of the working group. Finally, the conclusion was that a more *ecological* and systematic approach is needed rather than a unifying theory, which takes into account the existing subsystems, and which combines various theories focusing on each of these subsystems (didactics, instrumental approach, situated and distributed cognition, community of practice).

The issue of the theoretical frameworks leads to the third observation. If we take the triangle students' learning –teachers' teaching– tools' mediation as the point of departure, many questions arise (see Fig. 2). These questions, however, focus on one single notion: given a tool, the genesis of a fruitful instrument is far from self-evident, but is the result of a social process, guided by a set of tasks in a given institution. This is the core of the *instrumental approach*, which played a central role in our work. Several papers (Andresen, Boon & Drijvers, Dana-Picard, Delice, Guin, Hegedus, and Reggiani) refer to the instrumental approach to the use of tools in mathematics education. This theoretical approach offers a means to analyze the influence of both basic and complex tools. For example, the availability of a hammer may lead us to try to solve all kinds of problems –including problems for which a screwdriver would be more appropriate– by using a hammer. In that sense, the availability of the tool guides the choice of the problem-solving strategy (Drijvers & Trouche 2005). The instrumental approach is a theory on subjects' mediated activity:

it is a question of interaction between man and machine rather than co-action. It suggests answers to some of the emerging questions.

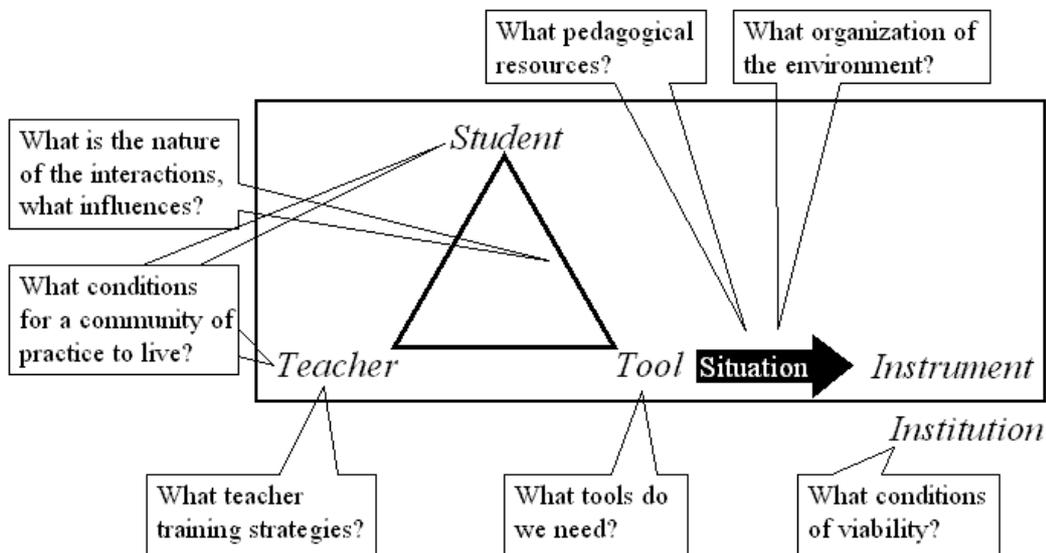


Figure 2 Questions in the triangle

One important question concerns the nature of the student – tool interaction. On this issue, the instrumental approach offers the essential duality of instrumentation and instrumentalization. The process of *instrumentation* refers to the effects of tool use on the student activity, whereas the process of *instrumentalization* refers to the appropriation and transformation of the tool by the student. Instrumental genesis, in this view, consists of the articulation of these two processes. This notion is close to the idea of shaping described by Hoyles and Noss (2003) who speak about the two-sided relationship between tool and learner as a process in which the tool in a manner of speaking *shapes* the thinking of the learner, but also *is shaped by* his thinking.

The two-way relation between tool use and learning is reflected in statements that came up in the group’s final brainstorm:

The instrumentation perspective stresses the influence of technology on doing mathematics: “Technology pulls rather than pushes”, “Technology can lead to bypassing the need for mathematical thinking”, “Doing mathematics with technology”.

The instrumentalization perspective stresses the influence of the student on the tool: “*The teacher and the students as designers of their tools in the process of learning*”.

In discussions, particularly on the paper of Hegedus, the notions of instrumentation and instrumentalization were deepened. It was stated that the phase of instrumentation does not precede the phase of instrumentalization, but that the two processes are deeply interrelated and take place simultaneously in interaction.

Concerning the question of pedagogical resources, the task design should take into account the instrumental genesis. Of course, as one of the participants stated, it is the task, and not the technology as such, that enables learning. However, the opportunities and constraints of the artifact guide learning, and therefore should be part of the designer's focus: how can the task capitalize on the tool characteristics so learning is enhanced?

On the issue of establishing a community of practice within the classroom, the notion of orchestration is fruitful (Drijvers & Trouche, in press). It stresses that the use of technological tools affects the didactic contract. Instrumental genesis should not only be an individual, but also a collective process. The conditions and environmental organization that are required for the community of practice to emerge are recommended as topics for further study.

The role of the teacher in technology-rich mathematics education

The second theme of this working group concerns teachers and their role in a technology-rich environment. Suggested questions for this theme are related to the problems of integrating technological tools in teachers' didactical practice:

What support do teachers need while integrating technological tools in their teaching?

What pedagogical resources are available or should be developed?

What would virtual communities of practice look like?

These problems initially appeared in French research projects on computer algebra systems and later on symbolic calculators. Artigue et al. (1998) showed that the integration of software (or calculators) was very limited, in spite of institutional efforts, school programs and the organization of teacher-training courses. In their analysis, the "pioneering" phase hindered the presentation of integration problems in an effective way, also causing important epistemological, cognitive and institutional problems to be underestimated.

The trend to only partially consider the role of the teacher becomes visible if the subjects of research projects on the teaching and learning of mathematics with technological tools are considered. In a recent study on a large corpus of published articles (1994 – 1998), Lagrange et al. (2003) found very few mentions of indicators characterizing the teacher dimension, whereas the focus was on the epistemological and cognitive dimensions. If the teacher is considered as a central "actor" of the integration, Lagrange's interpretation (2004) of this fact is that there is an implicit assumption: "new technologies and the associated didactical knowledge could easily be transferred to teachers by way of professional development and training". For him, the existing corpus of didactical knowledge is not sufficient to really help teachers integrate technology.

The results of some research projects (worked out after the quoted period), which were based on the analysis of ordinary classrooms and teachers' practice in technology-rich mathematical contexts, confirm this assumption. For instance, the

research project by Kendal et al. (2004) presents an example of both institutional constraints and opportunities introduced by technological tools (in this case symbolic calculators) from two points of view: the teacher and comparison between teachers' conceptions of teaching and mathematics and new inserted elements. These studies have led to stress: the development of different ways of organizing classroom activities and different approaches to learning with technological tools, the necessity to manage the option of having several solutions for a chosen problem, and several ways of using technology. The introduction of technological tools does not only add new variables in the classroom, but also changes existing conditions. As Zbiek (2001) emphasizes, in-service teachers have acquired "some degree of comfort" in their teaching practice, but the use of new tools provokes the search for a new equilibrium, that is, it brings their practice up for discussion again.

These studies show the relevance of the teacher theme when we deal with technological tools and their use in the classroom. For their integration, it seems to be necessary, on the one hand, to provide instruments to analyze teaching practices, and on the other, to organize teacher training by providing specific support. In spite of the fact that we took the teacher theme into account in this working group, there are relatively few papers covering it. This is evident from the triangle poster activity at the end of the working group, where few dots are situated near the teacher vertex (see Fig. 3).

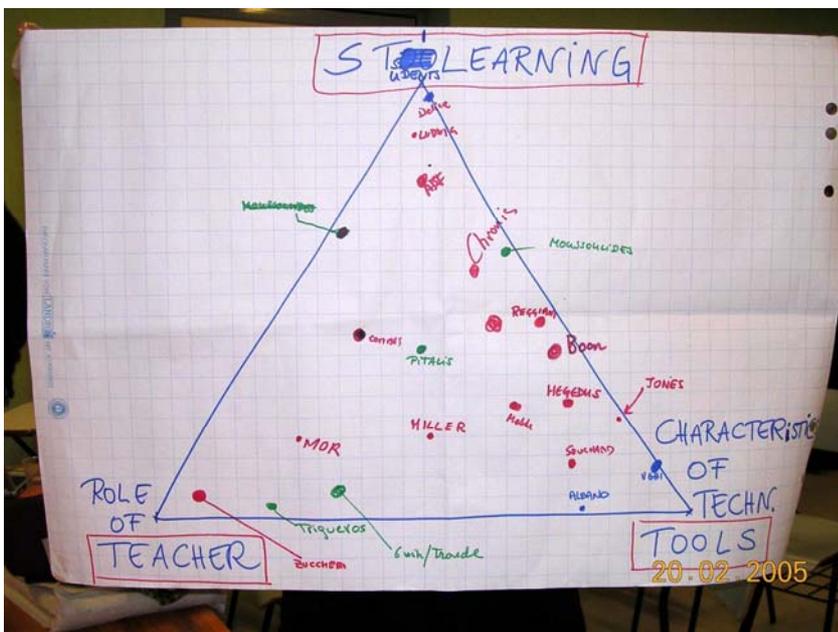


Figure 3 Locating contributions in the triangle

Although the teaching dimension was present in many of the contributions, only three papers really focus on this theme. The papers of Trigueros et al. and Zuccheri observe and analyze the use of specific technological tools in the solution of a chosen problem by teachers and/or in teaching practice. Trigueros & Garcia's paper concerns firstly the analysis of four teachers' strategies while using *The Balance* software (interactive software designed for the *Enciclomedia* project, developed within

institutional constraints and related to the concept of equivalence of fractions), and secondly the management of the same lesson by one of the teachers. The authors are also interested in the way the teachers use *The Balance* in combination with other teaching materials. Zuccheri's paper concerns the use of *Cabri II* dynamic geometry software in a teacher training course. This paper shows, on the one hand, how an experienced didactical proposal can represent a way to introduce the use of the software during the course, and on the other, how the activity as a whole can be a chance for the teachers to discuss their conceptions of mathematics and the use of technological tools in teaching.

Guin and Trouche's paper presents a specific distance in-service training organization, as well as its implementation and difficulties encountered in the process. This organization, based on an instrumental approach of pedagogical resources, aims to assist teachers using ICT in their own classrooms. The complexity of this collaborative virtual workshop stresses the difficulty of helping teachers to deal with technology-rich learning environments.

The characteristics of technological tools which can foster the learning of mathematics

Four lectures (presented by Mor, Mousoulides, Boon, and Miller) focussed on the characteristics of technological tools. Two of the lectures concerned a joint project using the visual programming microworld ToonTalk. Mousoulides & Phillipou investigate the interaction between two pupils who used WebReports as a platform for documentation and communication. This interaction suggests that the communication medium is essential for the learning of mathematics, as is building connections between different modes of representations. Mor et al, who use ToonTalk and WebReports as well, focus on "Guess my robot!", a game in which students explore number sequences. The theoretical framework integrates the notion of communities of practice with domain-specific epistemology. The authors identified three factors which influence the length and quality of interactions: facilitation, reciprocation and audience-awareness. Points of discussion were the issue of how to stimulate the emergence of a community of practice between pupils and the question of the specifics of the tool in use, ToonTalk, as compared to more general mathematical tools such as spreadsheets or computer algebra software.

The use of Java applets was a topic in the presentation by Boon (see fig. 4). Boon reported on a teaching experiment as part of a small research study on the use of Java applets for the learning of algebra. As well as the didactical background he discussed exemplary student behaviour, classroom observations and results.

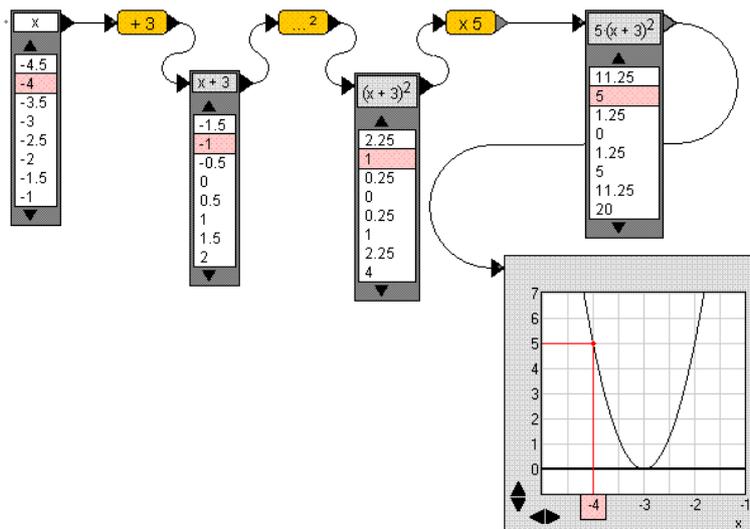


Figure 4 Applet combining different representations described in Boon & Drijvers

Which motivational effects accompany the use of interactive whiteboards in mathematics classrooms? This was the central question of Miller's presentation. Although it is not easy to distinguish presentational from motivational effects, a number of factors are considered by teachers and pupils to affect pupils' motivation. Interest and enjoyment were most evident in lessons where the interactive whiteboard, not the teacher, was the focus. However, the interactive whiteboard in itself is not sufficient to ensure that pupils are motivated; rather, it is the pedagogical stance and the quality of the teaching that enhance motivation.

A group session was dedicated to the question 'What tools do we need?' The leading questions are:

1. What kind of digital media do we need in mathematics education?
2. What are criteria for good digital media in mathematics education? Can certain digital media be advised against or recommended?
3. What would be a useful categorization of digital media for mathematics education?
4. What future development of digital media for mathematics education do we want?

Concerning the first question, one might wonder whether the use of technological tools makes sense for students, who have not, like we have, learnt mathematics without using technology. The use of technology does make sense, because being technologically literate is an important factor in our present society and culture. Technology plays an important role in mathematical research, so it should also be part of mathematics education. On the kind of digital media we need, it is noticed that a variety of tools is becoming more and more popular: interactive whiteboards, digital cameras, mobile telephones, et cetera. Opportunities for connectivity and

collaborative work are increasing. Tools offer flexibility in mathematical representation and allow for crossing borders between mathematical sub-domains. They enable students to “do things differently and to do different things” and may invite for challenging explorations. However, it still is not clear how this affects mathematics education.

Then the issue of criteria. There is a range of criteria for technology which are appropriate for use in the mathematics classroom, which sometimes may conflict with each other. As a first and important criterion for tools for mathematics education, it is pointed out that a tool should not take away responsibility from its users. A black-box like tool may not foster the critical attitude that is important in mathematics. However, powerful tools will inevitably confront students with features they do not know yet. We strive for black boxes to become transparent as much as possible and for tool use that has the character of student-machine interaction.

A second criterion is that of user-friendliness. A technological tool should be easy to deal with. It should allow for different types of use instead of one rigid approach. It should take only a short time to get used to the tool and the interface should be self-evident as much as possible: “if you need to push help, it’s not a good tool”. An immediate high-quality feedback should inform the mathematical process that students are carrying out. A third criterion is that the technological tool should represent sound mathematical knowledge in conventional standard mathematical notation. Finally, the adage “less is more” suggests that we prefer tools that do some things perfect to tools that do many things only to a limited extent.

The field of available technological tools for mathematics education is diverse. Structuring and categorizing this field is a difficult task. The following dimensions for categorizing have been discussed:

#### User-driven versus tool-driven

This dimension focuses on the intrinsic structure of the tool itself and the intentions and aims of its designers. The question is whether it is the tool which suggests the next step or the user who takes the decisions on what to do with the technology and how to do it.

#### Expressive versus explorative

This dimension addresses the user’s cognitive activity. Is the tool a means for the student to express his/her mathematical ideas? Or is it rather an environment for exploration of existing mathematical relations and structures? Expressive tools enable the students to be productive rather than reproductive.

#### Open versus closed

This dimension is in danger of being somewhat vague as long as it is not clear whether the adjectives “open” and “close” belong to the tool or to the task. Or is it the guidance of the teacher which restricts the activity? Still, some tools seem to provide more room for an open approach than others.

## General versus specific

This dimension concerns the scope of applications of the tool. Can the tool be used for a wide set of tasks, and be considered to be a general purpose tool? Or is it only useful for a limited number of specific tasks? A CAS is a more general tool than an applet for practicing solving linear equations.

The first three dimensions seem to be related. They share a focus on room for students to take their own decisions and to develop their personal ways of dealing with a mathematical problem situation. A final categorization simply reflects the main sub-domains in school mathematics, i.e. algebra, calculus, geometry and statistics:

Programming tools (e.g. Logo)

Graphing tools (e.g. the graphing calculator)

Dynamic geometry tools

Tools for algebra and calculus (such as CAS)

Tools for data handling and statistics (Excel, Fathom, Tabletop)

Finally, we address the issue of the future development of technological tools. Of course, there is a lot to wish for. Relatively ‘old’ wishes concern sound mathematical language, integration of various representations and mathematical sub-domains, room for dynamics and interaction, and challenging open environments that invite exploration and expression of mathematical ideas. More recently emphasized wishes concern the possibility for connectivity, communication, and collaborative and participatory learning, now that (wireless) internet communication has become widespread. Open source applications are welcomed because of the room they offer for customization for specific goals and settings.

The design of technological tools should happen in close collaboration between software engineers, math education researchers and teachers, so that new design incorporates progress in educational studies and meets teachers’ requirements. More specifically, the need for permanent availability is stressed, as well as the wish for pedagogically embedded computer algebra.

## Conclusion

To conclude this report, we briefly review the three themes and make some recommendations.

The findings on the first theme, the relation between tool use and learning, show that the theoretical framework of instrumentation and orchestration is recognized as powerful and promising. If it were to develop into a shared paradigm, this would facilitate communication within the researchers’ community. However, it is noted that the instrumental approach is not very accessible, and that it takes time to really experience its value. Furthermore, research on orchestration is relatively young. Therefore, we recommend a two-directional approach to research on the relation

between tool use and learning. On the one hand, ideas on instrumentation and orchestration should be elaborated and made more accessible to mathematics educators and researchers. On the other hand, we suggest that the articulation of this theoretical framework with other approaches, such as activity theory, socio-constructivism and theories on symbolizing should be further investigated.



Figure 5 The students, the teacher and the tool

On the issue of the second theme, the role of the teacher in technology-rich mathematics education, we observe that in spite of the relevance that is attributed to this theme, little research was reported in this working group. However, the above-mentioned notion of orchestration provides an interesting access to the theme of teaching and the role of the teacher. We recommend focused research on the changing role of the teacher in technology-rich teaching, on changing didactical contract, on the didactic repertoire that teachers need and on the different kinds of interaction between students as well as between teachers and students.

Finally, the theme of the characteristics of the tools. We notice an increasing interest in issues of connectivity and collaborative learning. We recommend further studies to investigate the potential of the C (for communication) in 'ICT'. However, there is also some concern. As an illustration, we note that the working group's virtual workspace<sup>1</sup> does not seem to lead to a real process of collaboration, while we as a group seem to be technologically literate and interested in our joint theme! The

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<sup>1</sup> See <http://merg.umassd.edu/cerme4-9/>

factors that determine the success of such distant collaboration would be very interesting to investigate.

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# MATHEMATICS AND E-LEARNING: A CONCEPTUAL FRAMEWORK

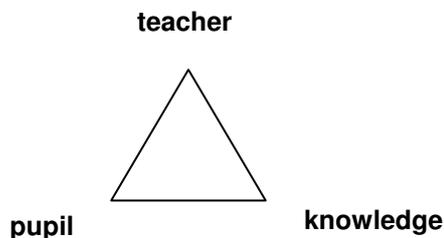
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**Abstract:** *This paper starts from the study of the epistemological statute of the didactics of the mathematics (Henry, 1991; D'Amore, 1999), which faces the phenomenon of learning from the point of view of fundamentals, in order to give useful and specific considerations for e-learning environments. Investigations on how the triangle teacher-pupil-knowledge changes are presented. Then the model of a-didactic situations (Brousseau, 1997) is analysed in the context of e-learning platforms.*

**Keywords:** e-learning, didactic transposition, a-didactical situation, learning design.

## 1. The triangle “pupil-teacher-knowledge” in didactics

During last twenty years the research in didactics of mathematics has analysed in various modes and with accurate details, what it is beyond the triangle (fig. 1) whose “vertices” are the pupil, the teacher and the knowledge (Chevallard & Joshua, 1982; Chevallard, 1985; D'Amore, 1999; D'Amore & Fandiño, 2002):



*Figure 1*

According to *didactics*, it represents a *systemic model* useful to situate and analyze the multiple relations among the three “figures” representing the “vertices” of the triangle. The complex nature of the systemic model comes from considering at the same time all the mutual relations among the vertices, including various implications of different nature.

For an accurate deepening of such topic we cite the synthesis in D'Amore & Fandiño (2002). In such analysis, the triangle has not an explicative and descriptive function of the education experience but, above all, methodological: each vertex of the system is the observer from which we look at the relations between the others, even if we

are conscious that none of the involved figures can be considered totally separated by the others.

Moreover the implicit effort is to render such scheme as more comprehensible as possible of the multiplicity of variables involved on the educational experience intended as problematic experience.

In such systemic model we can distinguish at least three categories of incident bodies:

- *elements* (that are the “vertices” or poles)
- *relations* among the elements (that are the “sides”)
- *processes* that identify the functional modalities of the system.

On the triangle the noosphere (Chevallard, 1992) insists, that is the external world, the society, the collection of the people which prepare the contents and the teaching methods, with their waits, their pressures, their a priori choices.

In the following a revision of such systemic model is presented (section 2), paying particular attention to the model of a-didactic situations in e-learning environments (section 3).

## 2. How the didactical triangle changes when e-learning platforms are used

E-learning environment can be used both in distance and blended education: the different management of the platform impacts in different ways on the vertices and on the relations among them.

The didactical triangle becomes w.r.t. such reference framework a more complex structure (fig. 2) with new vertices and different relations (more complete vision can be found in Albano, Balderas, Sbaragli, 2004). The vertices involved in the learning process under such environment are four: *the author, the tutor, the pupil, the knowledge*. Not in all the platforms the figure of the tutor is foreseen but we consider essential this presence in order to have an effective learning. We do the hypothesis of the following structure where the introduction of the ICT has a total influence, with different levels of deepness, on the four vertices and on different arisen relations and implications:

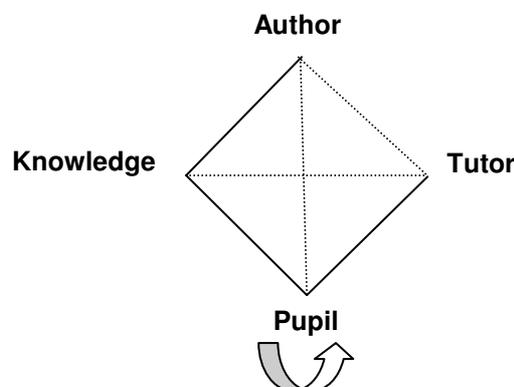


Figure 2

In our opinion such scheme concerns the complex system of the relations arisen among the figures interacting in the learning process when we use a distance learning platform, defining at the same time the specificity and the problematic aspects. Of course, we point out the influence of the noosphere.

We want to concentrate our attention to the distinction between the two figures: author and tutor.

• **The author.** In traditional teaching the teacher is author, tutor, evaluator of his/her course. In an e-learning environment it is possible to focus on two specific figures: the author and the tutor. The first is not yet a single figure, but with this name we mean a group of persons with different professional skills: the instructional designer/manager, the graphical expert, the ICT expert, the didactical (general and disciplinary) expert, the pedagogical expert, the sociologist, the knowledge domain expert, the communication expert... The richness of the involved figures in such a pole allows to create a variegated scenario of pedagogical waits concerning knowledge, of professional or ideological beliefs, of implicit philosophies (Porlán et al., 1996) that supplies with an enrichment of the platform. We consider that the comparison, the discussion, the thoughts that can occur among the different experts above, having diverse experiences, allow to take decisions about the content (*didactical transposition*)<sup>1</sup> (Chevallard, 1985, 1994; Cornu & Vergnioux, 1992) to be inserted in the platform and about the methodologies through which a certain content is introduced (*didactic engineering*)<sup>2</sup> (Artigue, 1989, 1992; Trouche, 2004), arriving in such a way to the construction of a rich and deep product.

The figure 3 shows a possible interpretation of the author together with its activities:

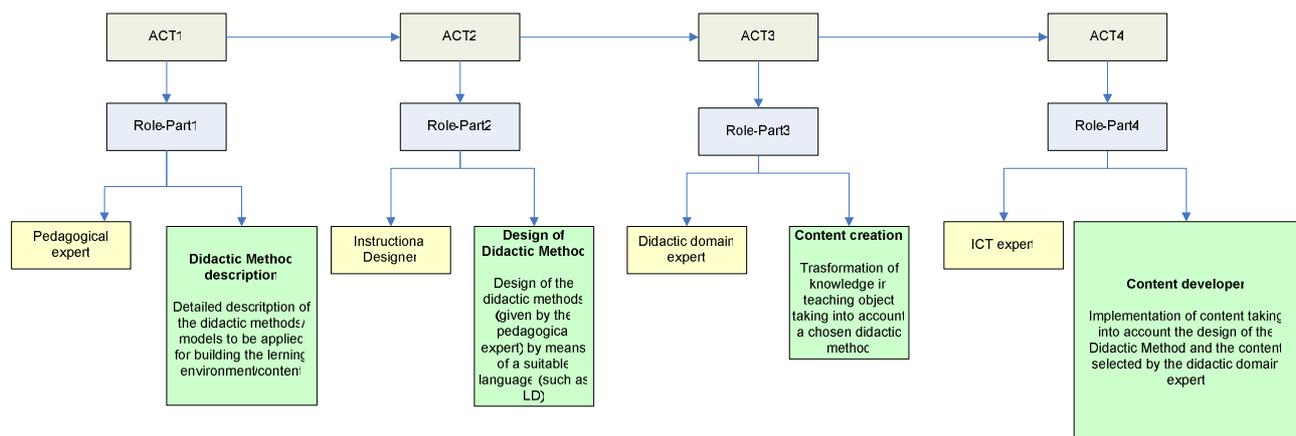


Figure 3

<sup>1</sup> The *didactic transposition* is intended as the work of transforming the knowledge in object to be taught w.r.t. to the place, the audience and the didactic finalities to be reached. Thus the teacher has to do a transposition from the *knowledge* (arisen from the research) to the *knowledge to be taught* (selected by the institutions) to the *taught knowledge* (chosen by the teacher as specific object of his didactic work).

<sup>2</sup> The studies on *didactic engineering* concern in particular the elaboration of didactic sequences, the creation of tools and didactic material organised coherently to the reaching of specific learning objectives. Actually with such word we refer to a research methodology of qualitative type (Sarrazay, 1995; Farfán Márquez, 1997).

• **The tutor.** The tutor represents the privileged figure of our structure having actual contact with the learners. When we refer to the tutor, we mean not a single people, but often we mean a group of people who take care of a certain number of students, ensuring at the same time variety of thoughts, proposals, besides the deepness of the relationships. The role of the tutor impact different areas: management/organisation, social and didactical (Cosetti & Pallavisini, 2002).

### 3. Didactic transposition in e-learning environment

According to Brousseau (1997) «*In modern didactique, **teaching** is the devolution to the student of an adidactical, appropriate situation; **learning** is the student's adaptation to this situation*».

In e-learning environment, the author and the tutor takes care of *teaching* and thus of the didactic transposition. This means that, each time learning needs occur, the author can create suitable Learning Objects<sup>3</sup> (LOs) that will be delivered to the students with which they interact, under tutor's guiding. In such a way the didactic model the LOs are created according to is transparent, it is not formally described and no possibility of reuse it exists.

In this section we want to give a rough formalisation of the didactic model described by the a-didactical situations, describing it according to IMS Learning Design. A-didactical situations that seem to fit very well the e-learning environments: the student is implicated in construction his knowledge interacting with a "milieu", properly designed by the author and the tutor in order to foster the devolution (interesting examples can be found in Laborde, 2001). We refer to various types of situation (fig. 4), distinguished w.r.t. the relation that may exist between a student and the milieu, according to the following model (Albano, 2004; Albano, D'Auria et al., 2004):

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<sup>3</sup> A Learning Object (LO) can be defined as "Any entity, digital or non-digital, which can be used, re-used or referenced during technology supported learning" (Learning Object Metadata Working Group of the IEEE Learning Technology Standards Committee - LTSC). We cite their characterisation adapted from the Wisconsin Online Resource:

- LOs are a new way of thinking about learning content. Traditionally, content comes in a several hour chunk. Learning objects are much smaller units of learning, typically ranging from 2 minutes to 15 minutes;
- LOs are self-contained – each learning object can be taken independently;
- LOs are reusable – a single learning object may be used in multiple contexts for multiple purposes;
- LOs can be aggregated – learning objects can be grouped into larger collections of content, including traditional course structures;
- LOs are tagged with metadata – every learning object has descriptive information allowing it to be easily found by a search.

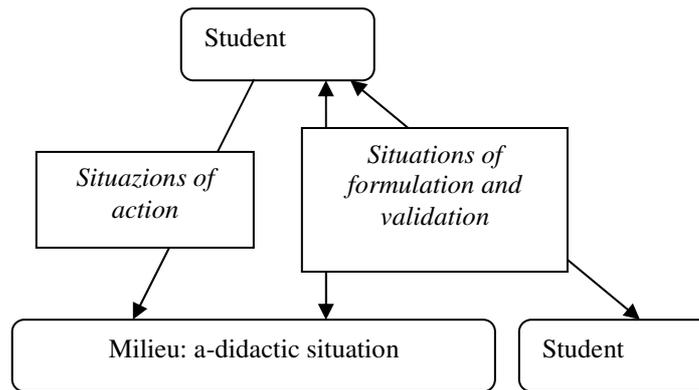


Figure 4

- *Situations of Action:* are those in which the student interacts with the environment, «If the exchange of information is not **necessary** for obtaining a decision, if the students share the same information about the milieu, the “action” is dominant.» (Brousseau, 1997). The sequence of situations of action constitutes the process through which the learner constructs strategies, namely “teaches to himself/herself how to solve the problem. In this sense Brousseau talks of “dialectic of action” since the student on one hand can anticipate the result of his choices and on the other hand the chosen strategies can be confirmed or not by the experimentation/interaction with the environment. The situations of action promote in the student the rising of a “model”, namely of a representation of the situation, which may be more or less implicit. On the basis of the model the student little by little constructs, he will do his following choices.
- *Situations of Formulation:* are those in which the student sends messages to the antagonist milieu with the intention of presenting an opinion. When the strategies are formulated, there are two strategies of feedback: one to the environment (milieu) that, once the formulated strategy has been applied, gives a response which can be positive or not; one to the other students he interacts with, who say if they have understood. The situations of formulation encourage the acquisition of explicit models and languages; if they have an explicit social dimension, we can talk of situations of communication (D’Amore, 1999).
- *Situations of Validation:* are the situations in which the messages exchanged with the milieu consist in assertions, theorems, demonstrations, both sent and received, namely the affirmations must be subjected to the judgement of the interlocutor who must be able to give a feedback, to protest, to reject a reasoning, to express some counter-examples, etc. The student is required to justify his assertions, to test their validity in a more formal and general manner than the simple observation of the results produced by the model implementation. In this phase an important aspect concerns the debate with the other students. These situations have to lead the student to evolve and revise his opinion, replace false theories with true ones, to organise the demonstrations.

At the end of these situations, after subsequent adjustments and refinements, eliminating possible errors, the students has achieved the production of his own personal knowledge (mediated), which needs to be institutionalised, that is to be accepted as knowledge socially valid for the domain experts. For this purpose in the situation of institutionalisation, the tutor makes explicit which was the institutional knowledge at stake and which other close examinations can be done.

### 3. An example of design

In the following we describe the Brousseau a-didactic situation using IMS Learning Design, so an a-didactic situation can be seen as a workflow of activities (fig. 5), and for each of them we will define roles, services and activity.

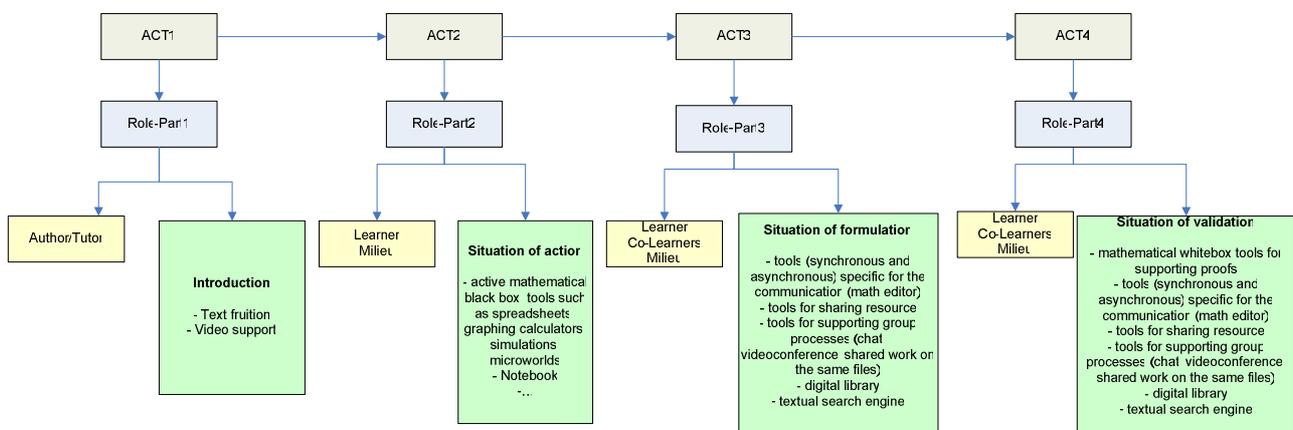


Figure 5

At first students are engaged in situations where the learner has to interact in active way with the milieu and he/she is not a passive receptor of the traditional learning. The student can be immerse in a “real” motivating and involving context, which foreseen some active phases and choices made and personally managed by the student, to whom the milieu replies. Such situations can be realised using “expressive tools”, that can be distinguished in pedagogical tools (e.g. Dynamic Geometric Systems (DGS), microworlds, simulations) and calculation instruments (e.g. Computer Algebra System (CAS), spreadsheets, graphing calculators, databases), properly arranged by the author/tutor. Here the milieu acts as a black-box: the students changes some parameters and observes how the environment modifies.

According to such considerations and looking at the attributes of meaningful learning, individuated by Jonassen et al. (1999), the situations of action can be realised environment having the following characteristics:

- “active” to allow the students to manipulate, even if virtually, objects able to react to the action performed by the learner;

- “meaningful” that is the learner has to be engaged in meaningful task so that they can effectively manipulate objects and observe the results of their manipulations: e.g. not just move a cursor or press a button, but modify parameters, shape, viewpoint, and so on;
- “dynamic” in order to change themselves on-the-fly according to the actions performed by the learner.

Considering the reactions of the environment to their manipulations, the learners construct in their mind some conjectures about which are the relevant aspects of the experience, about the relation cause-effect in that particular context. Seeing which are the parameters modifying the experience they hypothesise some beginning (sometimes superficial) relations among them, so they are able to predict the effect of some actions.

In (Hoyles, Noss, 2003) digital technologies are reviewed w.r.t. their impact in mathematics education. Expressive tools give the student many advantages, such as: to manage competences greater than he actually has (e.g. to make difficult computations, to plot, to apply algebraic transformations, etc.); to have a direct and immediate feedback; to use many semiotic registers (algebraic, graphical, numerical); to concentrate his attention on qualitative aspects rather than procedures. Note that the action in e-learning environment has an added value w.r.t. the paper-and-pencil: for example a figure sketched with a DGS is not static, but through draw mode allows to outlined all the figures preserving some geometrical properties, fostering the student to make conjectures.

Here we can individuate a cycle with three main steps (Tall, 1995): perception of an object (external activity), thought (internal process) in which the learner reviews what has been done and experienced, and action (external activity), where the “thought” of the learner is translated it into predictions about what is likely to happen next.

Once conjectures arise in their mind, learners are ready to formulate their hypotheses. Students are required to make explicit their own model of the experience, that is the implicit model that he/she has built “acting”, for example he/she is asked to make explicit the relations intervening among the variables at stake, to write a formula, to realise an algorithm, etc. In this sense, building a programme, by CAS as programming language at high level, allows new ways of modelling and representing mathematics. They are able not only to predict but now they must have the chance to test their own model, in order to clarify it. Thus we can say that the student also approaches an enactive proof: «In enactive mode proof is by prediction and physical experiment» (Tall, 1995). They can also interact with the shared environment as above, but here the milieu acts as an open environment, that is it replies by applying the received model and the student has the possibility to understand if the supposed model produces coherent results or not. This is the phase where the student, after the perception of and the interaction with external objects, starts to construct his own visuo-spatial prototypes becoming successively more verbal-deductive, leading to

iconic or visual proofs, representing not only a specific case but all the cases in a same class.

Since learning is a social construction, it is opportune that these situations are in particular situations of communication: the explicit models of each student can be shared and discussed with other students during virtual debates, forecasting a confrontation, in a collaborative learning process, through tools (synchronous and asynchronous) specific for the communication, the sharing of the resources, to support group processes (chat, videoconference, shared work on the same files).

Finally students are asked to collect their thoughts, refine all previous idea and present not just his/her point of view but he/she is required to defend it, to give some kind of proofs of his/her statement. In mathematics proof can be produced using CAS: attention should be given to new kind of proofs, such as those ones based on the use of logical value of algebraic operators and on the use of graphs. The final aim should be to produce a common vision together with its suitable justification, that will constitute the knowing. As support a virtual area can be organised (such as a discussion forum) asking the student to produce and share a document with his models and proofs. The debate with other students is considered essential: each student has to “contest” the proofs given by the others and defend his own theses.

As pointed out in (Hoyles, Noss, 2003), since most of the students interacting with digital technologies spontaneously articulate justifications of their actions along with explanations of why their actions produces the expected feedback (or not), such technologies might give the opportunity to produce a deep understanding of the topic, although we have to take also into account the obstacles that might be arisen (Drijvers, 2000).

Once the students has completed the described process, the institutionalisation allows the passage from *knowing* (as personal construction) to *knowledge* (as a socially shared construction).

#### **4. Conclusions**

In this paper we have presented how the learning process might change in e-learning environment, w.r.t. to the involved actors and the relations among them. In particular we have considered the didactic transposition which is the issue of the author and the tutor. The didactical model of the Brousseau a-didactic situations has been analysed and a rough design for its implementation in e-learning platform has been sketched. The impact of digital technologies has consequences as both new actors (e.g. author) and new meaning of existent ones. Moreover they seems to well fit the a-didactical situations, because they, suitably arranged by the author-tutor and as powerful tools containing knowledge, naturally foster exploration, conjecturing, explanation, verification and proof. The counterpart is represented by new obstacles that might be arisen that requires new pedagogical contexts.

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# SUPPORTIVE USE OF 'DERIVE'

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**Abstract:** *In this paper, a small group of upper secondary school students' work with differential equations in models, using Laptops with the CAS software 'Derive', is studied. In the paper, crucial points of their work are interpreted in terms of 'changes of perspective'. These terms relate to a certain notion of 'flexibility of conceptions', introduced by the author in an earlier paper. The aim of creating this notion is to restructure parts of well-known educational theory and turn it to account for teachers. It is analysed, how the changes are provoked and what role they play in the students' actual process of doing mathematics. The analysis may serve as part of the basis for subsequent preparation of guidelines, meant for teachers in the design of tasks and teaching sequences that support instrumental genesis.*

**Keywords:** Genesis of CAS as an instrument, differential equation models, flexibility of mathematical conceptions, upper secondary school mathematics.

## **Background**

As part of the evaluation of a Danish development project, where students in upper secondary school had laptops at their disposal for at least two years in their main subjects mathematics, physics and/or chemistry, I have made yearly group interviews with all the participating teachers (about 15 persons each year) and groups of participating students, three times in all. One broad aim of the project was to study the impacts of the use of laptops on the teaching and learning of mathematics. In the interviews, it was claimed by a number of students and teachers that getting 'a better overview and a deeper understanding' were common effects. When asked, the students and the teachers repeatedly specified these general claims by saying: 'because you can easily get series of graphs', 'you do not stuck in technical details', 'it is easy to see examples' or 'you do not have to remember a lot of techniques but may concentrate on the ideas'. Such claims point to certain potentials of special features, introduced or amplified by the use of laptops in working with mathematics. Thereby inspired, the aim of my Ph.D. project is to refine and conceptualise such special features and link them to educational theory. Subsequently, they should be articulated in a certain notion of terms, easy to understand and handle for teachers and useful to (partly) form design guidelines for teaching. The Ph.D. project started august 2002 and will be finished in summer 2005. The conceptualising of the experienced learning gains is guided by keywords such as diversity and dynamics of doing mathematics, tool use and 'webbing' or making connections.

## **Framework**

Taking the constructionists' view on learning as a process of constructing and modifying conceptions (Cobb et.al., 1992), I find the mental action: 'change of perspective' crucial for learning mathematics by working with it. This claim is

inspired by the idea of reification in an ‘extended’ version where cognitive development is based on a two directional process of taking distance and diving into the subject, respective (Sfard, 1991, Ackerman, 1990 and Douady, 1991), and further on the idea of creating knowledge, not as a one-dimensional hierarchical process but like building a web (Noss and Hoyles, 1996 pp 105-133). In this context, the term ‘perspective’ relates to different facets of the mathematical conception in question, constructed by the student. For example, a student may have constructed a conception of ‘function’, which implies only a process aspect (concerning calculations), a graphic representation and a computer language- representation. In themselves, these three would form narrow limits for working with functions so ‘adding’ more perspectives would be an obvious learning goal for this student. One perspective, desirable to ‘add’, is the object perspective, as a result of reification. Another example of a narrow conception of ‘function’ is commonly seen: students who identify ‘function’ with its algebraic expression are in some cases unable to handle tasks or problems in graphic representation, if the algebraic expression is unknown or omitted. A reified conception of ‘function’ in algebraic representation will not necessarily be mirrored in the graphic representation. In these cases, the mental activities of changing between the process and the object perspectives, and into and between the two representations, in both directions, seem to be crucial for the students’ ability to proceed in working with mathematics.

So, the first step in the process of conceptualising the experienced learning gains, is the introduction of the following specific use of the term ‘Flexibility’ of a mathematical conception in (Andresen, 2004a):

Definition:

The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception.

The concept of flexibility has different interesting aspects: the relevance of flexibility in a modelling context is for example distinct from the relevance of focusing at flexibility when working in a CAS environment. Therefore, changes in certain dualities of perspectives and between four specific representations have been chosen in order to make this definition operative and suitable for building learning trajectories:

*CHANGES OF PERSPECTIVE*

*Dualities of perspectives intrinsic to mathematics,*

1. Local and global position
2. General - specific
3. Analytic- constructive

*Dualities of perspectives linked to construction of epistemic knowledge*

4. The process - object duality.
5. Situated - decontextualised

*Dualities of perspectives linked to construction of pragmatic knowledge*

6. The tool - object duality

7. Model - reality

8. Model of - model for

**CHANGES BETWEEN DIFFERENT REPRESENTATIONS.**

Three main representations are considered: *graphic representation*, *analytic representation (or formal language)*, and *natural language*. A fourth, called *technical representation (or computer language)* is included as well, caused by the use of laptops. (Andresen 2004b)

The goal is not to categorise all kind of changes of perspective – neither is it to include every possible change. The basic idea is to pin out a few key elements, recognizable for teachers and useful for the design of teaching sequences and tasks that realise potentials of the computers and support mathematical activity by the students. The choice of these dualities of perspective and representations will not be discussed or further elaborated in this paper.

**Activities of mathematics and changes of perspective**

By this introduction of a definition of ‘flexibility’, and the implying underlining of ‘change of perspective’, I aim to stress that basic idea of mathematics as a human activity, which is also captured in the term ‘Mathematising’. This leads to an examination of possible links between changes of perspective and different interpretations of mathematics as an activity:

In the Dutch tradition, a distinction is made (Freudenthal, 1991, pp 41-42) between horizontal and vertical mathematising: activities, which lead from the world of life to the world of symbols (horizontal), are distinguished from activities that involve the shape and reshape of symbols and mechanically, comprehendingly, reflecting manipulations of symbols (vertical). These two different kinds of activities may relate to changes between different dualities of perspectives, but as the distinction between horizontal and vertical mathematising depends on the specific situation, the person involved and his or her environment, it causes no classification of the dualities of perspectives.

Examples of activities important for mathematising and possible links to changes of perspective, are:

- To structure, organise and reorganize mathematical ideas and conceptions at different levels of complexity. This may be linked to changes between process and object perspectives, whereas a reified process may be element of a structure.
- To build and test mathematical models: Building models involve changes from reality to model and from ‘model of’ to ‘model for’. Test of models may involve changes from general to specific, from global to local and from ‘model for’ to ‘model of’.
- To generalise mathematical experiences and strategies may involve changes from specific to general and from reality to model.
- To express mathematical relations and relations concerning mathematics, may involve changes between all the four representations.

To refine the field of mathematical activities for a closer examination of the links between activities and changes of perspective, the activities may be interpreted in terms of the eight mathematical competencies, stated in *Mathematical competencies and the learning of mathematics: The Danish KOM project* (Niss, 2002). Especially in the context of computer-use, the following statement in the KOM report should be remarked: “All competencies have a dual nature, as they have an analytical and a productive aspect. The analytical aspect of a competency focuses on understanding, interpreting, examining, and assessing mathematical arguments or understanding the nature and use of some mathematical representation, whereas the productive aspect focuses on the active construction or carrying out of processes, such as inventing a chain of arguments or activating and employing some mathematical representation in a given situation” (ibid. p 9).

This dual nature of the competencies resembles that duality of epistemic, respectively pragmatic processes of knowledge construction, on which the French theories of instrumental genesis are founded (see Artigue, 2002, Verillon & Rabardel 1995). The process of instrumental genesis implies, that the student develops means to the new tool in an appropriate and sensible way by building utilization schemes for instrumentalization (adapting the tool) and for instrumentation (finding out how to use the tool to solve a specific type of problems), (Drijvers, 2003, p 97). This leads to suggest a close link between the learning goals for processes of instrumental genesis at the one side and the educational goals, interpreted in these terms of competencies, at the other side. It is exemplified in a subsequent chapter on analysis of the data presented in this paper, how the process of instrumental genesis itself consists of activities that may be interpreted in terms of changes of perspective too. The key elements from the definition of flexibility, apart from the ‘intrinsic’ ones, are chosen to form two groups: Dualities of perspectives linked to construction of epistemic knowledge and dualities of perspectives linked to construction of pragmatic knowledge. This separation makes the use of ‘flexibility’ appropriate in a CAS- or computer context because it allows the inquiry to focus on tool use, interpreted as activities carried out with a specific purpose and aiming to solve a specific problem, opposed to less goal-directed activities. In general, the correlation between mathematical competencies as educational goals and the interpretation of learning goals in terms of ‘flexibility’ needs to be further inquired. Introduction of the conceptual tool ‘flexibility’ may serve partly to turn educational goals stated in the description of the eight competencies into learning goals, tangible for the teachers. The notion of flexibility offers conceptualisation of learning goals in a form that can be exposed into guidelines for design of teaching sequences.

### **Presentation and analysis of data**

The following episode is chosen from data, collected as part of the research project. The subject, differential equations, is suitable for working with changes between specific- and general-, and local and global perspective respectively. Especially, the complexity of the area fosters rich possibilities for such changes. The software ‘Derive’ was chosen for the whole development project since Derive is accessible at

all upper secondary schools in Denmark and the project aimed to develop results for a widespread use. The episode was chosen to illustrate how the use of laptop facilitates changes between local and global perspective, between general and specific perspective, and changes between graphic representation and the others. The analysis of the episode aims to find out how these changes are linked to the students' understanding of the problem in the task.

In this episode, the students' work takes as its starting point the following task:

*As previously remarked, the general version of the initial value problem which models the cholesterol level by a person is:*

$$\begin{aligned} \frac{dC}{dt} &= k_1(M - C) \\ C(t_0) &= C_0 \end{aligned} \quad (8.4)$$

*3. Solve the initial value problem (8.4)*

The following dialogs are cut from the transcription of video tape, recorded while the group worked with question 3:

*P1 Okay. (reads): Find the solution to the initial value problem*

*P1 Haven't we done that twice already?*

*P2 what?*

*P1 find the solution to the initial value problem – haven't we already done that? We did it with numbers. Look, what we did with numbers.*

*Here is something called  $dC$  over  $dt$  equals  $0,1$  times  $250$ . It is here*

(P1 found in an earlier task the equation  $\frac{dC}{dt} = 0,1 \cdot (265 - C)$ , with the solution

$$C := \frac{26,5}{0,1} - d \cdot e^{-0,1 \cdot t} \quad \text{where } C(0) = 180.)$$

In this first of two lessons, only two of the three group members are present.

As the students earlier in the project have solved the differential equation for specific values, it is obvious to P1 and P2 that there are links between finding the specific solution and finding the general expression for the solution: they have to think over, whether they did or not answer the question in an earlier chapter. They know, that 'to do it with numbers' (find specific solutions), is different from doing without numbers (find general solution) – in the last case, they describe the result as more 'cryptic' (in the next lesson, as demonstrated later in this chapter) and the two students are not convinced, that they really got a result or an answer.

The students answered the actual task 3 like this in the report:

$$\begin{aligned} C(t_0) &= C_0 \\ C(t) &= \frac{b}{a} - d \cdot e^{-at} \\ C(t) &:= \frac{k_1 \cdot M}{k_1} - d \cdot e^{-k_1 \cdot t} \\ C(t) &:= \frac{E \cdot k_2 + L \cdot k_1}{k_1} - d \cdot e^{-k_1 \cdot t} \\ C(t) &:= M - d \cdot e^{k_1 \cdot t} \end{aligned}$$

To formulate this, P1 and P2 make a change from general to specific perspective when they substitute the actual parameters into the general expression for the solution to the differential equation in question, picked out from their compendium of formulas. The following dialogue took place alongside:

- (...) P1 yes, but we wrote it – it should equal this, shouldn't it? (writes on the paper) See 1)-5) below
- P1 and this also fits with..ehh..shows that b.. and a equals  $k_1$  (has isolated a and b by hand,P2 now types in on the computer)
- P1 and what equals d?
- P2 d is just (mumble).. there is nothing...
- (...)P2 okay then we have e to the power of minus  $k_1$  times t
- P1 and we may substitute for M that we got earlier (uses paper and pen while P2 types on the computer)
- P2 I can just move this, can't I?
- P1 yes
- P2 what did M equal?
- P1 we wrote it up here (points at the screen)
- (...) P1 then we have to..ehh..solve it..
- P2 but we can not solve it!
- P1 yes, (points at the screen, types) now we can!
- P1 can you see it?
- P2 yes then we move the equation and then we solve it regarding to d

In this process, the students:

- 1) Recognize the type of equation as  $\frac{dy}{dx} = b - ay$ , (called 'the neutral form' in the textbook, referring to the letters x and y)
- 2) Transform the original equation to  $\frac{dC}{dt} = k_1 M - k_1 C$ ,
- 3) Identify a and b:  $b = k_1 M$  and  $a = -k_1$
- 4) Substitute these expressions of a and b into the general expression of the solution:

$f(x) = \frac{b}{a} - c \cdot e^{-ax}$  where c is denoted d for technical reasons: the 'c' is already used.

Then, the two students

- 5) Substitute the expression of M in terms of L, E,  $k_1$  and  $k_2$ , using 'copy', 'paste' and 'simplify' into the solution. It is not clear, why they do the last substitution.

A little later, they ask the teacher:

- (...)(talking with the teacher L who came to help the students)
- P1 and then: find the solution to the initial value problem. In all the others, we have answered that it was this model, haven't we? And then we have known some values so that the only unknown was C. But now we have no values so we cannot get further than to this one
- L ohh
- P2 I just mean – is that good enough?

*L ..and  $C_0$  was the initial value then, wasn't it?*

*P1 yes it was*

*L and the others were numbers.. yes that is the way you have to do it –*

*P1 we could graph the others...*

*L yes that is true. Is this the same type of function as the one you found in the others? Can you see that?*

*P1 But the others,.. the others we found, they looked like this, didn't they (sketches on the paper) and that is exactly the same. And then we just.. ehh. .took what we got.. we found a value for  $M$  and that, what we found, we just substituted and multiplied and then equaled to zero.*

*L yes*

*P1 and then we isolated  $d$*

*L which is an arbitrary constant. Then, I don't think you could do much more – how was the expression compared to the others you have got? Did you get others?*

*P1 no, we just got numbers*

*L but in this expression (points at the screen) – the whole first part is constant, isn't it?*

*P1 yes*

*L were the others shaped like this?*

*P1 it looks like this, the general solution*

*L yes, then it is the same*

This refers to the formulation of that answer to the task, shown above.

P1's remark: "And then we isolated  $d$ " may refer to the earlier task mentioned above, where the initial values were used to determine  $d$ .

It seems to disturb the two students, that they have no values to substitute in the general expression, aiming for calculating a concrete result. This observation reveals a process- and no actual object perspective by the students on the equation that states the solution. Besides, it leads to question the students' management of those tool-object perspectives, in which the general solution is seen as a tool for finding specific solutions. That is, seeing the general solution as a tool involves changes between general and specific perspective of the solution. As far as the two students apparently find, that the general solution might be of no use or at least may not count as an answer, they do not see the general solution in an object perspective, referring to the tool-object duality.

In the second lesson, the third group member P3 is present, and P1 and P2 show and explain to P3 what they have done, thought and written in the report:

*P1 here, we did the same as earlier on, (points to the screen) and take the general one downwards here and substituted, and then calculated. What is different here is, that earlier did we know the numbers, then we could get a value, and now we have only the parameters and then it gives such a cryptic one, doesn't it?*

*P3 yes*

Apparently, P1 again refers to the same earlier task as mentioned above.

P1 has now accepted, and tells to P3 as a fact, that the general solution (the ‘cryptic’ one) may count as an answer. As part of the explanation, P1 in own words and with fluency explains, how the general solution may be used as a tool for calculations. ‘Copy’, ‘paste’ and ‘substitute’ were used in this procedure:

The general differential equation typed in earlier in the text was marked with the cursor, copied and pasted in to the end of the document. Then the actual values are substituted for the constants and the equation is solved using the DSOLVE command. The dialogue shows that P1 is familiar with kind of a tool use of the conception of solution, but it is very close linked to the computer facilities, and the student does manage the actual change from general to specific perspective, at the very concrete level, closely linked to the computer.

The dialog continues like this:

*P1 but then, we have written such an interpretation, down there*

*P3 but normally, we do not want to get the  $d$*

*P1 yes we do. It is called  $c$  in the usual formula*

*P3 yes, but we do not always want to get that one*

*P1 it is the one we calculated in the other (cases, MA) and then we did ‘vector’ and found some more...*

*P3 yes but we just want to...*

*P2 but (points at the screen) it is this one, isn’t it?*

*P3 yes, this one, it is this one you want to get (point at the screen). And  $d$  is just a number and you only want to get  $d$  if you do not know  $d$*

*P1 but you do not have it*

*P3 yes you do, if you are given... if you know this one (points at the screen). And you know one point, so you can calculate  $d$ , exactly which solution. This is a family of solutions*

*P1 yes but really*

*P2 it does not matter if we have done a little too much*

*P3 yes but we have to write a comment*

P3 questions that goal of determining  $d$ , set up by P1 and P2. This leads to a discussion that reveals the three students’ knowledge of using ‘vector’ to graph a family of functions. That is, the students do manage a change from local (one solution) to global (a family of solutions) at the very concrete level, closely linked to the computer. Such use of ‘vector’ links the graphic representation to the algebraic and in the same moment, it makes the change between local and global perspective tangible. The use of ‘vector’ may involve a change from general to specific too, if a specific value is substituted into a general expression. P3, apparently, try to articulate the difference between the aim of determining  $d$  for the purpose of pinning out one specific solution within a family of solutions on the one hand, and the aim of determining  $d$ , just because it is an unknown entity in the equation on the other hand. P3, seemingly, has started to develop an object perspective of the conception of solutions, structured as a family. P2’s remark, that it does not matter if the group have

done a little too much, shows that this student doubts the relevance of the last part of the group's work.

### **Concluding remarks**

1) The important role of the dialogue between the students and the teacher is demonstrated in the episode: In the project, the changes between specific and general solution of differential equations, and between local and global perspectives, are provoked by direct questions and tasks. This makes it clear to the students, that the changes are prerequisite for answering the questions.

The students' consciousness of the links between specific and general solution are weak and not articulated in the beginning of the first lesson, when they are asked to find the general solution. Though, it is no problem for them to make the change from general to specific when the 'neutral form' is made specific for the general problem in question. The dialogue reveals how this consciousness grows from the students concrete work with the computer through the negotiations between the students and the dialogue with the teacher.

2) Computer use in focus draws attention to the distinction between 'process-object' and 'tool-object', distinctions between 'situated-decontextualised' and 'reality-model' resp. 'model of-model for' and to the changes of representation, especially shifts involving graphic representation and computer language. Out of these subjects, only the process-object and the tool-object aspects are considered in this episode. The 'process-object' duality is appropriate to describe the situation, where the two students aim at calculating some concrete results and hesitate to regard the equation as a result. In this case, the two students' development of a tool perspective on the general solution to a differential equation seems to depend on and prerequisite, that the students manage changes between general and specific perspective and between process- and object perspectives on the solution.

3) The notion of flexibility is useful to elucidate a phenomenon well known to many experienced teachers: 'Students may adapt procedures without knowing what they do', especially when working with computers. In the analysis, a seeming contradiction between P1's computer-mediated tool use of the general equation and the confusion about general solution is explained in terms of changes of perspective. This allows for an interpretation in the French theory of instrumental genesis, in terms of distinction between an instrumented technique (i.e. 'doing without knowing') and an instrumentation scheme mentioned above (Drijvers, 2003, p 100). In these terms, the student refers to the instrumented technique as shared knowledge between the students and makes the work proceed this way in the episode. Similarly, the use of 'vector' in the episode may be interpreted in terms of an instrumented technique, according to my analysis of the discussion concerning d. At last the students P1 and P2 do not demonstrate, that an instrumentation scheme is developed. Though, seemingly, the uses of these instrumented techniques and the discussions between the students support the formation of such ones in both cases.

4) In this context the notion of flexibility connects actual teaching episodes to Instrumental Genesis and thereby can lead to concrete advices for support to

formation of instrumentation schemes. Future analyses will enlighten how the building of and the use of shared knowledge of instrumented techniques supports the learning processes in groups of students, provoked by questions that encourage the students' changes of perspectives.

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# FROM DISTRIBUTED TO SHARED KNOWLEDGE – A CONCEPTUAL FRAMEWORK FOR A VIRTUAL SEMINAR IN TEACHER EDUCATION

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**Abstract:** *Life-long learning, self-responsible knowledge acquisition and cooperative work are aspects often related to the use of new technologies in education. The seminar “space geometry” was taught during spring 2004 combining different classes at four different universities. The main idea was to build a semi-virtual seminar with collaborating groups of distributed expertise. The task was to specify mathematic standards in space geometry while creating a matching website, and to prepare a face-to-face presentation of the groups’ seminar work. This report describes the goals, principles and assignments of the seminar, the methodology used and some results of the accompanying research study.*

**Keywords:** *virtual seminar, geometry, distributed system, Internet, jigsaw method, communication.*

## **Introduction and Theoretical Framework**

Taking a constructivistic view, learning is characterized by integrating new knowledge into an existing mental network of knowledge and building new cognitive structures (Terhardt 1999). This process has to be actively initiated by the learner and is always embedded in a special situation or context of acting. Learning cannot be separated from the situation in which it takes place. Learning is an individual, active, self-controlled and situative process.

It is also common knowledge, that our information and knowledge society needs new ways of learning, to develop a new culture of learning, a culture which places more importance on self-responsible knowledge acquisition and team work, group or collaborative work. Both individual and collaborative learning should be emphasised, individual learning should be integrated into collaborative or cooperative learning.

“Interacting with other students offers students the opportunity to gain different perspectives on a problem, to discuss different solutions and different problem-solving strategies, to get important hints, to argue about difficulties and to support each other with feedback and other forms of help.” (Salomon & Perkins 1998), p. 6

In the following we refer to collaborative learning in the frame of working with new technologies in teacher education.

Using new technologies does not automatically mean better learning results. New educational media have to be properly integrated into the learning context and students have to be well prepared for using these new media in an adequate way.

The Internet is a medium that supports collaborative work (Dillenbourg 1999, Stahl 2002). Virtual seminars with working groups in different places, different schools or universities are one possibility. There are a lot of difficulties with virtual seminars, e. g. interrupted communication (Sassenberg 1999) or information overload (Bruhn 2000). A major problem is also motivating the students to collaborate with students from other universities, if there are more than enough opportunities for face-to-face communication with students at the same university. Participating in virtual seminars is not a goal per se.

A further problem with virtual education and online-classes is the lack of social contact among the participants. For this reason, semi-virtual seminars or “blended-learning-settings” - a mixture of different kinds of teaching, especially the combination of face-to-face and virtual phases - are being more and more used (Derntl et al. 2003, Grabe & Grabe 2001, Reinmann-Rothmeier 2003).

We choose the “virtual jigsaw-method” because the acquiring of expert knowledge in the first step and the sharing or use of this knowledge to solve a given task is constitutive to the jigsaw method (Information on the jigsaw methods URL: <http://www.jigsaw.org/>, retrieved 2004-09-23). And it is also a method easily adapted to virtual learning (Rinn et al. 2003, Hinze et al. 2002).

### Seminar Setting

During spring term 2004 we taught a semi-virtual seminar “Space Geometry” with five classes at four different universities. The participants were advanced mathematics education students for voluntary and middle schools. The main idea of the seminar was to build collaborating groups of distributed expertise. Four of the five participating lecturers work in the area of mathematics education, one in the philosophy of education with a special interest in the didactics of new technologies. The table shows the four universities, the special local topics of the seminar and the number of participating students. There were two classes in Weingarten, one in mathematics education and one in didactics of new media. We will refer to these as the five *local groups*.

University	Lecturer	Special “local” topic	Number of students
Karlsruhe	B. Schmidt-Thieme	Open Classroom Settings	15
Ludwigsburg	C. Bescherer	Standards and Principles of Mathematics Education	21

### Working Group 9

Weingarten	M. Ludwig	Space Geometry and Real Life Applications	8
Weingarten	M. Henninger	Didactics of New Media	3
Würzburg	H.-G. Weigand	Computers in Mathematics Education	8

The lecturers chose five mathematical topics for the students to work on:

- Regular Polyhedrons,
- Nets of Solids,
- Intersections of Solids,
- Parallel Projection, and
- Spherical Geometry.

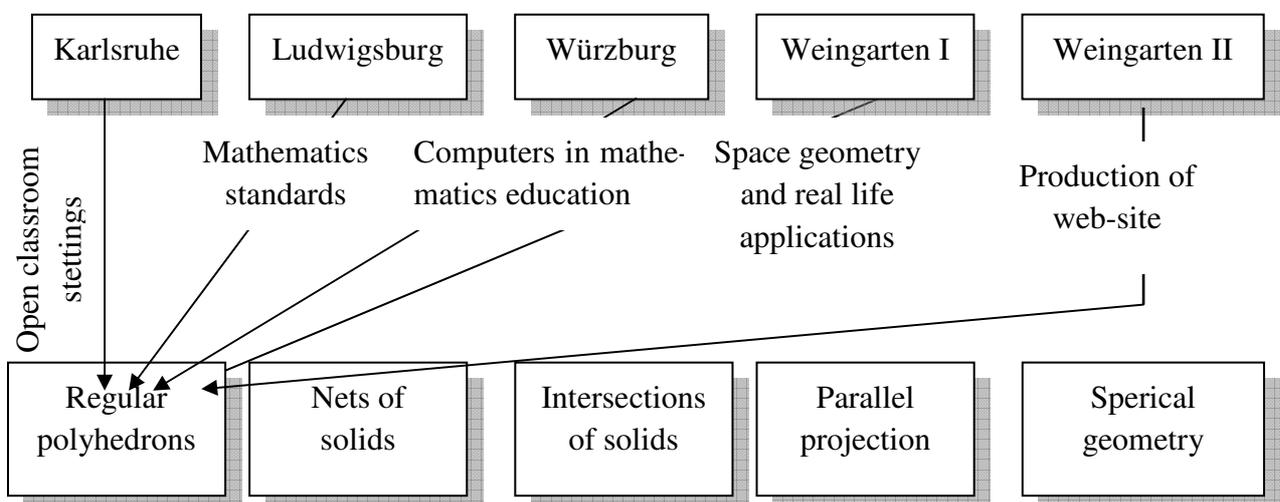
The task for these five “local groups” was the creation of a web-site that was to discuss the mathematical topics from four different experts’ viewpoints. These viewpoints were

- “Open Classroom Settings”,
- “Standards of Mathematics Education”,
- “Space Geometry and Real Life Applications”,
- “Computers in Mathematics Education”.

Initially, the students had to acquire expert knowledge in the local groups and then use it to develop the web-site. The main idea of the seminar was to use distributed knowledge to develop shared knowledge while collaborating via the Internet.

The second Weingarten group (Didactics of New Media) was to design and manufacture the web-site, given the contents and structure by the other groups. (In spring term 2003 we taught a similar seminar (Bescherer et al. 2003), the students had wasted too much time and energy in designing the web-sites without working on the content. We therefore decided to “outsource” the actual production of the web-sites.)

#### *The local groups*



*The topic groups*

*Figure 1*

Thus each student was actually a member of two groups: his or her *local group* and the *topic group* in one of the four universities (see figure 1).

**General goals of the seminar**

- The students should know the most important elements of space geometry related to school mathematics
- The students should decide on the goals of space geometry in the mathematics classroom
- The students should develop “media competence” by becoming acquainted with new technologies for web presentation of a topic, while communicating with others and using net technologies in mathematics education.

**Principles of the seminar**

The students work in topic groups of 6 to 8 members on one mathematical topic. The members of these *topic groups* come from the five different local groups; at least one participant comes from each local group.

The students of the local group concentrate on the special expertise or know-how of their local topic. They become “experts” in their special fields and then cooperate with the other groups to structure the web-site.

**Assignments of the seminar**

- Each topic group had to decide
  - on the mathematical content to be included in the site,
  - the goals and realisations of the (new) Baden-Wuerttemberg Educational Standards compared to the NCTM Standards (NCTM 2000),
  - the adequate and meaningful use of computertools in teaching their topic,
  - the best implementation of open classroom settings teaching their topic in school.
- Each topic group had to deliver the content (texts and graphics) for the web-site and decide on the structure of the web-site regarding their special topic. (Target group were teachers in training or on the job.)
- Each topic group had to give a presentation of their work at a face-to-face-meeting at the end of the seminar.

**Realization**

The average distance between the four participating universities is about 100 miles, so frequent face-to-face meetings were not possible. The seminar started with a face-

to-face-meeting in Ludwigsburg, where most of the students (with the exception of the Weingarten students) were present. After an introduction to the seminar concept and the internet-based learning platform Comvirement (Lerche 2004), the topic groups discussed and decided on a schedule for their work during the coming months. The intention here was primarily to give the students in the topic groups an opportunity to get to know each other.

The 14 weeks of the spring term were divided in three parts. In the *first weeks* they had to familiarize themselves with the learning environment Comvirement and the discussions via the Comvirement discussion board, and had to do the research on their local and mathematical topic, i.e. “Regular Polyhedron - place and importance in the mathematical standards”. Then they had to divide the contents, select the graphics and suggest the linking of their own pages.

Comvirement also facilitates –in addition to the discussion board with attached files- also an upload of files into special folders. The work during this “middle part” was mostly in the local groups and there were several face-to-face meetings among the local groups. The students were supported by the lecturers regarding their “local” topics, the organisational problems and preparation of the web-site.

During the last weeks -and especially days and nights just before the meeting- of the seminar, the web-sites were finished and the presentations for the face-to-face meeting at the University of Weingarten were prepared.

### **Research Questions**

The questions focused on three aspects:

- Do the students transfer their “expert knowledge” to their group-members? Does the students’ knowledge improve concerning the topics of the other local groups and also the other topics?
- Which ways or forms of communication do the students use? How develops the communication between the students of a topic group? What part of the communication is about contents respectively organisation?
- The quality (contents, structure and design) of the created product –the web-site.

### **Results and Observations**

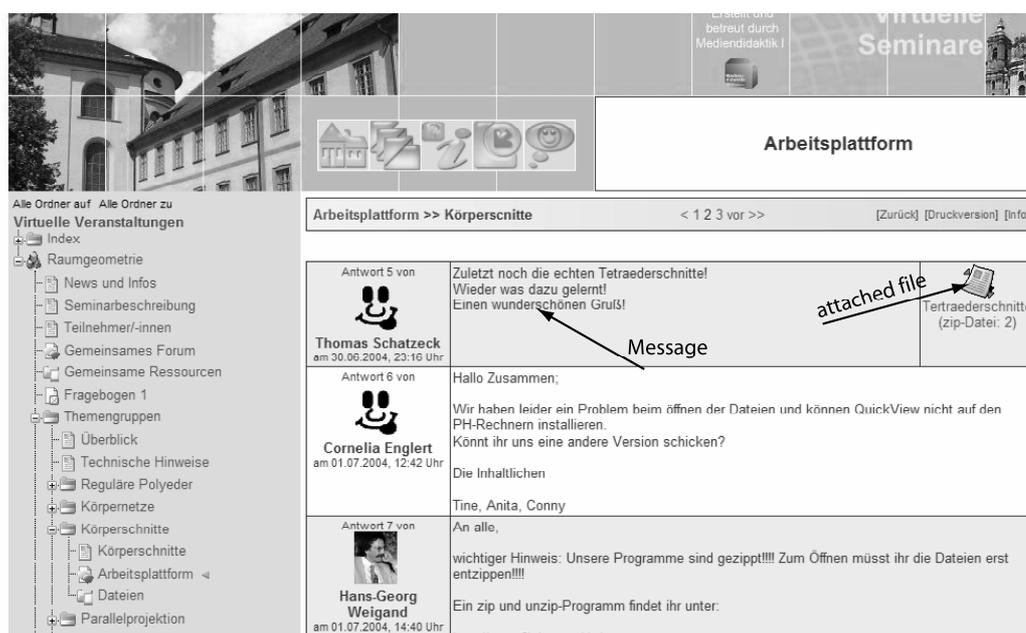
The following results focus on questions dealing with distributed and shared knowledge and the ways of communication in a virtual environment; observations on the use of the technology or organisational aspects of the virtual collaboration will be postponed.

Do the students really share their knowledge while cooperating as experts in completing their tasks? Or is this sharing process only successful in cases where the students really and consciously aim to achieve this goal? To get a first idea whether the sharing process worked, we designed two questionnaires with four items on each

of the five blocks according to the local topics. One was answered at the first face-to-face meeting in April, the second one at the presentation of the seminar work in July 2004. They contained questions like “Which computer software for mathematics education do you know? What were they used for?”, “What kinds of assessment are suitable for open classroom settings in mathematics education?”, “What is the difference between input- and output-oriented standards?”. The result was rather disappointing. There was no measurable increase of students’ knowledge in the other topics. We see one of the reasons for this negative result in the great involvement of the students in the work of their own topic groups. They were too busy working on their own projects that they weren’t interested in the other groups’ work. For upcoming virtual seminars we have to give students more time or even special assignments to follow the inputs of the other groups.

For the second set of our research questions –concerning the ways of communication–, we scanned through the messages including attachments and files in Comvirement. The topic groups used two tools for communication: e-mail and the discussion board of Comvirement. At the beginning of the seminar we emphasized the importance of the discussion board and asked the students to communicate this way. The e-mails were private, and naturally we had no access to this form of communication. Therefore our analysis of the communication only refers to the discussion board.

Comvirement allows setting up different discussion boards (see figure 2). We had a board for each topic group, for each local group and a common board for all participants. The local boards were used only for the information about organisational matters (which are of no interest here). Each topic board contained a discussion board and a file-folder. The discussion board supported the start of a new discussion strand or replying to a posted message. Files could be attached to a message or uploaded in the special folders.



## Screenshot of the platform “Comvirement”

Figure 2

Each message was marked for organizational, technical and personal contents. Although there were mixed contents in some messages, it was nearly always possible to classify each message in a first overview according to these categories. References to literature, URLs and files were also marked. In a second step, the message strands of each topic groups were analyzed with regard to the number of messages, "life-time" and the thematic progression between and within the strands. The attached files as well as the files in the special folders were counted, classified and compared regarding content.

The files in the folder were not analyzed further except for statistical information. The main difference between these two kinds of sharing files seems to be that the students attached drafts to the messages and filed finished documents in the folders.

The students used the following kinds of information to share their “expert knowledge”:

- Explaining explicitly a concept in an entry (i.e. the description of the software ‘poly’ or a list with variations of regular polyhedrons)
- Using catchwords, if they assumed the concept or content was well-known („Will you treat Euler’s Formula for polyhedrons and the theorem of Cauchy?“)
- Giving bibliographical references
- Giving references of URLs (web-addresses)
- Attaching a file
- Giving references to files or folders (sometimes with a reference in a message: “You will find our contents in the ‘Dateien’ folder.”).

The following table gives some data on the kinds of information and number of messages used in the topic groups. It shows for every topic the total sum of messages during the semester and the classification concerning the content of these messages. There are two interesting aspects: First, the development of the actual contents was mainly done in folders, which were sent in zipped versions attached to emails or downloaded from special folders. We didn’t suggest this method but see the big advantage of the joint working on *one* document. The idea “from distributed to shared knowledge” materialized itself in the growth of a common document. Second, the content of the majority of the messages (O-messages) was about organizational aspects, complain about the non-effective work in the group or the non-presence of some members of the group.

	Regular polyhedrons	nets of solids	intersections of solids	Parallel projection	Spherical geometry
Total sum of	116	196	84	136	137

Working Group 9

messages					
Explanatory message	5	2	10	4	3
Bibl. References	6	0	2	0	0
URL references	2	11	1	6	29
Messages with attached files	11	46	34	16	18
“O-messages”	92	137	47	110	87
Files in special folders	68	48	9	148	14

Another interesting observation is the ratio of bibliographical to internet-references. The six bibliographical references in the group “Regular Polyhedrons” are in one entry, the other references are distributed over all groups and topics. Over all, there are more references to the Internet than to printed media and definitely more references than common in traditional term papers.

There were also some questions or requests for special information to members of the topic group (“Could you tell us -on the basis of the standards- why this topic is appropriate right now (Grade 5 to 7)?”), but these messages were usually not answered. This led to a lot of frustration, and the students increased their communication via e-mail or even telephone. The discussion board was obviously not the only way of discussing among the members of the topic groups.

Most of the discussions started from suggestions concerning possible settings in the classroom like “For the lessons we assigned the different competencies to the different phases and learning stations.”

During the seminar the students had obviously serious problems with the organization of the communication or the knowledge transfer. They had to decide where to post the information or question, i.e. to start a new discussion strand or fit it to an existing one. Comvirement only shows the last eight messages; the previous ones are stored in different –but still accessible- web pages. Therefore, students often preferred to start a new discussion strand instead of scanning through all the old entries.

Besides this problem regarding the contents of the discussions, there is a more structural-technological issue: Because of the asynchrony of communication via discussion board, the time difference between posting the question and the answer was sometimes one week or more; in which case the work would stop as well. Some groups tried to solve this problem in setting dates for checking the messages in Comvirement. One group used a private chat or they just used the telephone.

Our third research question concerned the quality of the created web pages.

The structure of the sites followed the system Comvirement, it was quite simple, but clear. The second Weingarten group (Didactics of New Media) designed and edited the web-sites. The results are really quite good. Of course, the creation of professional pages would need much more time and skills. There is a crucial aspect missing in the web-sites. The links mainly refer to external sides and not to pages of the other local groups. There is not significant relationship between the sides of the different local groups. (This may also be a reason for the lack of students' knowledge of the other groups' topics.).

### **Conclusions**

The virtual jigsaw method is definitely a suitable way to achieve the transfer from distributed knowledge to shared knowledge. The products of this seminar -the web-sites- show that the students gained a lot of knowledge in the mathematical topics as well as in the local topics. From the viewpoint of a professional web-site design, there are a lot of deficits -naturally since teacher students are not web-designers. But most of the "mental work" regarding the content of the web-site has been done. The students' work toward this "product" web-site has started and furthers the learning-process. The process the students had to go through -acquiring expert knowledge in their local groups, selecting and formulating the important texts, structuring their own parts and connecting it to other parts of a web-site- is complex enough to pose a real challenge. On the other hand the outsourcing of the creation of the actual web-site and the clearly stated topics and viewpoints made the work manageable during the 14 weeks.

The students -in spite of all the frustrations of having to wait for answers, provided feedback that this kind of seminar is a lot of work but is worth it, because they really learned something. "In the end everything worked out in our group", was a student's comment. Or another one: "... and I think that, all in all, we really got something going".

The ways of communication in a virtual environment are complex and have to be practised as well as reflected to ensure that future teachers will be able to use this kind of virtual cooperation. We will develop further conclusions after our research data have been analysed.

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# CHAINING OPERATIONS TO GET INSIGHT IN EXPRESSIONS AND FUNCTIONS

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**Abstract:** *This paper reports on a teaching experiment that is part of a small research study on the use of Java applets for the learning of algebra. The didactical background of one applet is described, as well as exemplary student behaviour, some observations and results.*

**Keywords:** algebra, technology, educational tools.

## Introduction

Recently, the Freudenthal Institute carried out several ICT development projects. This has resulted in a considerable collection of Java applets and in knowledge on how to use these software tools to enrich learning. Prototypes of the software were tried out in the classroom and improved in close co-operation with teachers. These applets can be found at [www.wisweb.nl](http://www.wisweb.nl).

This paper reports on a small research study using one of the applets called Algebra Arrows. This applet was developed to support insight into the structure of expressions consisting of both numbers and variables, and to foster the learning of the concepts of variable, expression, formula and function.

## Research question and theoretical framework

The research project, entitled ‘Pedagogical opportunities of applets for algebra’, focuses on the following research questions:

1. How does the use of applets offer the students opportunities to develop thinking models and to practise skills in a motivating and varied way?
2. What pedagogical possibilities are offered by the use of applets in mathematics education, and how can the teacher exploit them?

The theoretical framework is based on notions from the didactics of algebra and from theories on tool use. Concerning the didactics of algebra, it is noted that an important difficulty of algebra is the double character of algebraic concepts as both process and object. Sfard speaks about reification, and Gray and Tall invented the idea of procept to indicate this dual focus (Gray & Tall, 1994; Sfard, 1991; Sfard & Linchevski, 1994). This difficulty plays a role in the understanding of symbols and formulas, which is part of so-called symbol sense (Arcavi, 1994; Zorn, 2002). We define symbol sense here as the understanding of the meaning and the structure of algebraic

expressions and formulas (Drijvers, 2003). Our intervention in algebra education aims at improving students' symbol sense.

Concerning the use of technological tools, the theoretical framework consists of the instrumental approach to using technological tools (Drijvers & Gravemeijer, 2004). This approach distinguishes the *artefact*, in our case the applet, from the instrument, which includes both the artefact and the accompanying mental schemes that the student has to develop in order to use the artefact for achieving a goal. The goal in the case of algebra education consists of the development of concepts and skills to solve types of algebra assignments. Following Rabardel (2002) and Verillon and Rabardel (1995), we speak of an *instrument* when there is a meaningful relationship between the artefact and the user for dealing with a certain type of task, which the user has the intention to solve. The tool develops into an instrument through a process of appropriation, which allows the tool to mediate the activity. During this process, the user develops mental *schemes* that organize both the problem solving strategy, the concepts and theories that form the basis of the strategy, and the technical means for using the tool. The instrument, therefore, consists not only of the part of the artefact or tool that is involved but can only exist thanks to the accompanying mental schemes of the user – in our case the student – who knows how to make efficient use of the tool to achieve the intended type of tasks. The instrument involves both the artefact and the mental schemes developed for the given class of tasks.

### **Research methodology and setup**

The methodology of design research is used because of the nature of the research questions, which aim at 'understanding how' instead of 'knowing whether'. Design research aims at shaping innovative instructional sequences, developing an empirically based local instruction theory and more general theoretical knowledge, and has its specific types of justification (Edelson, 2002; Gravemeijer, 1994).

After the design phase, in which the applet 'Algebra Arrows' and the student activities were developed, a teaching experiment was conducted in grade eight. Students worked with the applet for four lessons. During this experiment student activities were recorded by means of a video camera and screen capture software.

Data consist of video recordings of whole class teaching and of selected pairs of students working with the applet, audio registrations of mini-interviews on key assignments, and written answers on worksheets. Data analysis was carried out by qualitative analysis and coding of the data.

We now explain how the applet works and how it can be used, together with the didactical background that played a role in its development.

## The applet Algebra Arrows

### *Sequences in calculation procedures*

On a basic level the applet Algebra Arrows can be used to perform a calculation by making a chain of operations between an input box and an output box. The boxes and operations can be dragged into a working field and are connected to each other by mouse movements. The following is an example of such a chain.

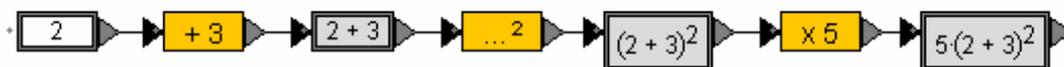


This representation of the calculation  $(2+3)^2 * 5$  or  $5*(2+3)^2$  visualises the calculation procedure, and shows the sequence of performed operations. If students are aware of the structure of the numerical expression, in this case  $5*(2+3)^2$ , and if they know the priority rules for arithmetic operations, they can ‘read’ the sequence of the operations from the expression and vice versa. If students are unable to do this, operations may be performed in the order in which they appear, i.e. from left to right. Tall and Thomas refer to this phenomenon as the “*parsing obstacle*” (Tall & Thomas, 1991). For students suffering from this obstacle, the expression  $2 + 3^2*5$  would fit better to the chain of operations shown above.

The sequence of calculation steps can be made clearer by inserting extra intermediate output boxes into the chain:



The applet also offers the possibility to display numerical expressions instead of



single numbers as results of the operations. This helps to establish the link between the structure of the expression and the calculation sequence:

### *A calculation procedure as an object*

What played an important role in the development of this applet is the idea that constructing an arrow chain to perform a calculation is a means to shift the attention from carrying out a calculation procedure to representing it. The task for the student is to construct the arrow chain representation. The applet then carries out the calculations.

Usually students perceive numerical expressions as tasks to be done. The result is a number. In algebra where expressions can be objects submitted to procedures of a higher order, this perception may be an obstacle. Tall and Thomas speak of the “*lack of closure obstacle*” (Tall & Thomas, 1991). We believe that representing a procedure in the way it is done in Algebra Arrows can be an important step towards perceiving an expression as an object.

Chain representations also foster the view on a calculation process as being independent from the specific numbers in the input box. One can fill in different input numbers in the same arrow chain:



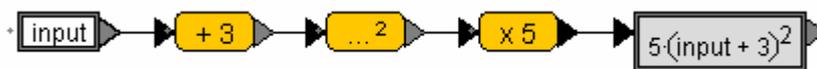
In fact, the above arrow chain represents an entire class of calculations and thus prepares for the concept of formula:  $result = 5 * (input + 3)^2$ .

The chain also involves a representation of the concept of variable by means of the empty place in the input box. This represents a variable as placeholder, an empty place in a calculation in which any arbitrary number can be substituted.

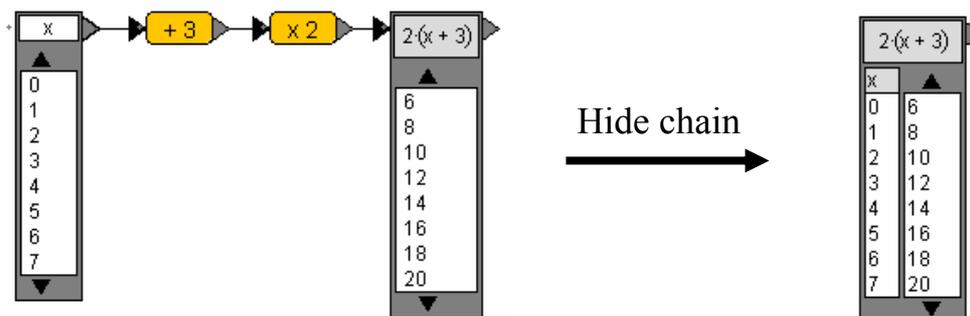
### Expressions and functions

The applet has more options that can support further steps in learning the concepts variable, formula and function. The didactical possibilities of some of these options were used in the teaching experiment of the research study, but these are not addressed in this paper. However, we show them briefly.

- Word variables can be used as input. In that case the result is a word expression:



- If the input box is either empty or contains a word or a character, it is possible to represent this variable input by a single table of numbers. The output box shows a table as well.



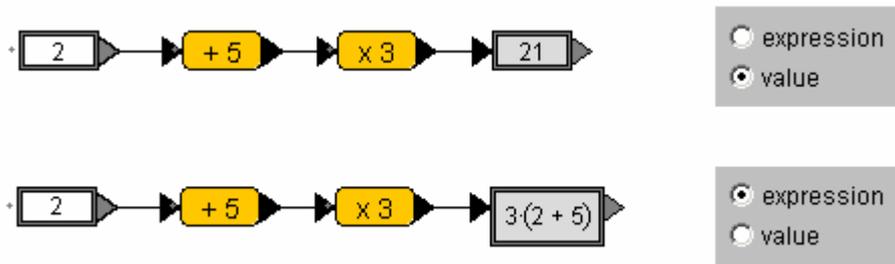
- A more conventional table representation can be made by means of hiding the chain.

With these options several learning activities were designed. For example, students were asked to find a chain of operations that creates a given table. In this case, the applet is an environment in which students can experience the meaning of performing operations on a set of numbers instead of on a single number. This is an important step towards the concept of function. Another activity was to find different operation chains for the same table. The purpose is to give students a meaningful notion of equivalence of expressions.

### A classroom observation

This observation is made in the first part of the teaching experiment, in which the transition is made from performing numerical calculation procedures with the applet to making representations of these procedures. The aim is to foster the object view on expressions and to prepare for the concept of formula.

At the start of the teaching experiment, the teacher demonstrated how the applet Algebra Arrows could be used to perform calculations. After that, the students worked on tasks on finding numerical expressions that represent a given arrow chain.



They could check their answers with the ‘expression’ option of the applet. The next step was to make arrow chains for given calculations, represented by numerical expressions. One of the issues we wanted to investigate in this teaching experiment was the conjecture that working with chain representations would foster the view on a calculation process as something independent from the specific input numbers. We hoped that the analysis of the student work on problem 3 (see below) could provide some evidence for this idea.

**Problem 3 From calculation to arrow chain**

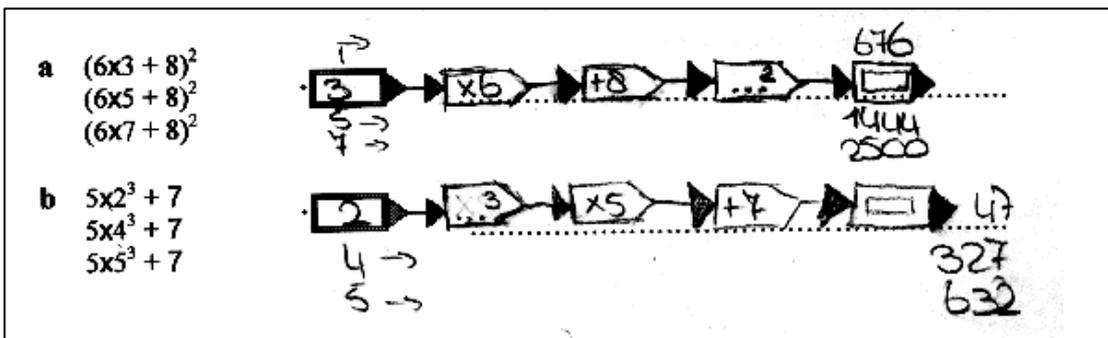
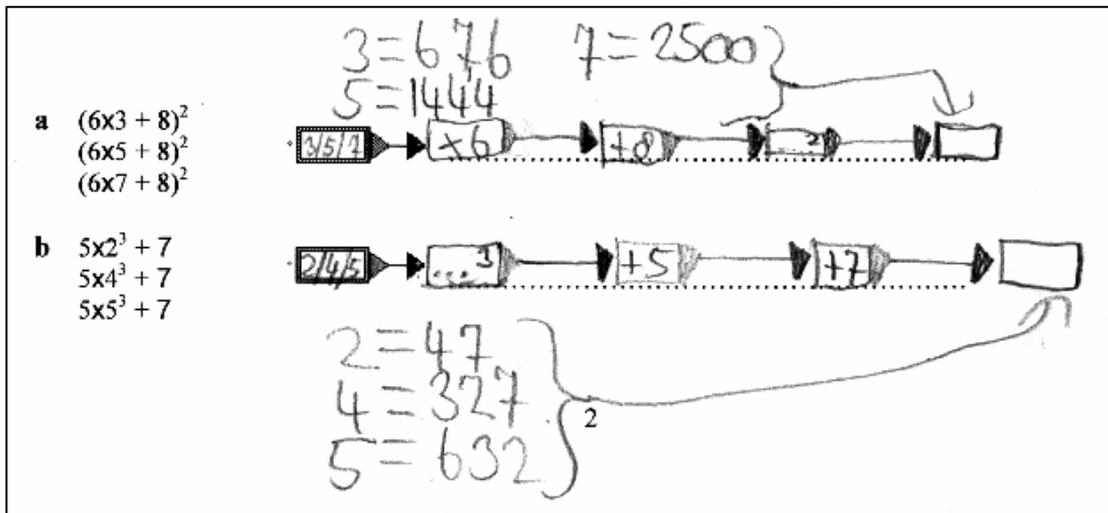
Below you see three calculations every time, that can be made using the same arrow chain (except for the starting number). Make this arrow chain. Use the option ‘expression’ to check it .

**a**  $(6 \times 3 + 8)^2$   
 $(6 \times 5 + 8)^2$   
 $(6 \times 7 + 8)^2$

**b**  $5 \times 2^3 + 7$   
 $5 \times 4^3 + 7$   
 $5 \times 5^3 + 7$

**c**  $7 \times \sqrt{\frac{3+5}{4}}$   
 $7 \times \sqrt{\frac{6+5}{4}}$   
 $7 \times \sqrt{\frac{9+5}{4}}$

In solving problem 3 students should be able to make a conceptual shift to see an arrow chain as a representation of a class of calculations. In the student worksheets we saw some nice examples of students who apparently were able to do this (see the examples below).



Yet, for many students this was rather difficult. They perceived the arrow chain as one calculation, strictly connected to the in- and output numbers in use. We hoped that by using the applet for single numerical calculations the students would discover that once they represented a calculation process, the same representation could be used for other calculations with a similar structure. It was noticed that the students were not yet quite used to the environment on a more basic level. For example some students used their pocket calculator to find the answer and then looked for an arrow chain that would provide the same result.

We recorded an interesting conversation between two students, Marja and Loes.

Loes: *If we start with 3, then you can do the same with 5 and 7.*

Marja: *But I don't understand what they expect us to do.*

## Working Group 9

Loes: *Well, you do 3 times 6, then plus 8 and then all squared, and after that you can do 5 times 6 plus 8 and then squared, and so on.*

Observer: *But now you have to make the arrow chain.*

Loes: *I will first look for the answer.. [uses her calculator]...the result should be 676.*

Marja: *The result of what?*

Loes: *Of the arrow chain.*

Marja: *Why?*

Loes: *Because that is this calculation [points at  $(6 \times 3 + 8)^2$ ].*

Marja: *But the next one has another result.*

Loes: *But we first look at the first one.*

Marja: *I don't understand what this is all about.*

Loes: *We should make the arrow chain.*



Marja: *Then we can make this one [makes this chain]*

Loes: *But that is not the arrow chain.*

Marja: *But this is an arrow chain, too.*

Loes: *But it is not the right arrow chain... [she makes the following chain]*



This example shows that Loes is able to look at the calculation process globally and can make the arrow chain representation, although at first she didn't use the applet to perform the calculation, as we had expected. For Marja, representing the calculations still seems too difficult. This was the case for many students, so the teaching sequence needed some redesign.

In the design of the teaching sequence we made the assumption that the students would perceive the applet as a simple calculator. We also thought that they would soon understand the differences between the applet and the pocket calculator they were used to.

This was not the case, so we try to analyse this problem by looking at the following calculation.

$$5x(2 + 3)^2$$

Using the applet, this calculation should be performed in this way:



To be able to make this chain, the user has to parse the expression, being aware of the priority rules. Many students though did compare the operation boxes with the buttons of their pocket calculator, on which the calculation is performed in this way:

$$5 \times ( 2 + \quad ) x^2 = 125$$

The expression is copied from left to right into an arrowchain. Insight into the structure of the expression is not needed, because the tool is thought to be responsible for parsing and applying the priority rules. As students were used to the pocket calculator for doing their calculations, they expected a similar way of operation from the applet. This was confirmed by the fact that some students asked where they could find brackets. In relation with these observations, it was striking that students could be set on the right track by asking how they should perform the calculation by heart, in which case they were forced to think of the priority rules.

Looking back we should have paid more attention to the differences and the similarities between the new tool and the pocket calculator at the start of the teaching sequence. We might have let the students make several arrow chains for single calculations first. By working on that we could have made the students discover that an arrow chain calculation is compatible with the following keystroke sequence on the pocket calculator:

$$2 + 3 = 5 \quad x^2 = 25 \quad \times 5 = 125$$

In fact, this kind of discovery took place. We heard a student say: "Oh, now I understand. An arrow chain doesn't use priority rules ". But this happened rather late in the teaching sequence.

The experiences give some feedback for the design of the applet. The results of each operation in a chain provide important information about how the applet performs the calculation. Perhaps the applet should be redesigned in a way that these intermediate results are always visible.



This might also prevent other misconceptions that we noticed. For example some students thought that the operations were carried out only if an intermediate output box was used, like the = button on a calculator.

### **Conclusions and discussion**

First, the activities with the applet help the students to focus on the structure of expressions and the related sequence of operations. Especially the applet option for displaying the result of an operation chain as an expression seems to stress the object character of the expression, which was addressed in the description of the theoretical framework of this study. Yet in the beginning the activities were confusing for many students, because they found it difficult to connect the applet activities to calculation activities they were familiar with. Even if they managed to link numerical expressions to operation chains, this seemed to be somewhat isolated from their existing knowledge about calculations, that was strongly connected with the use of the pocket calculator (see above).

This leads us to the second conclusion: the work in the technological environment, in this case the applet, needs to be closely connected to previous work in the traditional paper-and-pencil environment and to the familiar tools such as the pocket calculator, in order to enable transfer and to prevent the development of isolated knowledge and insight. In terms of the instrumental approach, the second component of the theoretical framework, this leads to isolated schemes, which are not interrelated to other knowledge.

In the first part of the teaching sequence, our intention was to make a smooth transition from performing numerical calculation procedures with the applet to making representations of these procedures to foster the object view on expressions.

Although it took more time and it didn't work as smoothly as we had intended, the results in the later parts of the experiment suggest that the activities, together with the individual interactions with the teacher and the reflections during whole class-teaching, finally transformed the applet to an instrument that was used to work on problems and could help in understanding the mathematical objects that were involved.

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# TECHNOLOGY-ASSISTED DISCOVERY OF CONCEPTUAL CONNECTIONS WITHIN THE COGNITIVE NEIGHBORHOOD OF A MATHEMATICAL TOPIC

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**Abstract:** *New technologies provide an efficient tool for broadening the mathematical landscape discovered by the students. Educators should develop compound activities; in order to enhance the epistemic value of the learning process and enlarge the student's knowledge of the internal connections within the cognitive neighborhood of learned topics. Actually three-fold activities are a suitable frame, involving handwork; CAS assisted computations and websurfing.*

**Keywords:** cognitive neighborhood, connections, exploration, technology

## **I. Introduction.**

Undergraduate courses are traditionally organized around specific topics, Calculus is taught separately from Linear Algebra, a course in Geometry has no ties with any other course, with the little exception of Analytic Geometry which uses some algebraic methods, and so on. For example, if a Calculus teacher tries to show algebraic properties of differentiation, through appropriate exercises, students often complain that it belongs to another course, not to the present one. One notable exception is given by courses in ODEs, which are often linked, firmly to Linear Algebra. Combinatorics is nowhere else than in a course on Discrete Mathematics and a course in Probability.

Suppose that the student has an access to sources of knowledge, beyond the adopted textbooks and lecture notes: the learning process becomes much more comprehensive. The additional sources include, among others:

- suitable websites, providing either ready to learn exposition or interactive activities;
- Computer Algebra Systems, via their computational and graphical features, together with pedagogical indications included in their commands (step-by-step execution of commands in Derive 6, indications in the solution process of ODEs in Maple, etc.).

The exploitation of these sources demands teaching and learning skills beyond the acquisition of notions and needed techniques via direct lecturing and practicing under the supervision of educators, the proposed activities aiming to an enlargement

of the students' mathematical world through personal research. According to the student's level, this research is either autonomous or driven by the educator's indications. Nevertheless teacher's intervention will be more rare than in the traditional way.

A few years ago, Cuoco and Goldenberg (1996) wrote:

“New technology poses challenges to mathematics educators. How should the mathematics curriculum change to best make use of this new technology? Often computers are used badly, as a sort of electronic flash card, which does not make good use of the capabilities of either the computer or the learner. However, computers can be used to help students develop mathematical habits of mind and construct mathematical ideas.”

It happens that even this level of use is not achieved, for various reasons. Among them:

- Despite the expanding availability of new tools, a great number of teachers still convey Mathematics in a traditional way, with frontal lectures and technical computations. A certain *pragmatic value* of the teaching is obtained, but the *epistemic value* generally not (Artigue, 2002, page 246). For this situation to change, teacher training has to include technological tools; this issue is not discussed here, but see for example Lingefjärd and Holmquist (2002), Baldin (2002) and Kyriasis and Korres (2002).
- Students often manifest a lack of interest for Mathematics, even when they learn a scientific curriculum, and consider Mathematics as a list of techniques for solving problems, i.e. only the pragmatic value, at a low level, seems to them worth of an effort.

Thereafter, Cuoco and Goldenberg (1996) claimed:

“The mathematics curriculum must be restructured to include activities that allow students to experiment and build models to help explain mathematical ideas and concepts. Technology can be used most effectively to help students gather data, and test, modify, and reject or accept conjectures as they think about these mathematical concepts and experience mathematical research.”

Among the newly available technological tools, we find CAS and the World-wide-web. Therefore new activities are needed, involving their usage, along with “older” techniques, and aimed at the following achievements:

1. Stimulate students' curiosity for interlaced techniques, using more than one of the newly available technologies.
2. Make Mathematics more attractive, and show them as a living field of knowledge by discovering new tracks.
3. Discover links between apparently different domains.
4. Last but not least, traditional libraries offer very small appeal to the average student. Searching the WWW makes him/her fonder of looking for documents relevant to his/her learning domain.

For this last point in particular, a suitable search of the WWW leads to new perspectives on old topics and helps to discover on-going research and interactive mathematical processes. The student is not passive; he/she can influence the teaching process, change the pace of learning, open connections, and discover new horizons.

This construction of a “compound cognitive process” fits Artigue's point of view: the paper and pencil part of the work, together with a possible CAS assisted part provide both efficient mathematical practice and conceptual insight into the mathematics involved in the problem under consideration (Artigue (2002), page 246). A web-search can provide an added value to the solution, and generally makes the learning process more efficient, with a broader perspective on the problem, its solution and what we will call *the cognitive neighborhood* of the pair problem-solution. This neighborhood includes domains in Mathematics related to the problem under consideration; the relation can be either obvious from start or be discovered during the student's autonomous work. The example developed in section II shows combinatorial identities belonging to the cognitive neighborhood of a parametric integral, two topics generally taught in independent courses. Actually we define two kinds of cognitive neighborhoods, one called *restricted* and containing mathematical domains, the other one called *extended* and containing also other items, in particular the instruments associated with the cognitive process at work. The artifacts are exterior to the human being, but the instrumentation process makes them an integral part of the cognitive neighborhood under consideration.

Sequences of definite integrals are often the central topic of exercises leading to induction formulas and/or closed formulas, as exposed by Glaister (2003) and Dana-Picard (2004a). After a finite number of iterations, various properties of the sequence of integrals are discovered, such as a closed expression for the general term of the sequence, which appears often to be of a combinatorial nature. Searching the web reveals various unexpected items; among them:

- a concrete meaning for the combinatorial properties of the given sequence of integrals;
- the history of the mathematical works having produced these combinatorial expressions;
- “real-world” situations with the same mathematical translation.

Such a task is generally built by the teacher, i.e. in this context, the teacher is active and creative; the student reproduces the teacher's working steps. After that, incite the students to search for related material, in particular using the WWW. Links to neighboring mathematical topics can be discovered. The student becomes more autonomous and develops more initiative. Actually, both the educator and the student are creative.

## II. An infinite sequence of integrals.

For any positive integer  $n$ , we define the definite integral

$$I_n = \int_0^{\pi/2} \sin^n x dx .$$

Using integration by parts, the following recurrence relation appears:

$$I_n = \frac{n-1}{n} I_{n-2} .$$

The sequence of integral splits naturally into two distinct subsequences respectively formed by the terms with even indices and by the terms with odd indices. Consider the first subsequence; by a telescoping process, a closed form is obtained for the general term of this subsequence (for details, see (Dana-Picard 2004a)):

$$I_{2p} = \frac{(2p)! \pi}{2^{2p+1} (p!)^2}$$

Now denote the rational coefficient by  $F_p$ , i.e.  $F_p = \frac{(2p)!}{2^{2p+1} (p!)^2}$ .

The first terms of the sequence are

$$F_1 = \frac{1}{4}, F_2 = \frac{3}{16}, F_3 = \frac{5}{32}, F_4 = \frac{35}{256}, F_5 = \frac{63}{512}, F_6 = \frac{231}{2048}, \dots$$

A search in the database named On-Line Encyclopedia of Integer Sequences (2004) provides a combinatorial interpretation for the sequence of numerators, but no interpretation for the sequence of denominators. Look at the sequence of denominators: with a slight modification, it can appear as the sequence of successive powers of 4; the first terms of the sequence ( $F_p$ ) are equal to

$$F_1 = \frac{1}{4}, F_2 = \frac{3}{4^2}, F_3 = \frac{10}{4^3}, F_4 = \frac{35}{4^4}, F_5 = \frac{126}{4^5}, F_6 = \frac{462}{4^6}, \dots$$

A new search in the database leads to the following interpretation of the sequence of numerators: for any positive integer  $p$ ,

$$F_p = \frac{1}{4^p} \binom{2p-1}{p} .$$

Actually, only a few terms at the beginning of the sequence are entered for performing the search; the database provides many other terms which can be compared to the values of  $F_p$  for greater  $p$ , thus obtaining a firm conviction that the interpretation proposed by the database fits the given sequence of integrals. Note that in this specific example, an index translation has to be performed for the above closed formula to be established. Moreover, the database proposes “real-world” interpretations, such as the number of walks of length  $p$  on a square lattice, starting at the origin, staying in the first and second quadrants, or the number of leaves on all ordered trees with  $p+1$  edges, and so on.

In conclusion, the following integral-combinatorial relation has been obtained:

$$\int_0^{\pi/2} \sin^{2p} x dx = \frac{\pi}{4^p} \binom{2p-1}{p} .$$

Another connection can be discovered: the sequence 1, 3, 10, 35, 126, 462, ... is described in the database as a *convolution* from the sequence of *Catalan numbers*. Convolution is a mathematical topic which deserves an effort to learn it. In another direction, real-world meanings for Catalan numbers are available in the On-Line

Encyclopedia of Integer Sequences (2004), in Dickau (1996) and many other sources. A biography of Catalan, with a description of his mathematical work, is to be found in (Mac Tutor 2003). Catalan numbers have also integral interpretations, as parametric definite integrals; one of them is given in the database, another one has been studied by Dana-Picard (2004b). We should mention that for this sequence, no modification of the immediate output has been needed in order to discover the nature of the sequence via the websearch.

### III. Computer assisted activities.

Examples of parametric integrals can be found where computation by hand of the induction relation and of the closed form for the given integrals is beyond the abilities of an average student. Usage of a CAS can help.

First, the general form of the parametric integral  $I_n$  is entered. In most cases, the immediate output is identical to the input and no pattern appears. Then the student substitutes special values for the parameter and computes  $I_n$  for small values of the parameter  $n$ . Suppose that the successive substitutions give answers without a visible general pattern. A web-search, through ad-hoc interactive sites like those mentioned previously, provides sometimes a remedy to this problem, by enabling the student to find either previous work on the same topic, or a pathway into further inquiry. The following frame can be accurate:

- i. Using a CAS, compute  $I_n$ , for  $n$  equal to 0,1,2,Λ ,10.
- ii. Look for a simple pattern in the output.
- iii. Connect to the On-Line Encyclopedia of Integer Sequences (2004); enter the sequence of numbers obtained during the first step (eventually, decompose a sequence of fractions into two distinct sequences, for numerators and denominators). This should provide a conjecture for a general formula for  $I_n$ . In the example above, the web-answer is unique. In other examples, there can be multiple propositions; further exploration is then needed in order to make a decision
- iv. With the CAS, check the conjecture for greater values of the parameter. Of course, such a process does not provide a *proof* of an explicit formula, only some kind of conviction is afforded. This is an example of Trouche's *théorème-en-actes* (Trouche, 2004a).

With some CAS, a pattern appears immediately for the general term of the sequence. In the example above, it involves the Gamma function (this function generalizes the factorial to non integer positive numbers; see (Thomas's Calculus, 2002) page 605; it is generally taught only in an advanced course. The output is:

$$\int_0^{\pi/2} \sin^n x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{1+n}{2}\right)}{n \Gamma\left(\frac{n}{2}\right)}$$

In such a case, the hope to bypass the lack of knowledge using the CAS is deceived: the student replaced his/her problem by a problem still worse from his/her point of view: he/she cannot understand the actual meaning of the output, the CAS is used as a blackbox and the pedagogical aspect of the work is lost. It is still possible to make substitutions into the obtained “strange” formula, for the sequence of numbers to appear and to be studied as described previously, but no conceptual understanding is afforded from a study of the general formula on display. Dana-Picard and Steiner (2004) point out the fact that the usage of such “high level” commands (here “high level function” could be more appropriate) does not help to building conceptual understanding. Or maybe the educator can catch this opportunity to reverse the trend, by giving a definition of the Gamma function and showing its first properties; this is part of the educator’s building of the theoretical discourse accompanying the technique. The “bad problem” becomes a motivating example for further discovery.

Nevertheless, the educator must pay attention to the danger inherent to the multiplication of the goals of an activity: maybe none of them is totally achieved. Moreover, the student is sometimes misled and thinks that a lack of conceptual understanding can be bypassed by multiplying technicalities with the computer. The usage of commands whose output involves the Gamma function should be postponed to a later task, after the present one has been fully performed. In other words, the cognitive neighborhood of a given topic can be too large for the student to be able to find reasonable pathways for an exhaustive exploration. The same remark is valid for the convolution mentioned with respect to the connection between our example and Catalan numbers. The educator must make the appropriate choices: which topic is at a “reasonable distance” within the cognitive neighborhood from the main topic under consideration, and which one is too far away at that time ?

At this point, we should emphasize the fact, already mentioned in the previous paragraph, that the theoretical discourse for instrumented techniques (Artigue, 2002) is intimately connected to the choice of the CAS: in this example, shall we explain the Gamma function (Mathematica’s output uses the Gamma function) or not (Derive’s display does not include the Gamma function)? Moreover, in the proposed activity model, the discourse has to include a subdiscourse aimed to master ways of web searching to broaden mathematical horizons, and not getting lost in this huge amount of more or less relevant sources of information.

#### **IV. Three-fold activities and exploration of a cognitive neighborhood.**

##### **1. Diagram presentations for the cognitive neighborhood.**

The frame of the author's courses is fixed by the institution where he teaches; the added value of extra tasks, not officially present in the syllabus but given as pilots, is received by most students as a “plus” in their education. After some adaptation process, their reaction is very positive and, for example, the best results of their

explorations are dispatched among their peers, generally via the electronic forum of the class. Moreover, some students use another CAS than the teacher; the comparison between the methods and their results is very enriching. This takes place generally in “private” conversations, not during plenary lectures or exercises sessions.

The solution of an old problem with new techniques has always a great mathematical value and a pedagogical interest. Therefore the introduction into the curriculum of compound activities, including traditional ways of doing mathematics together with the most up-to-date technologies, is important. As already claimed, the widening of the mathematical landscape provided by new technological tools reinforces the students' will for a deeper understanding of what they learn and stimulates them to further learning. Some time ago, a former student, whose name is Dor, came to the author's office, asking for extra mathematical material on a certain topic and for personal help. He said: “I learnt this material, but I still want more profound insight into what these objects are”. In Dor's words, this means: it is not sufficient in my eyes to know only what has been taught, I wish to understand more profoundly the nature of the mathematical objects under study, and the connections between them. In our words, Dor wishes to explore the cognitive neighborhood of his topic.

Consider the cognitive neighborhood of a given mathematical topic as included in a kind of space, which could be called “Mathematical Knowledge”. The topics within the neighborhood are related by “connections”, which can be represented by a diagram, as in Figure 1. Actually, even if we represent mathematical notions as vertices, we shall not represent the connections as edges, thus not obtaining a graph in the ordinary sense. Because of their non-uniqueness, the connections should rather be displayed as “clouds”.

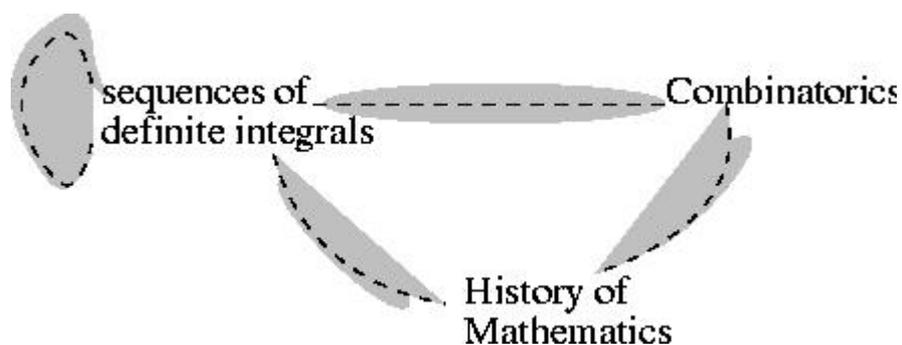


Figure 1: Restricted cognitive neighborhood.

In this diagram, a line crosses the clouds, to show the fact that a specific connection has been established during the proposed activity; other connections can exist, and actually do exist. The convolution mentioned in the example is an example of (unexpected?) connection among sequences of definite integrals; therefore we added a connecting cloud from this field to itself. The field “History of Mathematics” represents here all the general knowledge surrounding the Mathematics under study, such as whom is Catalan, how Catalan numbers appeared for the first time, and so on.

In fact, the cognitive process at work when learning Mathematics is not only composed of mathematical topics. In the three-fold activities described above, *tools* are used and the student's mathematical thinking moves in two reversed directions, performing an *instrumentalisation process*, together with an *instrumentation process*, as shown by Trouche (2004a). Figure 2 presents a diagram for what we will call an *extended cognitive neighborhood*, where not only the connections between mathematical notions and topics are on display, but also the artefacts which are to be used. The lower level in the diagram lies within the "Mathematical Knowledge" space; it appeared in section II that some of the internal connections are discovered and explored with the help of the diagram's upper level techniques, and would have been quite impossible to discover without, in particular, the websearch (remember that the sequences' database provided a lot of information from which even internal connections between different integrals were made possible).

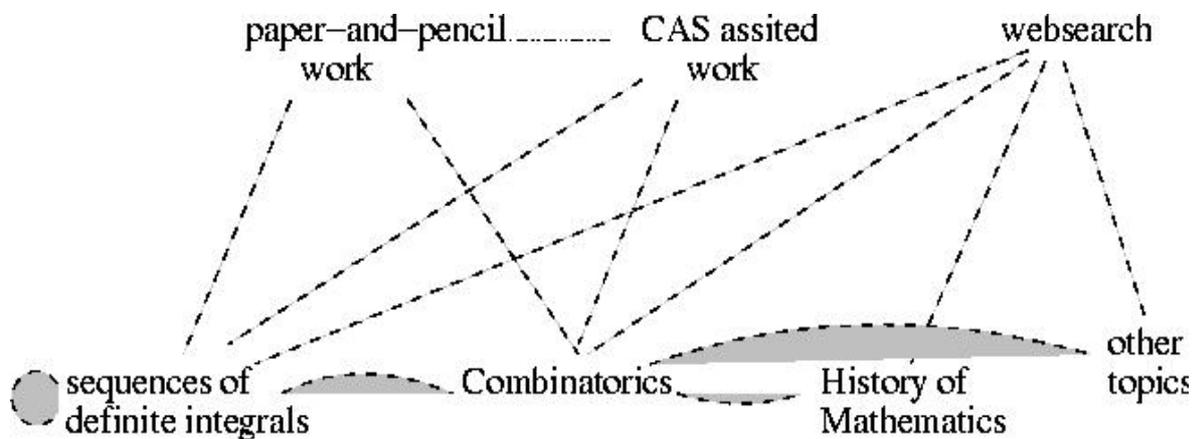


Figure 2: Extended cognitive neighborhood.

The actual construction of the bridges between the various domains in the restricted cognitive neighborhood is an individual issue. This construction is a dynamical process, not only for the instrumental genesis, but for human communication: sharing working experiences with classmates, with the educator(s), etc., can lead to mutualize all the components of the triple work-discovery-results. Each participant contributes his/her own experiences and finally, this mutualization enriches still more each one's extended cognitive neighborhood.

## 2. Advantages and disadvantages of compound activities.

Another advantage of this teaching-learning process is the student's self-teaching, at least part of the time. The learning process is composed of a synchronous part (as in a traditional process) and an asynchronous part (mostly the exploration of the problem's cognitive neighborhood), the importance of this asynchronous work being emphasized.

Finally, the author wishes to thank the referee for the following remark: the world wide web is not a "tool" by itself but gives a overwhelming amount of very different

tools (including CAS) and information. It can help to explore the cognitive neighborhood and offers a more profound understanding of the mathematical objects, but there is also a danger that a lot of superficial information is collected and students could lose the focus on the mathematical topic. An educator will lead the students' exploration according to a general scheme (this is a part of the instrumental orchestration described in (Trouche 2004b)), but the individual appropriation of the "compound tool" can be very different from one "explorator" to another, each student building his/her own assimilation scheme, transforming this tool collection into a "system of instruments" by his/her own instrumental genesis. Moreover the personal aspect of the possible explorations yield a shift in the teacher's role: he/she is not an ex-cathedra lecturer anymore, instead he/she involved in discussion of the students' discoveries and remarks; see (Monaghan 2004). Of course, we do not mean that the teacher becomes simply a "facilitator". Actually the students' discoveries can reveal new horizons to their teacher, both in mathematical matters and on pedagogical issues.

For such reasons (and others), too big an enthusiasm for this kind of mixed learning process must yet be tempered. The example of a compound mathematical activity that we described here shows an application of Lagrange's (2000, page 27) claim. The coordination of new techniques with the traditional ones will not change in a miraculous way the learning process. With new technological tools, some results will be obtained more quickly, but such a compound activity demands profound reflection from the educator, and "demands from the students time and efforts for their passage towards theory". "The difficulties encountered when implementing new praxeologies should not be underestimated".

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# TOOL USE IN TRIGONOMETRY IN TWO COUNTRIES

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**Abstract:** *This paper examines tool use in senior high school trigonometry lessons in England and in Turkey. It describes trigonometry in the two countries and student performance in several tests; Turkish students did better on symbolic tests and English students did better on ‘real life’ problem solving. Tools and related techniques are considered. ‘Cognitive functioning and tool use’ and ‘the ecology of tool use and related techniques in educational systems’ are discussed. Educational implications are drawn which include locating ethical questions in curricula change and possible ramifications of changing tool use in curricula change.*

**Keywords:** high school, trigonometry, comparative education, tool use.

## Introduction

This paper focuses on tool use in trigonometry lessons in two countries: England (UK) and Turkey (TR). It draws on data from a comparative study (Delice, 2003) of senior high school (16-18 year old students). This study examined curricula, assessment, classroom practices and aspects of student performance; in this paper we focus on tool use. There follows five sections: a brief overview of the study; selected data on student performance; a comparison of tool use in trigonometry lessons with comments on ‘techniques’ related to tools; a discussion which examines ‘cognitive functioning and tool use’ and the ‘ecology of tool use and techniques’; educational implications. A subsidiary theme of this paper concerns ethical issues in curriculum change regarding tool use, e.g. that ‘this tool’ should (not) be used.

## A study of trigonometry in english and turkish high schools

The study had two foci: (i) Student performance: finding unknown lengths/angles from diagrams, ‘simplification’ of expressions and solving word problems (ii) The contexts of learning: curriculum, assessment, classroom practice and teachers’ attitudes. Education research literature on the teaching and learning of trigonometry is virtually non-existent so we used an exploratory multiple case study methodology (Yin, 1998). Our approach could be called interpretative with a naturalistic mode of enquiry. We employed wide variety of data collection/analysis instruments. With regard to student performance four written tests were given to approximately 60 students in each country (from one UK school and one TR schools): algebra, simplification of trigonometric expressions, finding unknown quantities in right-angled triangles and solving word problems. Interviews and concurrent verbal protocols were conducted with a subset of the student sample to explore reasoning behind the answers in the tests. With regard to the contexts of learning data

collection/analysis included document analysis (curricula, examinations, textbooks), questionnaires, classroom observations and interviews with teachers.

At one level the research showed that ‘you get what you teach’, i.e. trigonometry in TR privileges algebraic aspects of trigonometry over ‘real-world’<sup>1</sup> problem solving’ whilst the opposite is, by and large, the case in UK and TR students, compared to UK students, did well in algebraic aspects of trigonometry and less well in solving trigonometric word problems. In the discussion section we argue that ‘you get what you teach’ is quite a complex affair. We now provide an overview of the curricula, textbooks, assessment and teaching of trigonometry in the two countries (space restrictions mean that this is a brief overview).

Both countries introduce trigonometry to 14-15 year old students and return to the topic when students are 16-17 years old. The focus of the study was the later stage but we make brief comment on the early stage. In the early stage the UK curriculum provides considerably more content: bearings; use of trigonometry in 2 and 3-D contexts (‘real-world’ problem solving); sine and cosine rules; graphs of functions for angles of any size. The TR but not the UK curriculum stressed surd forms of trigonometric ratios for  $30^{\circ}$ ,  $45^{\circ}$  and  $60^{\circ}$  triangles. Both curricula include solving for unknown lengths or angles in right-angled triangles and introduce trigonometric ratios through ratios of sides of right-angled triangles. The distinct UK foci on the use trigonometry in 2 and 3-D contexts and functions and TR emphasis on  $30^{\circ}$ ,  $45^{\circ}$  and  $60^{\circ}$  triangles continues into the later stage. In this stage the UK curriculum continues to focus on right-angled triangles whereas the TR focuses on the unit circle (which is surprising given the UK focus on functions). The TR curriculum includes substantially more theorems, formulae and identities than the UK. Trigonometric identities/formulae emphasised in the TR but not in the UK curriculum include writing trigonometric functions in terms of each other, half angle identities, sum and difference formulae, product formulae, writing the expression of  $1+\sin u$ ,  $1+\cos u$ ,  $1+\tan u$ ,  $1+\cot u$  in the form of products, writing the expressions of  $\sin 3a$ ,  $\cos 3a$ ,  $\tan 3a$ ,  $\cot 3a$  in terms of  $\sin a$ ,  $\cos a$ ,  $\tan a$ ,  $\cot a$  respectively (and same replacing ‘ $3a$ ’ by ‘ $a$ ’ and ‘ $a$ ’ by ‘ $a/2$ ’). The UK but not the TR curriculum includes differentiating and integrating trigonometric functions even though calculus is studied at the later stage in the TR. As calculus introduces new issues for learning, e.g. students’ conceptions of limits and infinity, we do not consider calculus in this paper.

Observed trigonometry lessons in both countries made considerable use of textbooks and textbook content mapped exactly onto curricula content. A notable difference between the two countries’ textbooks in terms of the questions and exercises set was that TR books ‘privileged’ (Wertsch, 1991, p.124) pure mathematics, e.g. surd forms, whilst UK books dealt with both pure mathematics and applications of trigonometry. We regard this as very important because this difference is not immediate from reading curricula documentation, e.g. ‘solve for unknown lengths and angles in right-

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<sup>1</sup> We put ‘real world’ in inverted commas to emphasise a certain scepticism that what is called ‘real world’ is indeed ‘real world’.

angled triangles' is common to both countries' curricula documents but it evidently means very different things in practice. Other textbook differences included: expected use of a calculator in UK but not TR (and vice versa for trigonometric tables); the angles used (often decimals in the UK but invariably multiples of  $15^\circ$  in TR textbooks); the regular use of cotangent in TR.

Both countries have terminal national high-stakes examinations. In the UK this is called *Advanced level mathematics* (A-level) and is independent of universities. A-level mathematics is modular and modules include pure mathematics, applied mathematics, statistics and discrete mathematics. In the TR this is the *University Entrance Examination* (UEE) and mathematics is not subdivided into modules (it is invariably pure mathematics). Questions on the UEE are all multiple choice questions whereas there are no multiple choice questions in the A-level examination. Calculators were allowed in A-level examinations but not in the UEE. There were considerably fewer trigonometry questions in the UEE compared to A-level (which is surprising given the greater trigonometric content of the TR curriculum). Questions in the UEE (but not in the A-level examination) invariably involved surd forms.

Observed UK lessons had 15-20 students whereas TR lessons had 38-45 students. UK classrooms were 'dedicated' mathematics classrooms whereas TR classrooms were general teaching rooms. UK but not TR classrooms were equipped with posters, sets of calculators, a computer and an overhead projector. Lessons in both countries consisted of teacher explanation and student practice. Observed TR trigonometry lessons centred on simplification, solving equations and inequalities and solving geometric problems. This was a feature of observed UK trigonometry lessons too but there was considerable emphasis on 'real world' problems. TR teachers were observed to encourage students to employ a number of ways to solve a problem (e.g. using different identities) whereas UK teachers provided students with a fixed set of steps to solve a problem. This difference in encouraging different solution strategies, was particularly noted with regard to drawing diagrams for word problems, i.e. UK teachers directed students' diagram drawing actions whereas little direction was forthcoming from TR teachers.

### Student performance

It is important for the reader to appreciate the significant differences in the performance of students' from each country. Space, however, is restricted. We thus sketch global performance and provide one illustration from the algebra, trigonometric simplification and 'real world' context tests.

The algebra test, 16 questions, was designed as a base-line test of students' algebraic facility (since algebraic fluency is useful in manipulating trigonometric expressions). The majority of questions asked students to 'simplify' an expression, the others required a solution. TR students did better than UK students: 71% correct answers compared to 44%. UK students experienced particular difficulties with algebraic fractions, often cancelling inappropriately, e.g.

$$\frac{a^3b - ab^3}{a^3b + 2a^2b^2 + ab^3} = \frac{-1}{2a^2b^2}$$

The trigonometric simplification test, 16 questions, asked students to ‘simplify’<sup>2</sup> expressions. All students found this difficult but TR students did better than UK students: 33% correct answers compared to 18%<sup>3</sup>. Students from both countries experienced difficulties with expressions involving exponents. In follow up interviews some UK but no TR students transformed trigonometric expressions into algebraic expressions, e.g. replace  $\sin a$  by  $x$ , and then converted the answer back into a trigonometric expression. This did not help, however, when the algebra was incorrect, e.g.  $(\sin^2 x)^2 - (\cos^2 x)^2 = (\sin^2 x - \cos^2 x)^2$ .

The ‘real world’ problem test presented 6 word problems. Students were expected to draw a diagram but were not instructed to do this. UK students did better than TR students: 63% correct answers compared to 46%. TR students had difficulties and had particular problems dealing with 3-D representations and diagrams where more than one right-angled triangle was required. One question, for example, asked students to find the distance between two people seen from the top of a 15m tower, one due west at an angle of depression of  $31^\circ$ , the other due south at an angle of depression of  $17^\circ$ . 32% of UK but only 5% of TR students got this correct. In interviews TR students stated that the difficulty was in producing the correct diagram, something they were not used to doing in class.

### Tools, techniques and activities

It was clear from observations of trigonometry lessons that, with regard to the tools and techniques used and activities undertaken, there were distinct national differences<sup>4</sup> that transcended individual differences which may be put down to, say, teacher characteristics. We describe similarities and differences below but first deal briefly with what we mean by tools, techniques and classroom activities and their importance in the teaching and learning of trigonometry.

A tool is a material artefact which has a purpose: to perform a task or set of tasks (though a tool may be appropriated for a purpose not originally intended, e.g. using a calculator as a straight edge). We stress the materiality of tools as some people, e.g. Douady (1991), speak of ‘conceptual tools’ and socio-cultural education literature, e.g. Daniels (2001), is replete with the term ‘psychological tool’. Whilst we appreciate both the analogy these authors are making and the importance of ‘cultural tools’ (mediational means (Wertsch, 1998)) in mathematical development we feel, like Trouche (2003), that it is important to distinguish between the physical and

<sup>2</sup> ‘simplification’ is, in our opinion, a misnomer as the resulting expression is rarely more simple!

<sup>3</sup> The situation was not quite as ‘bad’ as these figures suggest. Coding for both the algebra and trigonometric simplifications test had four categories: correct, incorrect, not attempted and partially correct. Partially correct answers were those where the student approached the question correctly but stopped short of the expected simplification. Partially correct answers were more common in trigonometric simplifications (21% [UK], 24% [TR]) than in algebra (16% [UK], 7% [TR]).

<sup>4</sup> We respect the people and teachers of both countries, recognise that there is diversity within national practices and have no wish to typecast people or institutions. Nevertheless, there were, in our observations, significant international similarities and international differences that it is legitimate to speak of “distinct national differences”.

psychological components of human tool use in mathematics. Although we stress the materiality of tools our concept of tool remains wide, e.g. we consider an algorithm as a tool. The materiality of an algorithm is less immediate than the materiality of a calculator but it nevertheless exists in the materiality of its spoken or written form (without a sign form it cannot exist). The fact that any algorithm can be programmed by a computer attests to its materiality. Tools used in trigonometry classes include calculators, trigonometric tables, formula sheets and algorithms. The tools used in doing mathematics impact on the mathematics that is done, e.g.

$$\text{Find } x, \text{ between } 0^\circ \text{ and } 90^\circ, \text{ such that } \cos(x) = (\sqrt{3} + 1)\sqrt{2}/4$$

can be solved with a number of different tools but each tool carries with it a distinct mathematical solution.

‘Technique’ in UK-mathematics-education-speak often refers to manipulation, e.g. expanding  $\cos(\alpha-\beta)$ , but Artigue’s (2002) considerations capture our conception,

“technique” has to be given a wider meaning than is usual in educational discourse. A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work. ...[they have] *pragmatic value* ... focusing on their productive potential [and] an *epistemic value*, as they contribute to the understanding .. (ibid., 248)

The pragmatic value of the  $\cos(\alpha-\beta)$  expansion is that we can use this expansion for any values of  $\alpha$  and  $\beta$ . The epistemic value may reside in interpreting the identity geometrically. ‘Seeing’  $(\sqrt{3} + 1)\sqrt{2}/4$  as an expansion of  $\cos(45^\circ-30^\circ)$  involves both pragmatic and epistemic values.

NB The reader maybe wondering whether  $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$  is, to our way of thinking, a tool or a technique<sup>5</sup>. Our answer is that it is, as it stands, a tool. A tool without an agent is neutral; this tool, on its own, simply performs a sign transformation. This tool may be used in a technique but the technique includes mediated reasoning in context.

Activities consist of tasks and motives (motives are essential as tasks are literally meaningless without motives and the same task carried out for different motives represents two distinct activities). Activities in school mathematics classrooms have, in our observations, ‘cycles’ such as those observed by Magajna (2001, p.73) in Slovenia “The observed task structure consists of several nested cycles. The most prominent level is the cycle of exercises and exercise-like pieces of theoretical explanations.” We now proceed to a description of tool use in trigonometry lessons in the two countries.

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<sup>5</sup> Artigue (2002) and, especially, Trouche (2003) distinguish between ‘cognitive structures (schemes) and cultural systems (techniques)’. Whilst we consider this an important distinction we do not consider schemes in this paper because: we see certain problems in the scheme/technique distinction (see Monaghan (2003) for summary details); a consideration of these problems would take this paper beyond 10 pages; we feel we can say what we wish to say about tool use in trigonometry in this paper without reference to schemes.

Apart from pens, rulers and compasses, which were common to both countries, considerable differences in tool use in the two countries was noted. In the UK, but not in TR, calculators and formulae sheets had widespread use. Trigonometric tables were not used at all in UK and their use in TR was marginal. Calculators are commonplace in UK classrooms but were not seen to be used in TR classrooms. Many aspects of classroom activity interrelate here in a dialectical, not a causal, way. In the UK angles of any magnitude are used in tasks. In the TR, however, the focus is almost exclusively on angles which are multiples of  $15^\circ$  and subsequent surd forms. Calculators have sine, cosine and tangent keys and these functions are privileged in the UK system.

Tools are socially invested with power and authority and are imbued with ‘cognitive values’ (Wertsch, 1998). In the UK curricula, syllabi, textbooks and high-stakes examinations stress calculator use and non-use. In TR only calculator non-use is stressed. All the UK teachers interviewed could be described as having a positive attitude to students’ use of calculators for checking answers, computations and solving problems. TR teachers interviewed, however, stated that calculators were costly and/or made students lazy.

In the UK formulae sheets are used extensively in trigonometry classrooms and in examinations. They are not used at all in TR. This impacts on classroom activity. There were a number of occasions in UK classes where teachers and students searched formulae sheets to find an appropriate identity to simplify a trigonometric expression and all observed individual UK student seatwork directed at simplifying trigonometric expressions involved examining formulae sheets. This obviously did not happen in TR classrooms where there was considerable emphasis on deriving trigonometric identities. Interestingly, when the subject was broached, everyone interviewed (UK/TR, teacher/student) reacted negatively to the word ‘memorisation’ (‘ezber’ in Turkish) but stated that they endorsed remembering key identities. The effect of using or not using formulae sheets on learning is a matter of debate<sup>6</sup> but we do not take sides on this debate; our point here is simply that their use, or not, impacts on the mathematics students do.

Trigonometric tables are not used in the UK because calculators render them obsolete. Their use in the TR is a curriculum objective. However, their observed use in TR classroom was only noted when the focus of the lesson was on how to use trigonometric tables. TR teachers stated that since angles other than the special ones, e.g.  $30^\circ$ , are not used in the UEE, they do not work with a wide range of angles and thus do not use tables other than teaching their students how to use them – an interesting case of the encapsulation of school mathematics (Engeström, 1991) and the pedagogic irrelevance, in one school mathematics institution, of an historically important mathematical tool.

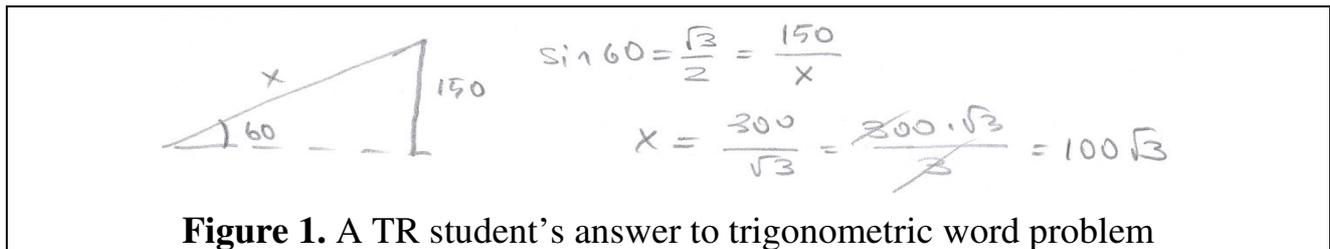
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<sup>6</sup> See <http://www.qca.org.uk/ages14-19/subjects/5660.html> and §6.4(h) of the downloadable pdf.

We now comment on tools, techniques and tasks. An important ‘player’ in this interlocked triangle is the calculator (its presence or absence). UK students used their calculator in all observed lessons. Calculators generally use decimal notation and do not emphasise fractional forms. For example, a UK student attempting the question

$$\text{If } \tan A = 3/4, \text{ find } \tan 2A$$

used his calculator to find  $\tan^{-1}0.75$ , multiplied this by 2 and then found the tangent of this number. He did not consult his formula sheet, so we assume that he did not consider using the identity  $\tan 2\theta = 2 \tan \theta / (1 - \tan^2 \theta)$ . Contrast this with a TR student’s answer to a word problem (fig. 1) where the solution is kept in surd form.



We consider both solutions ‘natural’ given the different classroom practices the students were used to. We explore this further in the following section.

## Discussion

### *Cognitive functioning and tool use*

There are a number of ways that the differing performances of TR and UK students can be interpreted: TR students are ‘better’ at algebra and at trigonometric simplification; UK students are ‘better’ at ‘real world’ applications of trigonometry; ‘you get what you teach’, i.e. if teaching privileges algebra (or applications), then students do better at algebra (or at applications). The first two interpretations are correct (for our sample as a whole) but are naïve and beg the question as to why certain students are ‘better’ at certain things. The third interpretation is also, in our opinion, correct but it is a surface explanation; with due respect to the groundbreaking work of Kendal (2001 – see also Kendal & Stacey, 1999) on teacher privileging, teacher privileging is only part of an activity system. As Lemke (1997) notes, people function:

...in micro-ecologies, material environments endowed with cultural meanings; acting and being acted on directly or with the mediation of physical-cultural tools and cultural-material systems of words, signs, and other symbolic values. In these activities, “things” contribute to solutions every bit as much as “minds” do; information and meaning is coded into configurations of objects, material constraints, and possible environmental options, as well as in verbal routines and formulas or “mental” operations. (ibid, p.38)

We consider wider ‘ecology’ issues in the next subsection, for now we focus on tool use and cognition and consider further the work of the two students immediately above (UK student finding  $\tan 2A$  and TR student solving a word problem). The question the UK student answered did not specify how to obtain  $\tan 2A$ . It was clear by his actions that he was very familiar with his scientific calculator as the actions he

performed (keying in  $\tan^{-1}0.75$ , multiplying this by 2 and finding the tangent of the result) were performed quickly and without a mistake. It was, in our opinion, an efficient technique using this tool. It is worth noting that many mathematics educators in each of our countries (see, for example, (LMS, 1995)) would see this as an ‘inferior’ solution to a solution using, say, the identity  $\tan 2\theta = 2 \tan \theta / (1 - \tan^2 \theta)$  and would further claim that the decimal answer does not possess the mathematical aesthetic of the fractional answer,  $24/7$ , likely to be obtained from using this identity. Whilst respecting the rights of such people to their opinions it is, we feel, important for education researchers to adopt a neutral position on tool use and point out that their arguments are ethical, not mathematical, arguments.

The TR student’s answer is also, in our opinion, an efficient solution. This student studied mathematics in classes where a calculator was not used but where  $\sin 60^\circ = \sqrt{3}/2$  was a trigonometric tool (yes, we mean a ‘tool’ and not a ‘technique’; the technique, the solution strategy in this case, employed this tool). This trigonometric tool was something he was very familiar with from mathematics lessons (where angles in trigonometric problems are always multiples of  $15^\circ$  and usually either  $30^\circ$ ,  $45^\circ$  or  $60^\circ$ ). Again this student’s answer could be criticised by some mathematics educators (those disposed towards ‘authentic assessment’) but, again, we would counter that this is an ethical, not a mathematical, criticism.

Does the work of these two students indicate differences in cognitive functioning? Yes, tool use transforms mathematical reasoning. In the case of the  $\tan 2A$  question, in particular, the calculator-based and identity-based techniques used to obtain an answer are quite distinct; we do not feel, however, that we can say more than this, in particular that any ascription to higher/lower cognitive functioning can be made. Tool use, however, does not exist in a vacuum and it is to the wider environment that tools reside in that we now turn.

### ***The ecology of tool use, techniques and institutional constraints/enablers***

Our immersion in this comparative study generated in us a sense of awe as to how trigonometric tool use and techniques fitted together within educational systems which were, somehow, ‘complete and complementary’. We first attempt to show the reader how we see these activity systems and then draw implications. The systems can be viewed from various perspectives; we choose to present it from the perspective of the student.

The student in a trigonometry lesson has to do certain things (understand something, complete an exercise). What they are to understand or perform is culturally and historically presented to them and exists within several wider communities of practice than their classroom (the school, the national education system). The things they are to do (specified in a curriculum document which is itself a product of an historical development) in the TR and in the UK differ. They are to do these things with culturally sanctioned tools and associated techniques. There is a dialectical, not a causal, relationship between the tools, techniques, tasks and the curriculum of each

country and the evolved ‘ecology’ of each system is complementary: the tools/techniques ‘fit’ the tasks/curriculum and *vice versa* (this does not mean that there are no internal contradictions within each system). From his/her position inside the system these tools may appear ‘natural’ but there is no question that they are really ‘natural’, they are historically situated artefacts. The student is not a lone agent; the teacher directs the student’s activity and actions, and the ‘voice’ of others, e.g. curriculum designers, is present in the teacher’s voice. All are subject to various rules of behaviour; these may be generated from outside the system, e.g. ‘thou shalt not use calculators’, or within the system, e.g. scanning formulae sheets to find an appropriate identity. Tool use plays an important part in characterising differences in these two systems but it does not, on its own, determine these differences (it acts on, and is acted on by, each system)<sup>7</sup>.

### **Educational implications**

An educational implication of this consideration of two trigonometry systems is that trigonometry in the TR and trigonometry in the UK are related but distinct trigonometries: classroom activities differ; there are considerable differences in the tools and techniques used; mathematical actions related to tool use differ; and the rules of behaviours regarding activities and tool use differ. International comparisons of student performance, e.g. Trends in International Mathematics and Science Study (<http://nces.ed.gov/timss/highlights.asp>), can be used by ministries of education to encourage greater emphasis on examination performance. The fact that a curriculum area in two countries are very different is further evidence for educators that argue that international comparisons cannot easily be made.

Regarding ethical issues there is, to us, no question that one system is ‘better’ than the other, they are simply different. We believe, though this would be the subject of a different paper, it is important that ethical issues of curricula change are discussed. We believe that comparative studies like this, with an activity systems focus, can help to locate where ethical questions lie. Activity systems are not static entities; they develop, sometimes through negotiation, sometimes through open dispute between participants with differing perspectives. There are some in the UK, e.g. LMS (1995), who wish for a system more like the TR system and some in Turkey (Altun<sup>8</sup>, 2002) who want more ‘authentic’ classroom activities. Some of those in the UK who wrote the LMS (1995) document would further wish and remove the calculator as a tool in advanced trigonometry<sup>9</sup> (Gardiner, 1995<sub>a, b</sub>). From an activity systems point of view this would be a severe disruption with potentially significant and difficult to foresee consequences; changing the tool would impact, at least, on techniques, tasks and rules

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<sup>7</sup> The description in this paragraph more or less accords with Engeström’s (1987, p.78) oft cited triangular representation of activity systems.

<sup>8</sup> ‘Teaching Mathematics in Secondary Schools’. The author argues for the use of calculators and computers in Turkish classrooms but notes that their use may weaken students’ manipulative ability.

<sup>9</sup> This phenomena is not restricted to the UK. See, for example, <http://www.coe.ilstu.edu/jabraun/students/cowdery/titlepage.htm>

of behaviours. We are not saying that tool use should not be subject to change, simply that it is a good idea to locate one's ethical rationale for change first and be aware of the potential ramifications of tool use change.

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See <http://www.mathstore.ac.uk/come/events/reims/index.html>

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# STUDENTS' CHOICE OF TASKS AND TOOLS IN AN ICT RICH ENVIRONMENT

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**Abstract:** *This article reports from a three year development and research project with students of age 10 – 13 where the aim was to develop the students' competence to use computer tools and be able to judge and choose for themselves what tools to use for a specific task. After a period of work on a set of tasks where the students could choose which tasks and tools to work with, the students were given a questionnaire to answer about their preferences, what tools they used and for some new tasks their judgement of suitable tools. The results indicate that a majority of the students could make reasonable choices, they liked challenges, a little difficult tasks but not too difficult. But some liked easy tasks that they could manage to solve.*

**Keywords:** ICT, tool, students' choices, attitudes, reason, lower secondary.

## Background

The students should “know about the use of IT and learn to judge which aids are most appropriate in a particular situation” according to the curriculum plan of Norway (KUF, 1999). Furthermore they should develop knowledge and understanding of the subject matter, be able “to find solutions by explorative and problem solving activities and conscious choice of resources”. The mathematics plan expresses a constructivist view of learning and the need to emphasise conversation and reflection. Starting points ought to include meaningful situations and realistic problems in order to motivate the students. We find similar recommendations in other documents about teaching and learning, e.g. the NCTM Principles and Standards (NCTM, 2000).

The project reported in this paper is situated within a social constructivist view of learning, with the aim to give students opportunity to develop their knowledge and understanding of computer tools according to the intentions in the curriculum plan. More precisely the aim was to develop students' competence to use ICT tools and choose suitable tools for specific problems. To achieve this, the project intended to build supportive learning environments that would give the students opportunities to cooperate in their problem solving, choose suitable tools for given tasks, and to some extent develop their own tasks in a specific setting. The research focuses on students' appreciation of the ICT tools and evaluates to what extent the students make reasonable choices of tools for a given problem.

By ICT tools in this context I think of computer software that are open and flexible, not made for specific topics or limited to pre-designed tasks, which makes it possible to perform tasks that are planned and decided by the user. Such tools can be used for different kinds of problems and provide learning situations where the students can

experiment with mathematical connections, find patterns and be stimulated in their development of mathematical ideas, communicate and discuss ideas.

In the project I see computers used in the sense of reorganisers rather than amplifiers; cognitive tools that influence the way we develop mathematical concepts (Dörfler, 1993; Pea, 1987). Computer tools provide an especially powerful support to learn mathematics, but our view of the tools influences the way we use them. ICT tools will according to Dörfler (1993) be a part of the cognitive system and expand and extend the students' cognition. This has implications for the kind of software that we choose as tools in mathematics classes. Suitable software have a communicative power, give the opportunity to develop conceptual fluency, provide an environment for exploration and investigation, integrate different representations and stimulate reflection (Hershkowitz et al., 2002). Criteria for software were considered for the CompuMath project, a large development project integrating ICT in the curriculum. In line with this we used general tools which can support a variety of solution methods; a spreadsheet, e.g. Excel, a graph plotter named Grafbox and dynamic geometry, Cabri. We also included the use of Internet for collecting information.

### **The ICT competence project**

A main emphasis in the project was to develop an ICT rich learning environment with cooperation between students and options to make their own choices. The aim and basic strategies for the project was discussed in project meetings with the teachers, held every term. Experiences from the classes, use of the ICT tools and further ideas for teaching were discussed and developed in cooperation. The project leaders, an experienced teacher and I, provided some ideas and material for use in classes, both new and existing material. To some extent the teachers also developed material, like prepared spreadsheets or Cabri models for tasks on files, and made them available for the rest of the project group, i.e. the teachers and project leaders.

Three schools (S, F and L) with seven classes and six teachers including the co-project leader participated in the project over a period of 3 years following students during classes 8 to 10. The teachers all had some experience using computers in their classes, but mainly using spreadsheets. On their request they were given a short course on the use of Cabri in addition to information in the meetings. The teachers were responsible for how they implemented the ideas and material in their classes or if they used other ideas. The rationale for this is when the teachers are in charge, the situation will be more realistic and sustainable. They know what other factors in school to take into account. On the other hand this gave less control in the project.

### **Methodology**

The design of the project was a combination of development and research. During the first two years the main emphasis was on developing the students' competence with ICT tools. Activities were project meetings, development of material and teaching ideas and I visited the classes to support the teachers, observe and get to know the students. I took field notes from the visits and collected some students' work. In order

to have an overview of what was used of material and tasks in the classes, the teachers reported on a simple form each term and in the project meetings.

In the last year, the project was more research focussed with systematic data collection. Students' works were observed using audio and partly video recordings and students' solutions on computer files and in some cases on paper were collected. Some students in school F were interviewed and I also observed some students at their official oral examination that is set at the end of the 10<sup>th</sup> year in school. After a special period of work close to the end of the project, a questionnaire including some items particularly related to the work was administered via the Internet.

In this paper I will present results from the questionnaire and discuss how they relate to the aim of the project. Only to a limited extent results from observations will be included since this part of the analysis is not completed.

### **Close observation and questionnaire**

Close to the end of the project, during several lessons over 2–3 weeks, all the students worked on a small booklet of 12 tasks, with a variety of problems to be solved, some traditionally formulated and some with open themes and no questions. The tasks had different levels of difficulty. The selection of tasks was prepared to give variation in which ICT tools would be suitable or if just paper and pencil or a calculator would be the best choice. For some tasks different tools could be used. A draft version of the booklet was discussed at the previous project meeting to ensure that the tasks would be appropriate for all the classes.

The students were given the booklet at the start of the period and were given freedom to choose which problems to solve, in whatever order they liked and what tools to use, ICT tools, a calculator or pencil and paper. They could choose to work alone or in groups and discuss the solutions of the tasks with their peers or teachers.

The activities in the classes were observed, and some discussions audio or video recorded. The students noted on a log-sheet what tasks they solved, what tools they used and any comments and students' solutions on file were collected.

About a week later the students were asked to fill in a questionnaire about their attitudes to their work on computers and the use of ICT tools. The software provided a specific code for each student and they answered individually. Some questions were related to the tasks in the booklet, asking which of those tasks they liked or disliked to work on, what tools they used and why. The questionnaire also presented four new tasks where they had to read and judge which tools would be suitable to use.

### **Strategies for competence development**

In order to build the students' ICT competence we developed some strategies for the teaching, partially from previous experiences, other projects and during the present project (Fuglestad, 2004).

Some basic features of the software had to be learned, step by step, but preferably in the context of some problems the students worked on. For example a set of small

booklets was produced for introduction to the features of Cabri, introducing new menu items when they were needed in the problems to explore.

We expected motivation and relevance to be crucial for the students to engage and develop tasks further by generalising and asking questions themselves. From students' engagement and concentration on their tasks, we observed that when a problem was challenging or interesting they were willing to engage quite hard ignoring disturbances from other students and working during the break. This was particularly commented on by some visiting observers and was revealed in the answers to the questionnaire to be presented later.

Opportunities for choices had to be planned in order to stimulate the students to make their own choice and to explore their own questions. Tasks in textbooks and for written examination often tell students what tool and methods to use with hardly any choices. In the project they were given open tasks and situations with no questions, for example a special theme or project with information and data in a special context. In this way students had the opportunity to set their own problems, develop the tasks and use ICT tools when that was suitable. In some tasks the students could develop the problems further or were challenged to look for a similar more general problem.

An open working situation appeared in some cases to be difficult for some of the teachers to use and the strategy had to be developed over time. This strategy was more in use in school S, where the teacher was very conscious of an open working method with a long experience of using computers. He was also the co-project leader.

Other elements of the strategy were to solve a problem with several tools, judge and discuss different solutions. This was used to a less extent than expected, because of limitations in the teachers' previous experiences and use of software. It appeared that when the teachers had limited knowledge of the software the students did not get the opportunity to use it. Another reason for limited use of some software was that the teachers did not expect Cabri and Grafbox to be part of the final official examination at the end of the school year, but a spreadsheet would be available for use.

Reflection and discussion after a period of work, discussing different solutions the students presented was agreed in the project group to be an important part of the teaching. To what extent this was practiced in the classes varied and is hard to judge. The teachers' role and the way the teachers interacted with students during their work were important to support the students' development of their independence and self reliance.

Although we discussed elements of the strategy in project meetings, some limitations appeared in practice. Teachers' previous and in some cases limited experience with computer tools and their habits highly influenced this. The teachers indicated that they needed help to develop competence with the use of ICT tools, both the tools themselves and the way to manage the situation, not give too much of the solution but ask questions and stimulate students' own exploration. The development of teachers'

competence was not a major focus in this project, but the work in project meetings revealed the potential for this in the cooperative environment in project meetings.

**Results from the questionnaire: What students liked to work on and why?**

The students were asked to choose a task they liked to work on, give their comments to why they liked it, what tools they used, why they used this and what they think they learned from working on this task. A corresponding question was asked about a task they did not like to work on, and why.

The most popular tasks turned out to be 1, 3 and 4. The first was a very simple task, about combining the value of two stamps 2.50 and 1.80 to give the amount 20.40 for sending a package. The task is easy but not just pure routine and requires some trial and error or judgement of numbers.

Task 3 was more open and dealt with planning the sales from a kiosk at a sports event on a warm day lasting from 10.00 to 16.00. The information given was about the environment, 400 children and 200 adults were expected to attend, and the students had to plan a selection of items to sell. They had to find or judge prices for buying and for selling and calculate what they earned, also judging how many of each they expected sell. An overview of what is needed can be prepared on a spreadsheet, but there are also a lot of judgements of numbers of items and prices as for example how to calculate cost of a cup of coffee or pieces of waffles you have prepared.

Task 4 is about calculating interest and status on a bank account given the start capital and the interest, and some more questions related to that. The task is less open than task 3, but requires some judgement and use of a table to get an overview. The task is suitable for use of a spreadsheet to make the table and change parameters.

It is interesting to notice that tasks 3 and 4 that received highest score on *like* the tasks also scored high on *dislike* but obviously from different students. A closer look into the variation of choices revealed differences between schools. The students were split into groups of different schools: School L had a high score on like task 3, for school F the highest score was on tasks 1 and 4, but for school S the highest score was task 12 which was not at all popular with the other schools. Task 12 was an open task about mobile phones; prices for different subscriptions were given in the form of a price list from the companies. To set the tasks and perform some meaningful calculations and comparisons was a part of the problem. Only one student from the two other schools (with 138 students) ticked he liked this task, but 5 from one class in school S (25 students). The most reasonable explanation for this seems to be that this class had more experience with this kind of task and were more used to an open working style where they had to set their own tasks.

The questions about why students liked the tasks show a pattern of two seemingly opposite reasons. Some students liked the tasks because they gave some challenges, they were not too easy (39), they gave a new problem not just the same again and again, and the students could make some decisions themselves (11). Quotes were translated from Norwegian as closely as possible to students' wording (quotes italic).

*Because we could use our fantasy and decide very much ourselves. We decided how much to buy and what the price for selling would be etc. (Task 3)(School F) Similar answers were given by several students for this task, combined with comments like: It was realistic and something we could use - can recognize the situation.*

*It was fun, but it could have been more difficult. (Task 3) (School F)*

*The task was firstly very good to solve. Logical thinking and simple methods. But also the same time some challenge. (Task 3) (School L)*

*I liked the task because it was so quick to do on a computer. (Task 4) (School F)*

*This was a task I could work on and it is possible to use other information than in the booklet. If it had been very difficult it would have been not so much fun, but this was appropriately difficult and not too easy. (Task 12) (School S)*

*This task was quite demanding, but we found ideas how to solve it. It was sometimes a little difficult but very fun when we managed! (Task 6) (School S). Similar answers to this one was given for some of the more demanding tasks.*

On the other hand students liked the task because it was easy and they could master it: *Because this was one that I managed to solve. Others: It was easy.* This answer was given for task 1 by many students.

Why did students dislike a task? Either the task was too simple or boring (29) doing several times the same: *Because it was simple, make it more demanding.* Or another: *There was not much to find out and we had a similar task before.* Some students thought it was too difficult and they could not manage (47): *It was too difficult and in the end we gave up.* And some asked for help but got too much: *Because at first we did it in a stupid way. Then the teacher came and showed how we rather had to do it, and suddenly it was like the teacher really did most of it.* (Task 9) (School F).

The students like challenges and variation and they like tasks that are a little difficult but not too difficult. I interpret these answers to reveal high motivation. On the other hand, some students also like easy tasks and prefer tasks they can manage to solve. There seems to be a polarisation between a group that like challenges and another that likes easy tasks. But for both groups, the answers reveal they like tasks that they can manage to solve. This was confirmed by answers to tasks they do not like; those that they cannot solve. This result is hardly surprising at all, but is a reminder to plan carefully combinations of challenges and tasks that the students can manage.

### **Choice of tools**

Some students, not many, preferred to use paper and pencil, and tried to avoid using computers overall. This was observed and also came out in the questionnaire. On the other hand, for task 1 paper and pencil or mental calculation is a reasonable choice.

A student described his solution like this when talking to the teacher: *I looked at the point forty and found I had to use three times 1.80 to get that, so it was easy to find the solution just by trying with different numbers in my head.* This is a good example

of mental calculation and valuable to the discussion in the class to highlight this method alongside with computer use. To be able to choose, the students have to see alternative solutions.

On Task 1 the choices of using a spreadsheet came out with about the same frequency as paper and pencil and calculator (7 of each). Of the 34 students that ticked they liked Task 3 in the questionnaire, the majority chose a spreadsheet only (16), combined with paper and pencil (6) or calculator (3). Task 4 was chosen by 28 students, and 15 chose a spreadsheet only and another 7 a spreadsheet together with paper and pencil or a calculator.

In order to judge if the students made appropriate choices, I looked into their choice combined with their reasons for their choice. It is of course possible to judge superficially, if we have some table or numbers we use a spreadsheet if we have a geometry problem or a figure that can easily be drawn we have to use Cabri. The reasons students gave can illuminate the quality of their choices. In most cases they commented that this software is most appropriate, straightforward and easy to use and make efficient calculations (for about 45% – 60% of the answers). Some reasons for choosing a spreadsheet: *Because you can use formulae and it is very put up in a good way everything becomes easier when you are going to calculate.* (School S) Another answer: *To use a spreadsheet is the simplest way to solve it. Click and drag.* (School L) Here the student refers to copying a formula in order to make a column with the formulae needed.

*A spreadsheet is good because it is easy to register and it will be tidy and so because of the formulae you have made. Also it is easy to change the results. I use also paper and pencil to make a quick estimate or so, and calculator to calculate approximately.* (School L) Comments on use of formulae suggest that this student is using the spreadsheet, but he thinks of using several tools in combination and it is not quite clear whether he performs all the calculations on a spreadsheet or also uses a calculator partly for this.

The overall impression, I think, is good; in a majority of cases the judgement is appropriate. However, this does not tell if the students are able to prepare an efficient and good spreadsheet for the solution. For this we need to look into their solutions. From observations and inspecting some solutions on file, I found a variety of levels of solutions but mainly correct. This part of the analysis has not been completed.

### **New problems and the students' choices**

In the last part of the questionnaire the students were given four new problems to look at and judge which tools can be used, what they would choose for themselves to use and why. The tasks were selected to make paper and pencil, calculators and computer tools reasonable choices on different tasks. In a short survey it was necessary to limit the size and degree of complications in the problems used.

In the introduction to the new tasks the students were asked to read, perhaps make some notes, but requested not to complete it. Then they were asked what tools can be

used, what they would prefer to use themselves and write comments why they made this choice.

Due to limitations for this paper I can only report results from one of these tasks. The task was as follows:

New task

Kim is going to visit his grandfather for a month in the summer vacation. The Go Cart Course “Svidd Gummi” has these offers:

Price per round for non members 150 kr

Membership card 300 and price per round for members 120 kr

A months card: 2 000 kr, with maximum 5 round per week.

What offer will be worth-while if he drives?

a) 8 round b) 12 round c) 16 round d) 20 round

Possible solutions to this task can involve making graphs using a graph plotter or tables and diagrams in Excel and compare the different functions that lay behind the price offers. The task can be solved in the simple way by just calculating the four offers for the four cases. The students’ answers may also indicate what kind of solution they are thinking of.

The dominating choice for this task is Excel only in 65 cases and Excel together with paper and pencil or with a calculator in 19 cases. Grafbox was chosen only by students in one class, but in that class by 6 of 25 students, in school S. This reflects the teachers’ choice of which tools they mainly introduced in their lessons. The teachers in the other classes commented they found graph plotter difficult to use and demanding for the students. A Grafbox- solution requires that the students are able to express each of the price offers as a linear function and enter the corresponding formula into the program. Although the topic “linear functions” is part of the curriculum, the students apparently found it more demanding to find the expression than to perform calculations in a spreadsheet. This result also highlights how the teachers’ choice influences the students’ learning. Other choices include combination of calculator and/or paper and pencil in 35 cases.

The reasons students gave for choosing Excel reveal some insight, but there were also many cases with very little indication of the main pattern of the solution. Many answers indicate the use of formulas to calculate, but with using the same simple set up several times, and not preparing a table of the functions. *Simple, setting in a couple of formula and then you have the answer.* (School S) *There is so much to write so it is best to use the computer.* (School F) *It is easy to put in formulas and then just change the number according to how much he drives.* (School L)

Other answers include the use of a table, which may give the indication of comparing functions in different columns, but the ways of expressing it was not very extensive: *Setting all in a table. Easier to calculate.* (School F) Or another: *I would have chosen this (Excel) because then I could put up tables, and change the numbers in order to*

*change the numbers* (School L) and a third: *Because you can copy a formula, but have to shift some numbers. The simplest would be a spreadsheet I think.* (School F)

In spite of clumsy wording, not very clearly expressed, these explanations have grasped some important features of a spreadsheet. The students seemed to say that when the task is more complicated, with several numbers to calculate according to the same rule, a spreadsheet is helpful. From the choice of Excel as the only tool, it is clear that they see Excel as a more appropriate tool than others in this case. But most of them seem to think they will calculate only the cases given in the task and not make a more general comparison like the one we will have with a graph plotter.

A graph plotter requires a more general solution. The students have to discover the model and express the corresponding linear expression for the functions involved. Reasons students gave for this choice were the following: *Easy to put into Grafbox and see the answer at once.* Another one similar: *Here I can make graphs and see what will pay off.* And a third slightly more general: *Here you can have an overview of how much to pay for according to how many rounds you drove.* (All in school S)

Again, it is hard from the reasons they gave to see how they will solve it, i.e. if they plot more than one graph at a time, but still they gave some indication of features of the graph plotter. From observations in this class, and inspection of some students' computer files, I found some of the students liked Grafbox very much and could handle this kind of task well. I would expect some of them to manage this task easily.

### **Summary and conclusion**

Looking back at the results, to what extent have the students constructed an understanding of the ICT tools that enable them to make reasonable choices of tools for a specific problem? From the analysis presented here these main point came out:

Student gave clear indication what they liked: challenges, not too much of the same and problems they can master.

Most students gave reasonable choices of tasks. Some students (about 18% of the answers) gave reasons for their choices that indicate a good understanding with reference to features of the software or methods they used or planned to use. Observations and inspection of some students' computer files support this.

In many cases the reasons for choices were short with limited indication of how they solved or planned to solve the problems. In some cases indicators like it is about percentage, about interest calculations or a geometrical figure trigger the choice. The choice may well be relevant but the depth of the choice might be questioned. Answers like "it is easy" or "it is the best choice" (46 - 60% of answers) can be judged as superficial but the fact that the tool chosen is appropriate can also indicate a good choice and we can only expect fairly short answers in this setting. Whether the students can solve the tasks efficiently with the tool they chose cannot be answered in full at this stage. This needs deeper analysis of students' solutions of the tasks with data from observations and files and is outside the limitations of this paper.

I suggest the competence to make reasonable judgement and choices of tools gives an indication of to what extent computer software will act as cognitive tools for the students. Their choices and reasons requires thinking about planning solution of a task, i.e. thinking on a meta level according to Dörfler (1993). This can be a fairly demanding task but I think the results indicate students have constructed relevant knowledge of the tools, with a potential for further development in further work.

The students that are used to an open, experimental learning environment liked challenging problems better than the others and revealed a better knowledge of all the ICT tools than other classes. This indicates the potential of this learning environment but the fact that they also had more experience using computers confine a conclusion.

Although the teachers' role was not a major focus in this research, teachers' influence on the students' knowledge and choice of tools was revealed in the students' answers and there appears to be a clear connection to the teachers' priorities.

Implications for practice and further projects: Give the students challenges, not too easy and not too difficult tasks. Give help and support but do not take over the solution. Open learning situation where the student can cooperate and set own tasks in a context give potential for development and should be further developed.

Further research and analysis of data from this and a coming project is necessary to gain in depth insight into students' choices and thinking about the use of ICT tools.

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# THE BALANCE

## AN INTERACTIVE ACTIVITY TO WORK WITH FRACTIONS

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**Abstract:** *This study has as its main purpose the analysis of teachers' and students' work with an interactive program called The Balance designed for Enciclomedia, a national project in Mexico. This software was designed as an assistant in the teaching of fractions. The analysis of the observations of teachers working with The Balance during an interview, and of one of the teachers working in the classroom was done using Duval's representations' theory as a conceptual framework. During interviews it was found that teachers' knowledge of fractions is algorithm dependent, and that when The Balance was used in class it acted as an important factor in the emergence of the concept of equivalent fractions in a rich spontaneous environment.*

### ANTECEDENTS

Although there is agreement on the inherent complexities of teaching and learning rational numbers (Hunting & Davis, 1997; Mack, 1998; Hecht, 1998; Hunting et al., 1998; Moss & Case, 1999; Tzur, 1999; Cramer et al., 2002; Litwiller, 2002), there is no consensus about how to facilitate the understanding of the concepts related to these numbers and their operations (Behr, et al, 1997; Taube, 1997). The major body of research in this area is focused on identifying the experiences children need to develop meanings for rational numbers (Taube, 1997).

Research carried out on preservice teachers suggests that while they have built up knowledge about fractions, they experience difficulty in explaining them to children, and are unable to clarify why the algorithms work (Selden, A & Selden, J, 1997, Lubinski et al., 1998, Chinnappan, 2000). Some studies suggest that teaching for understanding is not easy, and that even experienced teachers find it difficult to help students to develop a conceptual understanding of fractions (Ball, 1990; Meagler, 2002; Putnam & Reineke, 1993; Wilson, 1994). Research results inform us about the problems teachers have when teaching this important concept. For example, Moss and Case (1999) describe four principal problems that they identified in common teaching methods: teachers emphasize algorithmic procedures and pay much less attention to the development of a strong sense on the meaning of rational numbers; teachers do not use in their teaching what students have learnt about numbers and their informal knowledge about fractions before being introduced to the concept of fraction; representations used for fractions do not convey easily the difference between whole numbers and fractions; and teachers use rational numbers

notation as if it were transparent to students, when this has been shown not to be the case. In another study, Tirosh (2000) concluded that teachers need to pay more attention to the analysis of students' mistakes, and to take them into account when teaching this concept. Hristovitch and Mitcheltree (2004) report that teachers' understanding of fractions and decimals is not strong enough, and that they demonstrate difficulties to relate fractions and decimals.

Several research studies have built upon students' and teachers' difficulties to learn the concept of fraction (Davis & Thipkong, 1991; Behr et al, 1997; Cramer et al., 2002): The Rational Number Project Curriculum (Cramer et al., 2002), includes activities to work with rational numbers that emphasize translations within modes of representation: pictorial, manipulative, verbal, real world and symbolic. Results of its implementation showed that when these materials were used students responded better and could use efficiently use their memory in tasks involving fractions. More important, students approached the tasks conceptually by building on their constructed mental images. Other studies concerned with the same project led to the conclusion that children's learning about fractions can be optimised if they are involved in the use of multiple concrete models. These models help them to develop mental images needed to think conceptually about fractions. Children benefit from opportunities to talk to one another and with their teachers about their fraction ideas as they construct their understanding of fraction as a number. These authors conclude that teaching materials should focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer et al., 1997).

## **THEORETICAL FRAMEWORK**

Research studies in mathematics teaching at an elementary level show that children need the support of concrete contexts to provide meaning to concepts. The beginning of the abstraction process at this stage profits from the establishment of a clear relationship between real situations and mathematical concepts (Aleksandrov et al., 1973; Hitt, 1996). Computer simulations can help to provide scenarios where teachers and students can work on situations which otherwise would be difficult to handle in class. Simulations can show fundamental aspects of the concept and may satisfy certain requirements that research literature has found to be important, for example: a) enable the visualization of the real situation (Jimoyiannis & Komis, 2001), b) act as a bridge between previous and new knowledge, c) include different forms of representation of the same concept (Li et al., 1996).

Computer simulations can be considered as a representation register (Monzoy, 2002) because they satisfy an important characteristic of Duval's definition of representation registers (Duval, 1998, 1999), namely, the main properties of the mathematical object represented can be manipulated when working with the representation. For example, when working with dynamic geometry, students can directly act on figures to find invariants, or general statements about their transformations. Duval's theory of representations discusses the cognitive activities that can be fostered by the use of different representations when a mathematical

concept is to be understood. For him, the coordination of several representation registers is fundamental in conceptual understanding as it helps learners distinguish between the conceptual object and its representation, and to recognize a conceptual object in any of its different representations. Coordination between different registers is not spontaneous. Students need a specific strategy to make the translations between representations possible. The strategy includes the specific cognitive activities that students need to do. For instance, when a student observes variations within the same register, the student discriminates between significant units in that register, and the concomitant variations in another one. These theoretical referents can be used the strategies that teachers and students use when working with a computer simulation are studied.

From his research, Kaput (1994) concludes that a computer environment is useful to widen the sense of the treatment of a representation. The response capability of the interactive media can produce directly representational elements that are difficult or impossible for the individuals to produce, and that can be acted on by the student to change the state of the representation. Based on the same theoretical approach, Monzoy (2002) considers that the possibility offered by the computer to work simultaneously with several representations makes the learning experience richer by providing a concrete background for the identification, treatment and conversion activities required by Duval's theory. When designing computer activities it is important to take into account the concept that will be made apparent in different representation registers, the didactical strategies needed to work with the specific concepts, and the assessment of the tools and its use.

## RESEARCH QUESTIONS

In the first part of the study, the following questions were studied:

- What are the teachers' strategies when solving this lesson on fractions?
- How do the teachers use the Balance together with other teaching materials?

The questions posed for the second part of the study were:

- Can teachers use The Balance to facilitate the concept of equivalence of fractions to emerge in a natural way within classroom activities?
- Do teachers use several representation registers in their classrooms?
- How do teachers help students make translation and conversion activities?

In this paper we analyze the results concerning teachers' understanding of the goals of the lesson on fractions, and of the concept of equivalence of fractions itself. We also studied teachers' strategies when using The Balance, and those of one of them when using it in his classroom.

## METHODOLOGY

*Enciclomedia* is a project that intends to complement and enrich the free Mexican mandatory textbooks for all elementary school subjects by including an

electronic version of all the materials online with links to computer tools designed to help teachers in their classrooms. The first version of Enciclomedia has begun to be tested in 20 thousand classrooms throughout the country.

*The Balance* is an interactive software designed for *Enciclomedia*. It is linked to a lesson on fractions in the 6<sup>th</sup> grade book (Block et al., 2000). The Balance was designed as a help in the teaching of the concept of equivalence of fractions following the idea presented in the textbook. Students are to complete different mobile toys with several levels (see Fig. 1). In each activity, some boxes of the toys have fractions while others do not. This activity is challenging to teachers and students.

The research project intends to study how teachers and students use *The Balance* while they work with this lesson, and if it helps teachers and students in achieving the lesson's goals. This paper reports the first part of the project where four teachers, one male and three females, with an experience of at least seven years were interviewed while they worked with *The Balance* to solve the lesson's activities. Afterwards, one of the teachers, the most experienced one, and the one who showed a better understanding of the concept of equivalence of fractions, was observed in class.

The teachers were interviewed one at a time. During the one-hour interview they solved the activities of the lesson on paper explaining their reasoning. They also responded to the interviewer's questions. The teachers also used *The Balance* and commented with the interviewer about what they observed, the difficulties they had, and the way they thought they could use the program in their class. One month later, the most experienced teacher strategies while working with the lesson were observed and videotaped. The tape was later transcribed and analyzed independently by two researchers in terms of the theoretical framework described above.

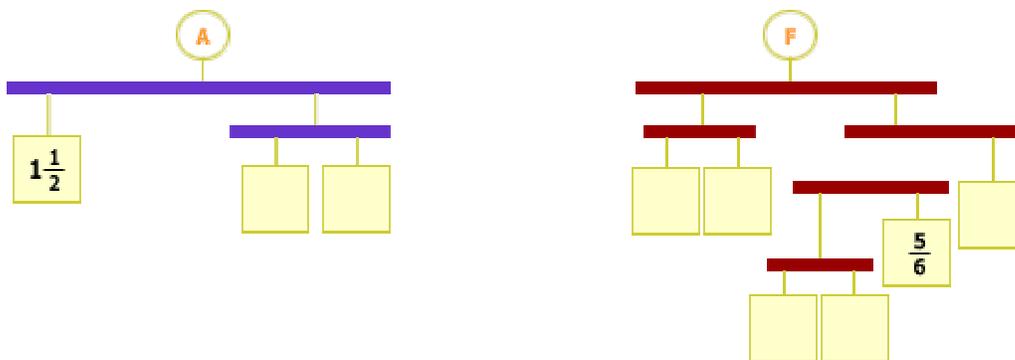
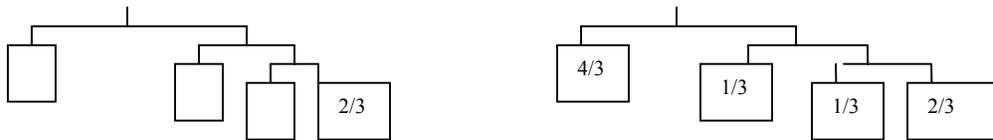


Figure 1. Mobiles on fractions

## RESULTS

The results of the study can be divided in two parts. One concerning those obtained during the interviews of the teachers and the other concerning the classroom use of *The Balance*.

**Results on teachers' understanding** *Working Groups*: The strategy used by three of the teachers when solving the lesson's exercises on paper consisted in filling the empty boxes of each mobile toy and adding them up to be sure that the main arm of the toy was balanced. When the proposed mobile toy had a number in one side of the balance these teachers would fill the empty boxes on that side with a number of their choice, add them up and fill the other side in a way that the uppermost arm was balanced. Their attention was focused on the equilibrium of the main arm and not on the other levels of the mobile toys since they thought that was the purpose of the exercises. For example, when solving the activity shown in next figure, one of the teachers explained *"I fill the three blanks on the right with any number, I add them up and then fill the blank or blanks on the left side"*.



Even though these teachers had read the teachers' guide provided by the Ministry of Education (Block, 2002), only one of them understood that all the arms of the mobile toys had to be balanced. Their strategies were the same, and they struggled a lot when working with complex activities. They used both, whole numbers and fractions, to fill the boxes. It is important to note that the solution of the book's activities on paper, cannot give any feedback to teachers and students about the goal of the lesson, since the drawings of the toys do not move.

These teachers exhibited different behaviors when they first used The Balance. Their reactions varied. One teacher calculated her answers on paper and concluded: *"1 1/2 is equal to 1+1/2"*. When the interviewer asked her to use The Balance, she tried to verify her answers with the tool, but immediately realized that The Balance was not in equilibrium. She exclaimed: *"1 1/2 is equal to 1+1/2, then why I can't get the equilibrium position?"* She then proceeded to explore with the computer tool and reconsidered her work until she was able to find the correct answer. Another teacher thought there was something wrong with the computer program. She insisted she had checked her answer on paper, and was sure it was correct. When told about the necessity of the different arms of the toy to be balanced, she struggled but was able to complete the activity correctly. In the case of the third teacher, the interviewer had to point out that indeed, the first arm of The Balance was horizontal because  $1 \frac{1}{2}$  is equal to  $1 + \frac{1}{2}$ , but the second arm was not horizontal because 1 is bigger than  $\frac{1}{2}$ . The teacher who understood the goal of the exercises followed the same strategy. He first solved the activities on paper and then used The Balance to verify his answers.

During the interview all the teachers struggled when working with slightly more complex activities. They showed dependency on memorized algorithms that are not easily applied when all the arms of The Balance need to be in equilibrium, and had difficulties to explain their procedures. When asked how they would use the balance in their classroom all of them responded they would let the children work on

the activities on paper and then use ~~The Balance~~ <sup>Work Balance</sup> to verify and discuss their answers. One of them mentioned that she would start with a concrete manipulative such as ribbons or wood blocks to review the concept of equivalence of fractions, plus some work on drawings of fractions before working with the book's activities. They all made emphasis on the need to plan the lesson very carefully because of its difficulty, to insist on the rule of multiplying both parts of the fraction by the same number to get an equivalent fraction, and to be prepared to answer students' questions. They all commented on the dynamic movement of The Balance as a help to give feedback to students to find correct answers. For instance, after some time trying to complete an activity with the computer version of The Balance, one of the teachers exclaimed *"Before working with The Balance I knew the algorithm to find equivalent fractions, but after working with The Balance, I feel that I have really got the concept because I can see it"*. This teacher found a way to explain how she had done it: working from the boxes at the bottom arms up to the upper ones balancing every arm. Teachers' strategies varied from the most simple activity to the more complex ones. In the later, they usually went back to writing the numbers and applying their procedural knowledge to find possible answers, before trying them on the computer tool. Only one of the teachers was able to solve the activity directly using The Balance, showing he was flexible at doing translations and conversions between different representations. These results pointed out that the emphasis teachers give to algorithmic procedures, as reported in the literature, might be related to the fact that they do not have a rich conceptual understanding of equivalence of fractions.

The data show that the use of the tool was important for the teachers. Through its use, they were able to compare, translate and convert between the two different representation registers they used, numerical and computer tool, and reconsider their work. The movement of The Balance in response to users actions gave them instantaneous feedback, and helped them reconsider their strategies and the numbers they used. All the teachers commented at the end of the interview that they were able to develop new strategies that they could then use in class to help their students understand the concept of equivalence of fractions. One particular benefit they all considered was the possibility to verify the results obtained while using the algorithms. Another benefit mentioned was the possibility to help students reflect on how to equilibrate the toy, even before doing the calculations. One of the teachers said *"After working with the balance for a while, students will find a way to balance all the levels of the mobile toy, I can see it."* These data confirm that this lesson is difficult for teachers. A concrete manipulative would be a good help to teachers and students, however a manipulative for this kind of tasks would be very difficult to construct and handle. The computer environment showed to be an effective and innovative way for the teachers to work with the lesson. The possibility to reflect on their actions with The Balance helped all of them to solve complex activities included in the lesson using the new strategies they developed.

Teachers' difficulties while solving the activities suggest that their conceptual knowledge is fragile. They need to constantly check what they are doing, and they are

not fluent in finding equivalent fractions. Working Group 9 activities they do are not straightforward. They also show that The Balance, designed to explore possible solutions, gives instantaneous feedback, and can be an effective tool for these teachers to work with, and to construct new meanings for the concept of equivalence of fractions.

**Results on the classroom use of The Balance:** The teacher worked with his class of twenty four students. He had not used the interactive tool with his students before, and the students had not worked with these specific exercises. The teacher started the lesson by asking the students to solve the first exercise represented in figure 1 as mobile A. He did not review their previous knowledge. This confirms previous literature results about teachers who tend to take for granted students understanding of concepts about fractions that have been taught before. The class was organized in groups of six students. Students found the activity difficult, but two of the groups, were able to solve the first and simpler activity correctly. The strategy followed by the other two groups of students was the same. They tried to divide  $1\frac{1}{2}$  into 2 fractions and obtained, in one group,  $\frac{1}{2}$  and  $\frac{1}{4}$  for the two boxes and in the other  $\frac{1}{4}$  for both. The teacher wrote all the students' answers on the blackboard and then introduced each one into The Balance. When the answers were not correct the mobile was not in equilibrium but students were not surprised, they supposed something was wrong with their procedure. A discussion with the whole group helped students explore using new numbers, as well as realize the need to use the same fractions in the pair of boxes, and for the sum of fractions in one level to be equivalent in order to balance each arm.

During the discussion, the teacher used and compared results obtained by the students in different representational contexts. Some students drew diagrams on paper to solve the problems, other groups of students depended on written calculations. The teacher wrote the students' responses on the blackboard and compared them by using The Balance while discussing with the students why some answers were correct and others were not. The use of the tool inspired the teacher to ask a question he had never asked before: "*is this the only correct answer for this problem?*" The whole group discussed, and students decided it was not. They decided to use equivalent fractions to find new correct responses. Their comments referred always to The Balance. The strategy followed by the teacher was close to that used in his previous lessons, when he could not use the interactive tool: Starting with work on the lesson in a symbolic way and checking answers. But, according to the results of the observation and his own comments after class, the development of the lesson was different, students discussed more and were more interested. The use of the tool provided the class with the possibility of using a "concrete" and new representation that could be used together with the symbolic and the graphical to do conversions and translations between registers. The use of these representations together with natural language helped the conceptual meaning of the activity to emerge.

The students saw in The Balance an interesting exploration tool where they could explore and verify their original answers. As they tried different activities, they

Working Group 9  
were able to discuss with other students and with the teacher, and to ask questions until they were satisfied. It is interesting to note that student's strategies, independently of their correctness, were based on splitting the fractions, using always rational numbers, while those of the teachers mixed whole numbers and fractions.

## CONCLUSIONS

The results of the study show that most of the teachers have difficulties with the concept of equivalence of fractions. However, the use of the Balance helped them to reconsider their work and their strategies. The tool helped them to understand the purpose of the lesson in the textbook and to find successful strategies to solve the activities. Teachers used the tool mainly as a means to verify their procedures. They showed a tendency to use whole numbers in some boxes and rational numbers in others.

When The Balance was used during the interviews, teachers could not understand why it behaved differently than expected, probably because they were not able to relate the activities of the lesson with their conceptual understanding. The Balance helped the teachers reflect on the concept of equivalence of fractions.

When The Balance was used in the classroom, the teacher and his students used several representation registers while working on the lesson: geometrical, symbolic and the dynamics of the tool. It seems that the learning experience of the class was enriched when the Balance was used, new discussions arose, and new ways to think about equivalence of fractions were discussed. Based on the data, we believe that the use of The Balance helped teachers and students to focus on the mathematical concepts. It helped the teacher in the classroom to realize the need to be attentive to events in a way that enabled him to see more than what he expected to see and to use it to enhance students' learning. More research is needed on this respect and on the understanding of the role of visual and dynamic representations in the understanding of this concept as suggested by Stylianow and Pitt-Pantazi's (2002).

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# DISTANCE TRAINING, A KEY MODE TO SUPPORT TEACHERS IN THE INTEGRATION OF ICT?

## Towards collaborative conception of living pedagogical resources

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**Abstract:** *The school integration of ICT remains still rather weak, even in highschools. Changes in users' practices required by such integration have probably been underestimated: teachers are obliged to question and change their professional practices. It turns out that standard training sessions towards ICT have been found to be unsuited for supporting teachers in overcoming their difficulties with this integration. This document describes a specific distance, in-service training organization, as well as its implementation and difficulties encountered in the process. A structure of resources has emerged from this collaborative virtual workshop towards the conception of pedagogical resources. Such long-term organizations could turn into efficient supports for teachers using ICT in classrooms.*

### 1. Towards integration of ICT

This paper is intended to focus on several aspects linked to the integration of ICT into the teaching and learning of mathematics. To focus on the problem of integrating technology into classroom practice requires theoretical and empirical research towards what can be an efficient use of technology: specifically this requires devising *situations* (Brousseau 1997), implementing them in classrooms and modifying them in order to be efficient and viable in these classrooms. It also requires conditions which would allow these situations to be *reproduced more widely* and training strategies which could enhance the integration of ICT into teachers' practices. Both could be realized through a collaborative conception of living pedagogical resources in teachers' *communities of practice* (Wenger 1998), communities already existing or to be built.

In spite of many institutional actions and the enthusiasm of pioneering teachers, integration of ICT into teaching mathematics in secondary school is only slowly increasing in France, despite the rapid evolution of technological tools and equipment (Trouche, in Guin & al 2004). Similarly, a survey in Catalonia and France in 2003 (§4.4) gives evidence that more than 50% of mathematics' teachers *never use ICT* (including internet and calculators) *with students*. This situation, which seems to be worldwide and not specific to secondary school teaching (Jones & Lagrange 2003), recently incited researchers to find reasons for this lack of computers use.

A collaborative research project lead by 4 French research teams (Lagrange & al 2003) studied a comprehensive corpus of 662 published work: it was found that most

papers were essentially focused on epistemological issues and on the learner. Only 5% of studied papers were related to conditions of integration in every day practice, in terms of the *viability* of technology within schooling institutions. Consequently, few papers have taken into account the conditions of viability of ICT in classrooms and the influence of the teacher. As turns out, initially, most researches were focused on *potentialities* of tools in mathematics education and has often underestimated radical changes in users communities of practice (students, but also teachers) that this integration requires. Nevertheless, we have noticed a *progressive awareness* of the complexity of teaching with ICT, as more recent studies on teachers in real teaching, under traditional norms, closely related to our “French” approach (Monaghan 2001).

Artigue (1998) pointed out other obstacles in ICT integration and that common training strategies in France do not help teachers to overcome. Standard training sessions towards ICT are rarely designed out of teachers’ practices, and training strategies are essentially based on the transmission of «expert resources». Moreover, its short period (about 3 days), isolated from school practice, does not allow a *continuous* support to be provided during the necessary adaptation of resources to each teacher’s usage context. *Therefore, which type of resources and training organization would further the integration of ICT?*

## **2. Towards instrumental approach**

The main features of Rabardel’s theory will be briefly evoked in this section because they are, in our opinion, crucial to tackle issues not only related to student’s activity with artifacts, but also related to teacher training and conception of pedagogical resources with ICT. This theory lies within the field of cognitive ergonomics, which is linked to an *ecological* view (Gibson 1977) of human activity with artifacts. Rabardel’s theory is based on the *theory of activity* and the idea of *mediation* due to Vygotsky.

First, artifacts are necessary mediators in human activity and the activity mediated by instruments is always *situated* (Rabardel 2001; Trouche 2004) and *distributed* (Hollan & al 2000). Second, there is a clear distinction between the technological *artifact* and the *instrument* that a human being is able to build out of this artifact. It goes through user’s activity worked out in a given context and through a complex *instrumental genesis*. This genesis combines two simultaneous and deeply interconnected processes: an *instrumentalization* process (focusing on the artifact) and a process of *instrumentation* (focusing on the subject). Finally, an instrument consists, on the one hand, of a part of an artifact and, on the other hand, of *schemes* which are psychological structures organizing the subject’s instrumented activity aiming to accomplish a given task. Instruments are both *private* and *social* entities, as schemes are social because they have characteristics that are both shared and widespread in communities. Therefore, Rabardel considers designing instruments as an *activity distributed by designers and users*, evoking the idea of *designing through usages*.

We analyzed students' activity with symbolic<sup>1</sup> calculators according to the meta study previously mentioned (Lagrange & al 2003). We brought out the necessity of a plurality of approaches: Rabardel's theory was combined by Artigue, Lagrange & Trouche (in Guin & al 2004) with other French didactical theories, especially, the *anthropological* approach of Chevallard (1999) and Vergnaud's theory (1996) on mathematical conceptualization. In the same way, Drijvers & Gravemeijer (in Guin & al 2004) argued for a relationship between instrumental approach and other theoretical perspectives on learning such as the *semiotic*, *symbolization* and *modeling* perspective.

These papers based on experiments in real classrooms show evidence of the *complexity* and *diversity* of instrumental geneses (complexity increasing with the complexity of artifacts). Moreover, Kendal & al (in Guin & al 2004) have pointed out the diversity of *teaching styles* with CAS, depending strongly on their conception of mathematics. They described the different methods of organizing the classroom and of devoting time to technology or mathematics. These papers also outline the *crucial role* of the teacher in dealing with scenarios aiming to build *coherent systems of instruments* from the diversity of students' instrumental geneses. Based on the evidence of these experiments, the success depends on *precise piloting* by the teacher beyond the careful choice of a didactical engineering. First scenarios designed by researchers required serious *reorganizations* to be viable even in experimental classes, with expert teachers. Consequently, the theoretical instrumental approach has been developed to describe the place of didactic intervention: *instrumental orchestrations* defined by their *configurations* (i.e. specific arrangements of the artifactual environment), and *exploitation modes* of these configurations, aiming to reinforce the *social* dimension of instrumented action and to oriente the construction of instruments' systems (Trouche, in Guin & al 2004).

Although computing competencies are necessary for an instrumented practice, these theoretical researches point out that integrating ICT into classrooms requires other teachers' competences. *The implementation* of situations and scenarios by teachers for their own use in given classrooms is another unavoidable step which is far from being obvious. Consequently, the integration of ICT has created serious difficulties for teachers involving a profound *questioning* of professional practices and requires radical changes in teachers' practices. The question is, which training organizations could improve the transition to pedagogical action and the conception of pedagogical resources that could be *reused* more widely in communities of teachers? According to Rabardel's theory, we will consider pedagogical resources as artefacts becoming *instruments* when integrated by teachers in their own practice. *Which type of pedagogical resources could facilitate their implementation in a given classroom, as well as their evolution within communities of practice?*

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<sup>1</sup> Calculators including formal computation.

### 3. SFoDEM, a distance training organization

The integration of ICT calls for new mechanisms of professional development which provide *continuous* long-term support for teachers in their efforts of pedagogical action. In this way, an *evolving network* of teachers was introduced in the USA to develop *usage scenarios* for geometry software, even before the means fully existed (Allen 1996). The relevant idea of these usage scenarios (Vivet 1991) acknowledges the necessity of taking into account the pedagogical organization of a class and the role of the teacher. Such usage scenarios may be considered as a first approach of didactical exploitation scenarios (§2). Another training organization has been developed around units integrating usage scenarios and *accounts* of classroom exploitations of these units by teachers in training (Guin, Delgoulet & Salles 2000).

The latter approach has been extended to SFoDEM through employing a distance platform (SFoDEM is piloted by IREM<sup>2</sup> and supported by institutions at the regional scale and the Ministry of Education). In the region of Montpellier, teachers are rather old; therefore, implementing new methods of teaching is particularly difficult because they have deeply established practices. Moreover, they have very few experiences of collaborative work, whereas ICT integration requires an effective collaboration between teachers to overcome its complexity. The IREM of Montpellier had a base of pedagogical resources and a training network towards the use of ICT. However, the usual 3 days training courses organized proved to be inadequate to face ICT integration in standard classrooms. Insufficient attention has been paid to teachers' concerns on ICT use in their own environment.

Therefore, the main objective of SFoDEM was to provide a continuous support for teachers in the *conception, appropriation and experimentation* of pedagogical resources to get over the crucial transition to the pedagogical act. This requires a collaboration to be built between teachers with different teaching experiences aimed to support their day-to-day practice. Various themes were chosen (transition from numerical to algebraic setting and ICT, graphic and symbolic calculators, experiments of teaching sequences towards dynamic geometric diagrams, simulation of random experiences and cooperative problem solving via Internet) to find invariants in distance training viable beyond the organization and these studied themes. This type of training organization requires to deeply re-think the *structure* of pedagogical resources. Resources should be designed which can be adapted in various environments with different configurations, moreover in order to facilitate the search of resources, the appropriation of them by users, the *mutualisation* and *reuse* of resources through the possibility to adapt them. Finally, to facilitate the *implementation* in the class of various software such as Cabri, Géospace, Excel, Derive etc. and various classroom organizations; essentially, resources have to *evolve* enriched by the experience of users.

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<sup>2</sup> Institut of Research on Mathematics Teaching (<http://www.univ-irem.fr/>)

SFoDEM is piloted by a leadership team of three researchers and its platform is managed by an administrator. About 15 trainers are involved in the training network and every year since September 2000, about 100 teachers volunteer to participate in this project. The *training committee* (composed of the leadership team, the administrator and the training network of trainers) manages the *coordination* of the five themes: first experiments on distance teaching have pointed out the necessity of compensating distance with an established structured and controlled organization and showed the crucial role of planning and *regulation*. Regulation is carried out at a global level by the training committee relying on a regular assessment with *barometers* based on questionnaires. The organization alternates face-to-face meetings and distance periods (the trainers of each theme have a face-to-face meeting each week, the training committee each month, and each theme -trainers and trainees- meets four times a year).

#### 4. Implementation and evolution

##### 4.1 First difficulties

First, this organization has rapidly revealed that schools equipment is frequently *inadequate* or *inaccessible*. Second, trainees were not adequately trained with the softwares involved, as nevertheless required to participate in SFoDEM. But mainly, there was a *reluctance* to take an *active part* in exchanges within this controlled organization and a reluctance to fill barometers, because evaluation is highly unusual in French teachers training context. Moreover, collaborative work is far from being spontaneous among French teachers. Thus, the trainers are charged to find ways to create a confident atmosphere, an active participation of trainees, enhancing the value of their work and *elaborating a community of practice* within each theme. Moreover, customary working modes were also deeply questioned within the training committee because usual trainers' strategies were essentially based on *imitation* strategies where trainees were asked to take the position of a student.

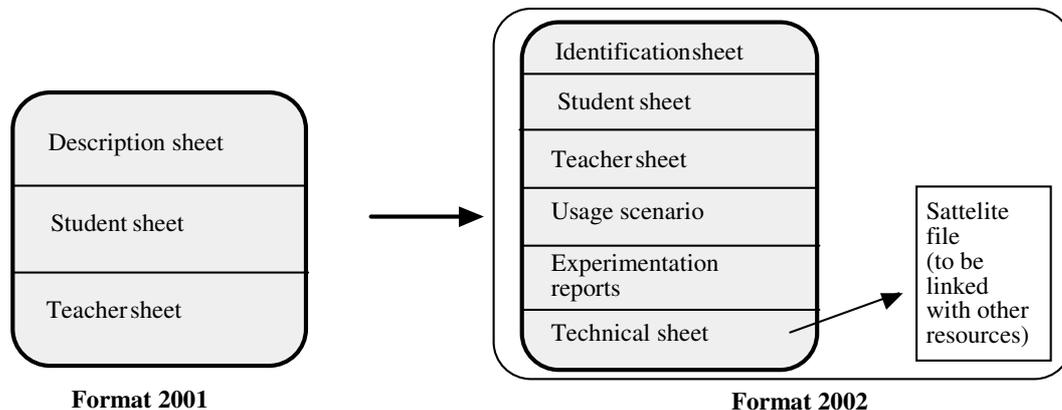
The first change was to *make explicit* rights and duties for all actors involved in the organization within *charts*. These charts are reference texts explaining in detail tasks and working modes of each community (trainers/trainees/leaders) and interacting modes with the others. Charts underline the fact that distance working modes require agreement with a strict schedule and the unavoidable act of writing down (and consequently, *making explicit*) *didactical choices* which usually remain tacit for teachers.

Moreover, initial resources provided by trainers, often expert resources, were too complex for an experimentation by trainees in their own class. Then, there was an evolution towards simpler resources, easier to implement and towards *virtual workshops* of trainees creating resources from initial ideas, named «germs of resources». This evolution may be considered as an evolution from a top-down approach towards a bottom-up approach. A face-to-face final meeting on various themes was organized between trainers and trainees in order to share resources

produced by trainees, and it showed the diversity of approaches for the integration of ICT. They really appreciated to have an overview on resources achieved in other themes. It was also the only way possible to *valorize* trainees' work and make it visible, because there is no institutional recognition for this type of work in France.

#### 4.2 The model of resources

Distance working requires, on the one hand, to make explicit essential information of pedagogical type or technical-type which remains *tacit* when used in a face-to-face environment (for example, configuration of material and software tools). On the other hand, it requires us to write down the resources apart from a particular software tool, separating technological and pedagogical levels, student and teacher documents. The model of resources was elaborated within the training committee from the available resources to adapt them for the needs of a distance organization. Moreover, this model was aimed to facilitate the evolution of resources after trainees' experimentations in their own class. Then, this model was afforded to trainees for rewriting resources to validate the model.



**Figure 1 – The model of pedagogical resources**

It is composed of *indissociable* elements which underpin the resource (Figure 1). An *identification sheet* describing the activity, its context in the syllabus, and the conditions of its implementation in the classroom (technical aspects and others). A *student sheet* describing the student's activity. A *teacher sheet*, with pedagogical objectives relating to the official syllabus and prerequisites, pointing out the pedagogical interest of ICT use for effective learning. A *technical sheet* facilitating the technical appropriation, describing software and configuration, directions for use specific to this resource with links to *satellite files*. These files may include information, technical and mathematical knowledge shared with other resources (the idea was to «factorize» information as soon as possible); a *usage scenario* describing the task for each unit, its approximate length, tools and devices utilized and the teacher role in the management of this situation. Several scenarios may be described for the same activity, according to the diversity of teachers' behaviours using ICT as previously mentioned. These scenarios will be modified according to *experimentation reports* completed by trainees after experiencing the resource in their class.

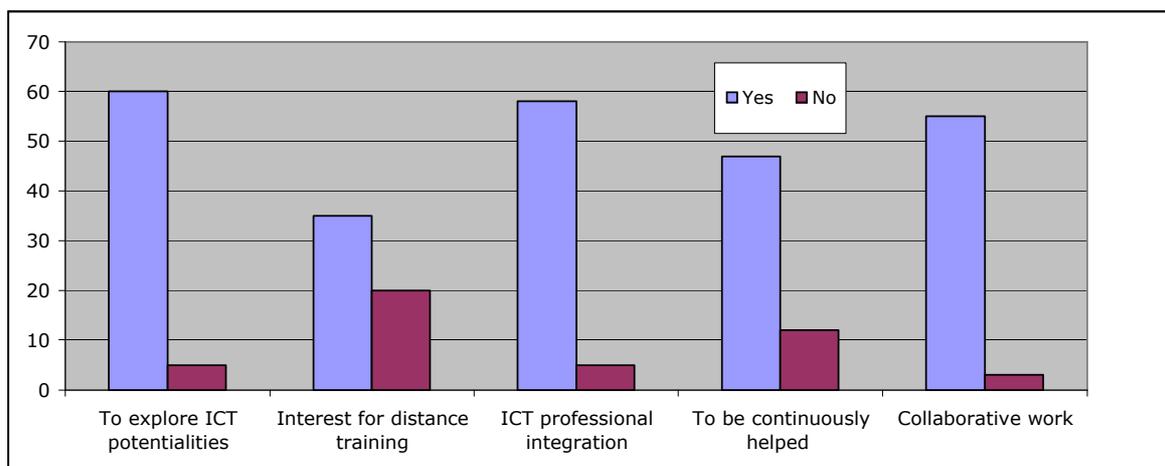
Thus, the resources design process is iterative, combining top-down (from a given model) and bottom-up (from users' experiences and experiments) approaches. The idea of *designing through usages* (§ 2) is at the center of this process where resources are considered as instruments built by trainers and trainees in the SFoDEM communities of practice. Such scenarios may become *germs* for future didactical exploitation scenarios (§2).

### 4.3 Some results

The results of the *experimental phase* of SFoDEM are available in a CD-Rom (Guin & al 2003) composed of examples with animated resources produced by trainers and trainees within each theme. This CD-Rom includes a presentation of assessment tools (essentially barometers, end of § 3), collected data, and trainers and trainees reactions to these experiences.

Through these barometers, the trainees were questioned about their interest in this distance training organization, their personal equipment, the facility for loading resources, their prior (before experimentation) analysis of these resources, and a posteriori analysis.

Mainly, it turns out from the analysis of these barometers that SFoDEM can be considered as a first answer to teachers' interests and needs (Figure 2): interest for ICT integration and need for a continuous support and a collaborative work.



**Figure 2 – Trainees's reasons (2001) for choosing SFoDEM (76 answers among 121 trainees)**

Most teachers have easily loaded resources and they consider that the content of the provided resources is clear, complete and useful for the class. However, there are still many who do not dare to use these resources in the classroom.

Finally, the analysis of the questionnaires essentially points out that the implementation of new working modes requires a deep individual involvement of trainers and trainees which cannot be expected in the short term. Furthermore, *working memory update* within each theme still remains at an embryonic state, despite the fact that it is essential for distance working. Nevertheless, the emergence of a common structure had positive effects on the evolution of resources (Figure 3).

The *operational phase* began in September 2002 to test the organization efficiency, modified according to the evaluation results: a *technical committee* of two people was added in order to relieve trainers from technical problems and from the mediatization of resources, allowing them to focus mainly on didactical aspects of resources. Due to the progressive awareness of the *coordination* central role in this type of distance organization, another person was also added to coordinate the three themes retained for the operational phase. From collected data in this new phase, one may notice a significant improvement: less retirement, a more important trainees' involvement, a better management within each theme of the working memory. Usage scenarios play a central role for resources implementation in classrooms while including supporting notes to help teachers put the unit into practice. Separation between pedagogical and technical levels ensures the resource *more independence* as regard to the technical environment. Experimentation accounts are essential to have resources *evolve* among teachers' communities. This evokes the idea of *living pedagogical resources* (Figure 3). Nevertheless, compared to the deep involvement required from teachers, the *lack of institutional validation and certification* of ICT-based teaching skills remains the main obstacle to the success of this training organization.

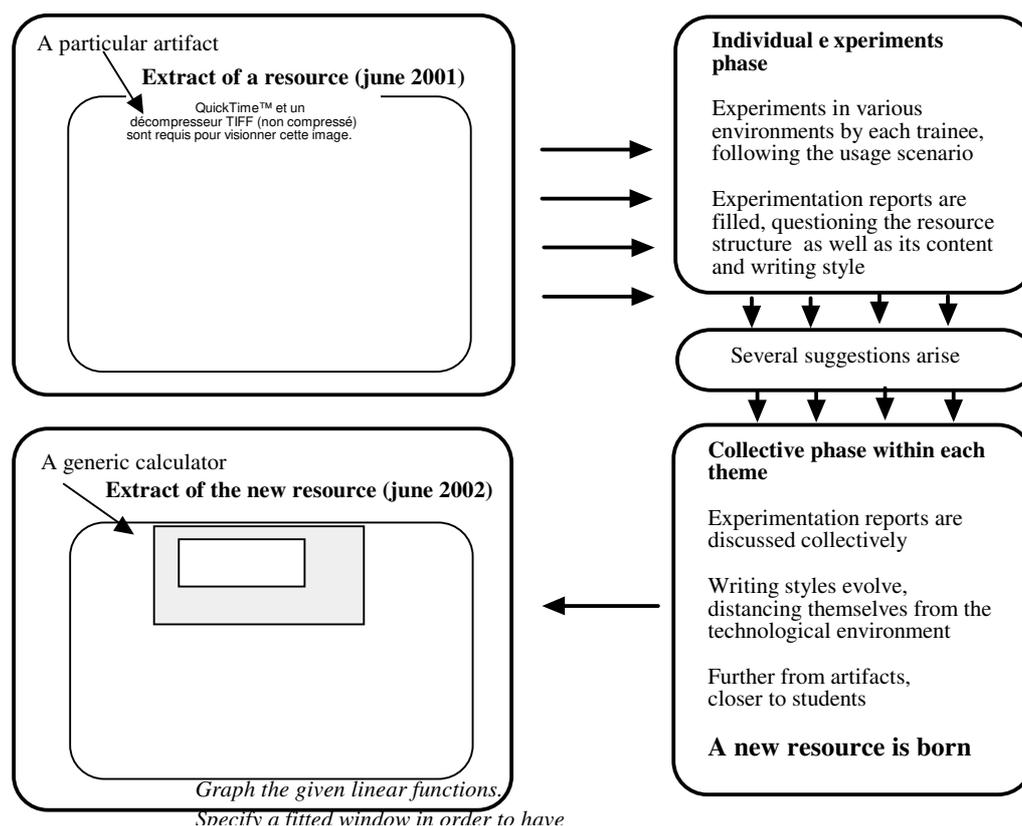


Figure 3 – Extracts from “birth and life processes” of a resource in a SFoDEM community

#### 4.4 Other projects

In the same way, the specificity of the European project INTERREGIII (2002-2004) lies on a *comparative study* of two European border regions (Catalonia in Spain and Languedoc in France) on mathematics teachers' needs towards ICT training. This study has proved two main points: the first one is that all mathematics' teachers have

a computer at home, but still more than 50% of them *never use* ICT (including internet and all types of calculators) in their own class. The second one is that 75% are interested in being involved in a distance training organization with *collaborative work*, even if it is completely new for them. Therefore, two *pilot projects* were designed with a common outline, while still taking into account the specificities of each region. The French project began in June 2004. The main objective was to bring teachers to use ICT as a mode of distance communication through creating a *virtual class* involving teachers and students in open problem solving processes. It appears that the continuous support to teachers involved in this project has constituted a precious help, for a real integration of ICT in each classroom (Combes et al 2005). Another project of the IREM of Montpellier, named AccESSIT (<http://www.irem.univ-montp2.fr>), is also devoted to mathematics teachers' support in ICT integration, but at university level.

## 5. Discussion

This paper describes some ways explored in professional development to help teachers who deal with technology-rich learning environments (theme 2 of WG9). All these experiments show the complexity of this task, but the *designing through usages* approach gives some possibilities to overcome various difficulties, essentially because these are *not underestimated*. Starting from teachers' practices, designers of training organizations need to consider *configurations* that would enhance a collaborative reflection on affordances and constraints of available artifacts, on mathematical situations design taking into account the educational context and on pedagogical resources characteristics required to *share* resources.

Each community will have its own response. These conditions are not easily met, but they are necessary to an environment where resources can be *alive* and where professional practices may evolve in the medium term. The usage scenarios, through the description of a precise piloting of a teacher, and experimentation reports are in *the very heart of the evolution process*.

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# DYNAMIC REPRESENTATIONS: A NEW PERSPECTIVE ON INSTRUMENTAL GENESIS<sup>1</sup>

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**Abstract:** *In describing a certain tool/instrument common to two software environments, “hot-spots”, we argue that instrumental genesis, relative to a student’s discovery of new mathematical concepts, does not just concern the actions of the user but co-action of the software environment with such actions. We focus on this theme because of its potential to impact a student’s long-term conceptual and procedural understanding of algebraic and geometric constructs as they begin to invest more personally in the construction of certain mathematical objects.*

**Keywords:** Dynamic Geometry, SimCalc MathWorlds, Instrumental Genesis, “Hot-spots”, Co-action, Representational Infrastructure.

## Examining Some Commonalities Across Dynamic Software Environments

For the purposes of this essay we concentrate on particular dynamic software environments including dynamic geometry environments (e.g. Geometer’s Sketchpad, Cabri II+) and simulation software (e.g. SimCalc MathWorlds, a dynamic algebra environment). There are various *anatomical* features that these types of software share including:

- Navigation - ability to move around the screen, move mathematical figures, scroll & zoom coordinate systems, scroll around simulation worlds,
- Interaction – click and hold and drag or manipulate objects,
- Annotation – marks, literals or numerals can be added (and adhere to) parts of figures and diagrams,
- Construction – mathematical figures or diagrams can be made piecewise through specific tools,
- Simulation – allow objects that are part of, or associated to, the figures or diagrams, to be animated, or model data and observe a simulation of these data,
- Manipulation – constructed figures or diagrams can be changed by interacting with particular features of the construction, while preserving mathematical rules within the construction.

One of the key infrastructural pieces of these software that allow many of these features to operate is the existence of “hot-spots”. These are points that can be used to

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construct mathematical figures, e.g. join two points with a segment, or construct a piecewise graph, and then used to dynamically change the construction. We shall now discuss the existence of such phenomena from a theoretical perspective with respect to user-environment interaction, using examples to describe how such interaction is a sustainable bi-directional process that has the potential to ground and develop certain mathematical concepts.

### **Links to Existing Theory**

Our discussion is centered on the examination, and use of, tools and artifacts within software environments. It aims to parallel and offer a different perspective of the complex construction of instrumental genesis (Verillon & Rabardel, 1995) with dynamic software focusing on impact of students' interactions with representational tools that are *infrastructural* as well as the software environment's co-action with such use. Related work investigating the use of Computer Algebra Systems (Drijvers, 2000; Guin & Trouche, 1999) distinguish between instrumentation – how tools affects and shape the thinking of the user – and instrumentalization – where the tool is shaped by the user. Instrumentation is more evolutionary as mental schemes (Vergnaud, 1996) emerge as users execute a task. As the task is completed the uses of a certain tool becomes internalized. Instrumentalization is a psychological process which develops ways of using, manipulating and shaping the artifact in use, an organization of use-schemes, a personalization and sometimes transformation of the tool, and a differentiation between the complex processes that constitute instrumental genesis and which are critical for teachers to master (Guin & Trouche, 2002). Whilst we agree with the constituent parts and definition of this process we believe that it lacks a deeper, fundamental role of the environment and co-actions with the user's intentions of a tool. These go past retro-actions which are fundamental to the user's future-actions, and are more appropriated to the environment, and the tool's reshaping subject to environmental factors established by the software. We aim to extend the theoretical notion of instrumental genesis to include the idea of co-action, a symmetric notion, which stresses the importance of the role of the environment the tool is being used in and the dialectic process between the user, the tool and the environment.

### **Static Tools and Infrastructural Tools**

Consider a traditional use of a tool: the hammer. A hammer is a static object which can be environmentally coupled with other objects to produce similar actions. Today, whilst similar actions exist in software environments, we propose that “hammers” do not always exist particularly in the world of dynamic figures. Of course selecting some text and pressing Command-C or Ctrl-C will copy text or pictures to a clipboard in most modern wordprocessors - a good stimulus-response software action.

We wish to focus on the role of “hot-spots” in our chosen software, but wish to extend the definition of them as tools or instruments. The “hot-spot” in our chosen

software environments is not an artifact of the environment but an axiomatic part of the system that allows “true” mathematical figures to be built. Dragging a “hot-spot” is not the same as “using a hammer to try to hit a nail” – note the verb use. A hot-spot will always be used for dragging (in various forms and for various purposes), a hammer will not always be used for hitting well. A hotspot will always be dragged and a hammer is never hit but instead used to hit. Will they ever be the same? Well, the hammer is still as effective as the hitter. The hitter hits a particular point. The action is directed by the actor. The local environment does not help with the accuracy or efficiency of the tool use, it resides with the user and practice. In addition, the action of dragging a hot-spot leads to the software environment reacting in some way. It is also true that hot-spots could be used in an ambivalent way, dragging without any understanding of what the hot-spot-environmental coupling is constructing or preserving, a form of catachresis.

We propose that in a dynamic environment with “hot-spots” the action is not owned, in fact, agency is a collaboration between user and environment, both are actors and re-actors. Both act and re-act on each other. Basically, a co-action is always in effect. It is because hot-spots are infrastructural that our focus is made more pertinent and is the main thrust of our essay. Let us elaborate with an example.

#### *Example 1: Dynamic Geometry*

Consider a construction of an equilateral triangle in a dynamic representational media such as Dynamic Geometry Software (DGS). Constructions that do not use measuring tools are called Euclidean constructions. Gauss proved that a regular polygon can be inscribed in a circle by means of a straightedge and compass alone if and only if the number of sides,  $n$ , can be expressed as  $n = 2^k \cdot p_1 p_2 \dots p_m$ , for a non-negative integer  $k$  and each  $p_i$  a distinct prime of the form  $2^{2^r} + 1$ , for  $r > 0$ . Some of the regular polygons that are constructible, according to this theorem, are those with 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, or 24 sides. Note that the theorem does not tell us how to do the construction, only that it can or cannot be done.

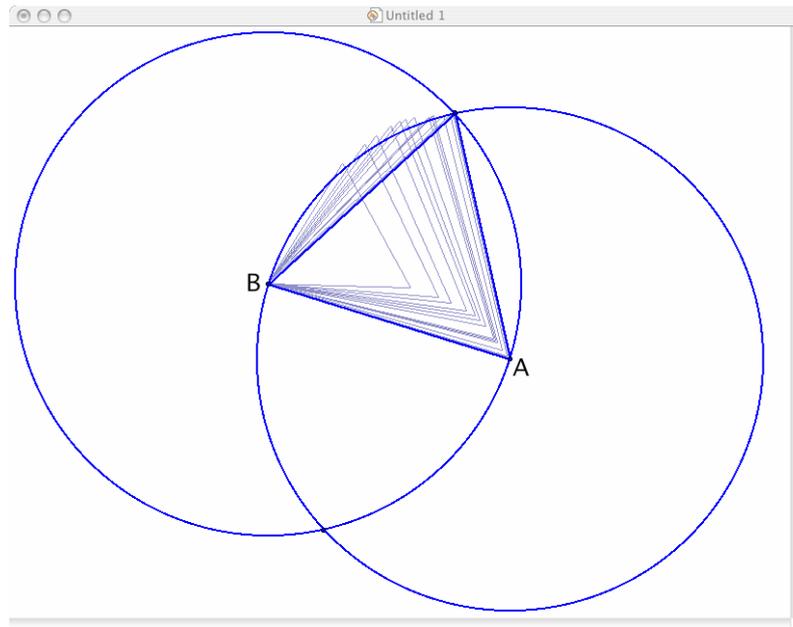
Euclid, in the first book of his *Elements*, postulated the construction of an equilateral triangle (a 3-gon) based on a line segment  $AB$ , where two circles are constructed with radii  $AB$  ( $A$  and  $B$  are centers for each of the two circles respectively), and the third vertex ( $C$ ) of the triangle is where the two circles intersect. Euler’s assumption (often debated) was that the two circles do in fact intersect (in fact twice). Now, a paper and pencil construction, with straightedge and compass, can be used to construct the triangle. But we only create one triangle. We have actively engineered the object, and our actions are more crystallized, but the medium is still inflexible.

We choose to focus our attention on dynamic geometry software (DGS) environments (particularly Geometer’s Sketchpad and Cabri II+) as they aim to develop spatial sense and geometric reasoning by allowing geometric postulates to be tested, offering “intelligent” constructivist tools that constrain users to select, construct or manipulate objects that obey mathematical rules (Mariotti, 2003).

Empirical work states how these features lead to improvement in student engagement through aesthetic motivation (Sinclair 2001), enhances students' ability to generalize mathematical conjectures (Mariotti, 2001) and aid students in developing theoretical arguments (Laborde, 2000; 2001; Noss & Hoyles, 1996).

Each DGS offers an environment where point-and-click Euclidean tools can be used to construct geometric objects that can be selected and dragged by mouse movements in which all user-defined mathematical relationships are preserved. In such an environment students have access to conjecture and generalize by clicking and dragging hot-spots on the object which dynamically re-draw and update information on the screen as the user drags the mouse, and in doing so, efficiently tests large iterations of the mathematical construction. Figure 1 below attempts to illustrate this dynamism through a snapshot of such a physical action. The sides of the triangle have been marked to leave a trace. The center A "hot-spot" is dragged from left to right, in doing so the circles enlarge, but the triangle's properties appear to be preserved in an array or family of similar equilateral triangles. Dragging the hot-spot illustrates how the Euclidean construction of an equilateral triangle has been correctly implemented in this sketch.

Indeed we have discretized this 'physical' motion of grabbing (A) and dragging the hot-spot. But what we have here is an illustration of where the user has not only actively constructed the triangle, but has the affordance of a flexible media where the diagram can be deformed, but the engineering preserved, through one dynamic action. The dynamic action allows a series of constructions to be instantly created as an embedded environmental automated process, and the medium can keep a trace of such constructions and actions, but more so, *co-actions* between the user and the environment.



*Figure 1: A trace of equilateral triangle constructions*

The “hot-spot” is a critical part of the construction. It is not just a spot (or dot) that can be moved (although the user started it off as such originally) but it fuses pieces of the geometric figure and becomes axiomatic to the figure – deleting it would delete pieces of the figure. In moving the hot-spot the figure dynamically re-constructs, so the hot-spot now has ownership of the figure, and in fact the hot-spot is intimately bound up with the mathematics of the figure, i.e. that such a construction will always produce an equilateral triangle. By marking the triangle we can see a discrete trace of the mathematical constructs inherent in the figure.

### **Reflection on What is Occurring in Dynamic Figures**

Here is the critical point: the hot spot is no longer directly owned by the user. It is an infrastructural piece of the environment from which the user is now receiving feedback. In fact it goes further than the existing theories of instrumentation and instrumentalization. The actions of the hot-spot and the figure being dragged by the user are now environmental (belonging to the software) in terms of visual feedback. So, the genesis of this figure goes from something personal, user actions, to environmental in terms of feedback. Following a construction, the diagram becomes more quasi-independent of its creator. Colette Laborde (in press) has made the point that the artificial realities of the diagram obey the rules of geometry that are preserved in the elements of the diagram, just as world objects obey the rules of physics in nature. But when an element of a diagram is dragged, the resulting re-constructions are developed by the environment NOT the user. So what becomes

important is that the environment provides useful feedback. We continue to use this point when we reflect on instrumental genesis later.

The tool, in this case a hot spot, that the user once defined as part of a construction, becomes re-shaped by the environment and the Euclidean rules that govern it. The actions of the user co-exist with the response or actions of the environment. The tool is also highly efficient (unlike our hammer example at the start) as it continues to fulfill its role without fault, as pre-determined in the construction, and so one might conjecture that this becomes an extremely useful learning tool because of this reason.

We can also discuss the physical use of hot-spots as a method to test the validity of geometric constructions (Mariotti, 2001). The dragging of well-constructed objects, to establish whether the mathematical constructs that underlie their engineering can be preserved upon manipulation offers another dynamic perspective on geometric diagrams and is referred to as a “drag test”. For example, the construction of the equilateral triangle in figure 1 is a “true” Euclidean construction as illustrated by the drag test. Such embodied actions of pointing, clicking, grabbing and dragging parts of the geometric construction allows a semiotic mediation (Pea, 1993; Brousseau, 1997) between the object and the user who is trying to make sense, or induce some particular attribute of the diagram or prove some theorem.

#### *Example 2: Dynamic Simulations*

We now offer another example from a second type of representationally rich software. Our example focuses on the coordination of piecewise linear position functions and stepwise constant velocity functions represented as graphs and motions in a software environment called MathWorlds ([www.simcalc.umassd.edu](http://www.simcalc.umassd.edu)).

MathWorlds supports the creation of graphs, which are visually editable by clicking on hot spots as well as being algebraically editable. These motions are simulated in the software so that users can see a character move whose motion is driven by the graphs they, or someone else, have constructed. Students can step through the motion, examine tables of values, and perform other operations in order to help them make qualitative and quantitative inferences about the motions represented by the graphs; all representations are linked.

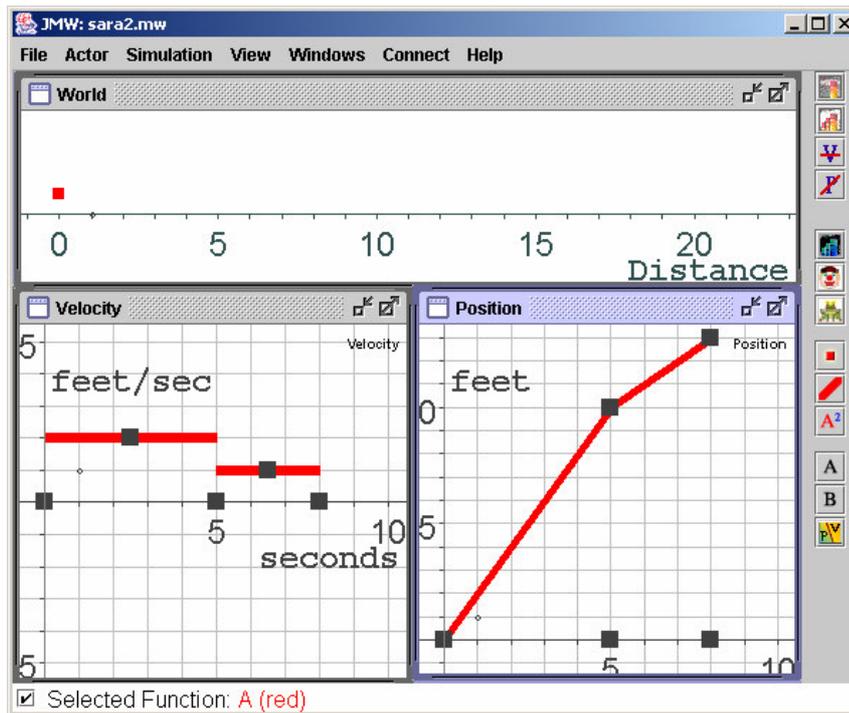


Figure 2: SimCalc MathWorlds: Rate and Accumulation

Software runs on hand-held devices (for example, the TI-83+ graphing calculator or the Palm) as well as desktop cross-platform PC's (as a Java Application). Figure 2 illustrates a screenshot from the PC version of MathWorlds. An actor A, is depicted by a red dot in the world (horizontal in the top third of the screen). This actor's motion is driven by the piecewise function graphically visible in two forms in the lower half of the screen.

Observing the Position-Time graph in the window to the right you can see hot-spots on the end of each segment, coordinates (5,10) and (8,13) as well as two hot spots on the time axis (5,0) and (8,0). We have parsed the two actions of vertical dragging, to change the slope of each piece (2 then 1 foot per second), from horizontal dragging to change the duration of each piece, to allow students to examine each covariate separately. So the hotspots here are tools for the user (to change the data that drive the actor's motion) but the software environment offers feedback (every time) which is consistent mathematical feedback. If the slope of the first piece is changed by the user dragging the hot spot at (5,10), the first segment of the Velocity-time graph changes also. So if the slope increases the constant rate piece increases by the same amount, fusing the relationship between position (accumulation) and velocity (rate) a fundamental Calculus principle that is being made accessible through executable representations in MathWorlds. Also, the actor cannot disappear for a moment of time and so the position pieces are continuous (and are forced to be so in the software). Once again the hot-spots are infrastructural and once the user has used the tool, this tool or instrument, which is embedded in the environment, executes a series of actions on the representation. As in the geometry example, the mathematics of the

construction are axiomatic to how the environment behaves and are part of the environment.

Kaput (2000) highlights how hot-spots are embedded in two of five innovations that constitute a representational infrastructure for the MathWorlds environment. These include definition and direct manipulation of graphically editable functions, and direct, hot links between graphically editable functions and their derivatives or integrals. Others include, connections between representations and simulations, the ability to import physical motion, and re-animate it, and the use of hybrid physical/cybernetic devices embodying dynamical systems. These are realized in a new media for carrying representational infrastructures.

So such “tools”, as hot-spots, are actually instantiated at an infrastructural level and are a product of new, dynamic medium.

Whilst this example highlights the functionality of such a tool in MathWorlds, actual classroom activities that we have devised make use of one or more actors that we have either pre-defined in an activity document and that the student has to interact with or are the product of the student’s work. An example of the first would be to make a motion for Actor A graphically that matches the motion for Actor B, except we previously hide the graph for Actor B. Here the use of the “hot spot” undergoes a shift in utility from being a tool for the user to an executable representation in MathWorlds.

### **An extended sub-example to networked classrooms**

Recent work (e.g. Hegedus & Kaput, 2003) has combined the use of MathWorlds with the latest advances in classroom connectivity, where multiple functions constructed on hand-held devices can be aggregated into the PC version of MathWorlds and projected onto a whiteboard. Now multiple representations can be executed in a social context, where students’ personal contributions make an interesting gestalt in terms of their collective motion or as a family of functions. These can be hidden and displayed as needed so that the teachers can focus the attention of the students’ work in meaningful ways. Varying constructions across naturally occurring groups in the classroom give rise to a suite of interesting mathematical activities, beyond the scope of this paper, but an important emerging example of how the new ingredient of networked classrooms (now including hand-held devices to computers) is leading to a new emerging environment characteristic, new representations of mathematical objects are being shaped and formed by multiple contributions. So the teacher can choose to interact with an aggregation of mathematical objects for a variety of pedagogical purposes because the environment now allows the interaction of multiple constructions.

### **A Different Perspective on Instrumental Genesis**

We are offering a different perspective on instrumental genesis, which adheres to the existing process in the utilization of tools as a relationship between instrumentation to instrumentalization as summarized at the start, but which extends it with respect to

the environment that the user interacts with. In thinking of hot-spots as not only tools or instruments but infrastructural to the software environments, as being intimately bound up with the mathematics that is preserved in the software (in the routines of the program), e.g. continuity, multiple representation of functions, Euclidean, then tools as instruments can be perceived in a slightly different way. For instrumentation, we additionally define it as *how co-actions with a tool shapes the user's actions and understanding of the use of such a tool within, and with respect to, an environment*. Instrumentalization is extended to *how the tool is shaped by the user (user's knowledge) and the environment*, i.e. when the tool is manipulated by environmental factors following a user-input. So an instrumental genesis can be extended to include simultaneous co-actions between a user's use of a tool and a software environment's use of a tool, the feedback and reaction of a user being a certain process of utilization, internalization of the how the tool is manipulated, used by the environment, and then re-used by the user.

### **Implications for Teaching and Learning**

The central theme of such dynamic representations is that the representational infrastructure offers a secure scaffolding that is grounded in the mathematical structure (axiom, definitions, rules) that are efficiently preserved when the representations are executed. The student as user has the support of rigorous scaffolding deep in the infrastructure that is extremely difficult to replicate in static, inert media. Mathematical constructions in algebra and geometry become more dynamic, motion based events, with explorations, conjectures and reasoning based around the aggregation of mathematical objects or co-actions of students and software environment.

The net effect is an impact on pedagogy, which leads to serious contemplation about the nature of the activities, the facilitation and well-structured questions that guide and nurture discovery. A deeper discussion of which is beyond the scope of this paper.

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# SUSTAINING INTERACTION IN A MATHEMATICAL COMMUNITY OF PRACTICE

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**Abstract.** *This paper focuses on an activity in which students explore sequences through a game, using ToonTalk programming and a web-based collaboration system. Our analytical framework combines theory of communities of practice with domain epistemology. We note three factors which influence the length and quality of interactions: facilitation, reciprocation and audience-awareness.*

## **Introduction**

This paper tells the story of an experiment to design a mathematical community of practice, in the course of the *WebLabs* Project, a 3 year EU-funded educational research project oriented towards finding new ways of representing and expressing mathematical and scientific knowledge in communities of young learners. Our work focuses on the iterative design of exploratory activities in domains such as numeric sequences, cardinality, probabilistic thinking, fundamental kinematics, and ecological systems. In this paper, we will focus on an activity called *Guess my Robot*, which is aimed at advancing students' understanding of number sequences. We use that activity to explore the following question:

*What are the factors that sustain interaction in a mathematical activity over a web-based collaboration medium?*

Our analysis is informed by the notion of 'community of practice' as it is used within the situated approach to learning (Lave and Wenger, 1991; Wenger 1998). The insights we gain from this analysis are fed into the next iteration of the activity design. Thus, we have built on our initial observations of communities to actively cultivate their existence.

Wenger proposes three dimensions of practice as the property of a community:

- Mutual engagement: a sense of "working together". Sharing ideas and artefacts, with a common commitment to the interactions between members of the community.
- Joint enterprise: having some object as an agreed common goal, defined by the participants in the very process of pursuing it, not just a stated agenda but something that creates among participants relations of mutual accountability; that become an integral part of the practice.

- Shared repertoire: agreed resources for negotiating meanings. This includes routines, words, tools, procedures, stories, gestures, symbols, and so on. Artefacts that the community has produced or adapted in the course of its existence and have become part of its practice. The repertoire combines both reificative and participative aspects. It includes the discourse members use to create meaningful statements about the world as well as the styles in which they express their forms of membership and their identities as members.

To these we add an epistemological dimension, in that we intend to encourage the formation of *mathematical* communities. That is, we are trying to generate communities of practice – both physically and virtually – in which there are agreed socio-mathematical norms, where it is natural to make conjectures, test hypotheses, offer counter-examples and so on. By restricting our attention to a specific domain of mathematical activity, we commit ourselves to make specific and concrete claims. Our focus on design provides us with a unique opportunity to go beyond explanatory observations. We can verify our claims by changing the activity system and monitoring predicted change.

### **WebLabs, ToonTalk, WebReports and the Guess my Robot game**

*WebLabs* utilizes two main media for its activities: *ToonTalk* (a programming environment) and *WebReports* (a web-based collaboration system). We see programming as playing a key role in individual and group learning. Students explore and test their conceptions of the phenomena through programming. Furthermore, by sharing programmed models, they can communicate ideas in a concrete yet rigorous form. We are programming with *ToonTalk*<sup>1</sup> (Kahn, 1996; 1999; Mor et al., 2004) a language used in the past with younger children to construct video games (Hoyles, Noss & Adamson, 2002). *ToonTalk* is a computer game, programming environment and programming language in one. In *ToonTalk* programs take the form of animated cartoon robots. Programming is done by training these robots: leading them through the task they are meant to perform. After training, programs are generalised by “erasing” superfluous detail from robots’ “minds”.

The individual and collaborative facets of learning are intertwined at all stages of our activities. The *WebReports*<sup>2</sup> system was set up to support both. The primary aim of this system is to allow learners to reflect on each others work by sharing working models of their ideas. The “atomic unit” of content in the system is a web report: a document containing formatted text, multi-media objects and most importantly – *ToonTalk* models. Reports are edited using a visual editor. Students can grab any model constructed in their *ToonTalk* environment, and copy it instantaneously into their report. These models are embedded in the report as images, which link to the actual code object. When clicked, they automatically open in the reader’s *ToonTalk* environment – which could be in another classroom or another country. The reader can then manipulate the object, modify it, and even respond with a comment that may include her own model. This last point is crucial: rather than simply discussing

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<sup>1</sup> <http://www.ToonTalk.com>

<sup>2</sup> <http://www.weblabs.org.uk/wlplone/>

what each other *thinks*, students can share what they have *built* and *rebuild* each others' attempts to model any given task or object.

Our activity design methodology exploits the affordances of the system. The initial discussion of a phenomenon can lead to the group's publishing a report on their observations, conjectures, and suggested path of inquiry. Finally, when a task or activity is completed, a concluding report will be published by either individuals or the group, to share conclusions with remote peers.

One of the experiments we have conducted in the course of the WebLabs project was a game called Guess my Robot. The activity we designed was based on the "Guess my rule" game, an activity well-known to many teachers and researchers as a way of encouraging students to discuss and compare the formulation of rules, and in particular the equivalence (or not) of their algebraic symbolism. It has also been employed in the context of Logo and spreadsheets (c.f. Healy & Sutherland, 1990). In its classical form, it has been used as an introduction to functions and to formal algebraic notation. As Carraher and Earnest (2003) have recently reported, even children in younger grades enjoy participating in this game, and can be drawn into a discussion of algebraic nature through using it.

We first experimented with the Guess my Robot activity in 2002/3 (Mor & Sendova, 2003). Our experience from this pilot informed both the design of the activity and of the collaboration system. In 2003/4 we expanded the experiment, with significantly greater response. This iteration included 33 students from 6 sites (in different European countries). There are several differences between our version of the game and other variations. Most notable is the media by which it is conducted, and the specific rules of game inspired by those. In our game, *proposers* (students) invent a rule for a number sequence and model it as a ToonTalk *robot* (procedure) that generates that sequence. They then collect the first few terms of its output in a ToonTalk *box* and embed it in a web report. *Responders* can click on the image of the box, and explore its contents in their own ToonTalk environment. They use a variety of tools to uncover the rule of the sequence: ToonTalk programming, Excel and (even!) paper and pencil. Once they succeed, they respond to the challenge by posting a comment on the report, which includes a robot they created for generating the same sequence.



**Figure 1: Rita's Guess my Robot page**

Figure 1 shows an example of such a challenge. It was posted by Rita<sup>3</sup>, a 14 year old girl from Portugal. This example will accompany us throughout this paper. Rita's challenge provoked several different solutions, which led to long threads of interaction, some of which included fairly sophisticated mathematical arguments. Not all of our data is so impressive: overall, 45 challenges and 33 responses were posted. However, only 17 of the challenges received any response at all. A lot can be said about those challenges and responses – their mathematical structure and its relation to the tools used; the forms of expression which evolved through the game; how students construct their challenges, and how they select a challenge to respond to; the evidence all these present on questions of meta-cognitive skills and practices and so on.

### **Data and methods**

The present dataset encompasses 33 students from 6 sites, 15 girls and 18 boys, ages 10 (2), 11 (10), 12 (16), 13 (2) and 14 (3). Challenges were posted between 26<sup>th</sup> December 2003 and 5<sup>th</sup> May 2004. The last response was submitted on 28<sup>th</sup> May 2004. Overall, 45 challenges and 33 responses were posted. Only 17 of the challenges received a response (obviously, some received more than one – a maximum of three per challenge). However, there are 114 comments altogether, up to 30 per a single report (3<sup>rd</sup> quartile at 3.25). The subject group is highly diverse. Each site had its own characteristics in terms of student selection, class setting, age, ethnic background, gender, and teacher-student ratio.

From a methodological point of view, one of the advantages of using a web-based collaborative system is that it is a self-documenting medium. All the challenges and responses posted by students, as well as any verbal comments, are archived and dated on the system. This data is abundant and easily accessible. Yet at the same time

<sup>3</sup> We use the aliases, or “handles” children chose for themselves in the web reports system. With the system's access restrictions in mind, we can use these as anonymized identifiers.

it is shallow: it does not record the classroom interactions or the problem-solving strategies used by the students. Analyzing this data cannot provide answers about personal and group learning trajectories, but it can point to interesting questions, such as:

- Students developed an ability to flow between different representations of the same sequence. In what ways does this ability affect their understanding of the mathematical objects they manipulate and the methods they use?
- The structure of the game requires participants to make conjectures, model them by programming, and test them. Does this facet of the activity influence students' mathematical argumentation?
- We identified several canonical structures of sequences which appeared in many challenges and in different sites. These structures are notably different than those taught in standard curricula. What are the epistemological sources of this difference, and what are their implications?

These questions are then explored by looking at field notes, session recordings and interviews across sites. In this paper we wish to focus on one theme, the issue of sustaining interaction in a mathematical game, within a web-based collaborative system. The next section elaborates this question.

### ***Sustaining mathematical interaction***

It is clear that sustaining the kind of interaction we seek is strongly contingent on the domain, the activity structures, and, of course, the tools that we offer to students. Nevertheless, as in any learning environment, the epistemological, cultural and social factors are intertwined. Thus, our answers cannot be detached from social and cultural considerations.

Asking how to sustain interaction implicitly suggests that it is a positive force. Yet this is itself a claim that needs to be scrutinized. In the case of Rita's challenge, the first responses were bare robots. As the interaction developed (in fact, in several concurrent threads) students went deeper and deeper into the questions that emerged from the situation: equivalence of models, solution strategies and even notions of proof. Participants shifted from the competitive and somewhat technical base level of the game to a collaborative effort of understanding the mathematical structure of their models, and sharing of analytical tools.

Assuming we accept sustained interaction as a desirable phenomenon, we need to look closely at the cases where it occurs and try to identify their unique characteristics. We should obviously pay closest attention to cases where the interaction is distinguished not only by quantity but also by quality. That is, quality of the mathematical and meta-mathematical discussion exhibited in the interaction. There are 3 main themes that have emerged from our preliminary observations: facilitation, reciprocation and audience-awareness.

## Facilitation

Our first conjecture regards the role of the facilitator. As Wenger et al. (2002) note, “*Alive communities, whether planned or spontaneous, have a ‘coordinator’ who organizes events and connects community*”. We assert that this role of coordination, or facilitation in our terms, is critical in maintaining the dynamics of the game. Facilitation takes on three forms:

- Technical: providing technical apprenticeship on how to use the system, e.g. how to post a response; pointing teachers and students to interesting postings.
- Pedagogical: setting new challenges to participants; noting the mathematical or computational aspect of postings to teachers and students.
- Sociomathematical: shifting the conversation towards mathematical content. In the terminology of Yackel & Cobb (1995), establishing the sociomathematical norms of the game.

At first, the Bulgarian students posted their response in a separate report. Yishay copied the text and the robots from their reports and posted them as comments on Rita’s challenge. He then e-mailed the teachers at both sites about this. Obviously, this is not a very interesting event to report. Nevertheless, none of the following discussions about sophisticated mathematical ideas would have occurred without it.

As an example of promoting sociomathematical norms, consider the following comment posted by the London researchers:

This is a question from the London team (Richard, Celia, Ken, Yishay and Gordon) to all three of you:

We think your robots will generate the same sequence for ever, but how can we be sure?

This question provoked students in both sides to think about the question of equivalence. The Bulgarians approached this question by working it out algebraically in a group. Rita considered this option, but thought that the rules of the game restricted her to using ToonTalk. Her solution was to construct a robot that compares two sequences by subtracting respective terms. She explains:

Clearly that this is not a prove of that robot produces the same sequence, that is only one conjecture, or either, I have 99% of sure that they are equal, but still did not can to get a demonstration.

One of the responses to Rita’s difference robot is an example of a pedagogical intervention. Gordon comments:

Wow - this is really great work! Did you know that you could actually create other sequences using the difference robot that you built? I.e. if the two robots you send off in the trucks don't generate the same sequence, then your difference robot will generate a sequence of non-zero numbers. Try it!

Gordon suggests a new challenge, based on the work that Rita had published. Unfortunately, at this point we have to report a lack of success. Rita responded politely, but did not pick up the challenge. Her teacher’s field notes reveal an explanation: she answered the comment, and was disappointed not to receive a response from Gordon. It was not a lack of interest in the mathematical problem, but

rather a suspicion that Gordon would not maintain the interaction on his side. We will return to this important observation later, when we mention the issue of presence.

Using a web-based medium eliminates constraints of organizational structure. An expert in London or Portugal can facilitate activity in a classroom in Cyprus. The WebReports system includes several features which aid facilitation. For instance, challenges are listed automatically, with the number of comments they received. The facilitator can identify challenges which have not been responded to, and use the system's messaging facility to invoke other participant's awareness to them. Whenever the facilitator identifies a common technical or conceptual problem, she can publish a tutorial which addresses it.

### **Reciprocation**

A second theme we identify is reciprocation. Under some circumstances, students feel a stronger obligation to reply than others. These circumstances may have a social element, for instance the sense of obligation is stronger when a comment is posted by a group of students or by a teacher. On the other hand, a very strong element in reciprocation is a socio-mathematical factor: participants sense they should "give something in return" for a positive experience, and solving a tough challenge is seen as such. Thus, participants' tendency to respond rises with the difficulty of the challenge. This conjecture addresses not only the frequency of responses, but also their quality: when the challenge was gratifying, students respond with more than their solution, adding unexpected levels of mathematical discourse to the interaction.

When Nasko posts his response to Rita's challenge, he adds:

Here is also a sequence generated by the same robot. Two questions:

1. What was the input of my robot?
2. Can your robot generate it?



Nasko's response dissects the *process* of generating the sequence from its *initial conditions*, giving rise to the idea that the same process can produce different mathematical objects.

Rita responds in two stages. First, she reciprocates on the social level – congratulating Nasko on his response, and sharing her original model with him. She explains to her teacher that she should respond immediately so as not to discourage him. Only then does she set on solving his challenge. After she does that, she reciprocates on a domain knowledge level, by posting her solutions.

The flip side of this phenomenon is that students do not respond to challenges they see as uninteresting. Sometimes, a student might pick up a simple challenge as a "drilling challenge", but will not invest in posting her solution. At the end of the

activity, we asked students to publish a concluding reflective report. When asked about the responses to her challenge, one girl responded:

I don't receive any comments to my sequence, because is to easy...

Reciprocation is so natural in classroom practice that it goes unnoticed: a teacher acknowledges a student's remark; students support each other's claims. In a web-based environment it raises tensions which we need to accommodate. Teachers need to actively seek students' contributions and react to them, less the students feel unnoticed. Other issues arise from the need to adjust to asynchronous communication: at the beginning of one session, Rita posted a comment and then sat back, waiting for a reply, growing frustrated by the minute. Her teacher had to explain that although she could see several members of the community on-line, they might be occupied with other activities and unaware of her comment.

On the positive side, streamlining the ToonTalk objects into the text of the reports had the effect of enriching students' interactions. When Nasko posted his robot as a response to Rita's challenge, she reciprocated by posting hers. This gave rise to the question of comparing the robots and asserting their equivalence. Since robots, as coded objects, are by nature formal structures, the discussion took a much more formal tone than may have been the case with bare text.

### **Audience-awareness**

Our last conjecture is perhaps the most socially-oriented. We find that two characteristics of a participant provoke response to her contributions: cordiality and presence. The first is almost trivial – participants respond more eagerly to friendly, inviting comments. The second is accentuated by the medium we chose, and in a way related to the issue of reciprocity. We find that participants prefer to interact with peers who project a strong presence. (e.g. appear on the “active users” list, post frequent comments, have a rich home page). Our conjecture is that this stems from the fact that participants are in fact interested in sustained interactions, and thus prefer to communicate with peers (or researchers) from whom they expect a higher probability of response. This entails immediate implications for us: participants are set back by one-off comments, and researchers should refrain from commenting if they do not intend to participate in subsequent discussion.

An example of this idea has been mentioned above: Rita did not attempt to solve Gordon's challenge because she suspected he might not be available to appreciate her response.

On the positive side, a team of Cypriot students replied to Rita's challenge nearly a month after the previous interactions. Because they identified themselves as a team, Rita felt a stronger commitment to her audience. She felt obliged to reply to the Cypriots, and to do so thoughtfully. The Cypriots volunteer an explanation of their solution strategy:

1. We copied Rita's numbers in Excel, to be easier to find relations between the numbers and especially the differences.
2. We found the differences between the numbers on that sequence.

3. We noticed that differences between numbers could be calculated if we multiply every one difference by 4.
4. So, we decided that we could work with formula  $4 * \text{number}$ .
5. To get Rita's sequence, we had to add 8 to the previous formula. The final formula is  $4 * \text{number} + 8$

Best

Cyprus Mathematics WebLabs Team

And Rita responds by taking the role of the facilitator, and elevating the discussion:

I can prove that my sequence and your sequence are equal with the process of algebraic representation used by Sofia group.

Rita's sequence:

$$A_1 = 2$$

$A_{n+1} = (A_n + 2) \times 4$ , but if I using the distributive property of the multiplication relatively to the addition I can write that:

$$A_1 = 2$$

$$A_{n+1} = A_n \times 4 + 8$$

That is the algebraic representation of the Cyprus's sequence. Then I can prove that two sequences are equal.

## Conclusions

In this paper we have explored the question of sustaining interaction in a mathematical activity over a web-based collaboration medium. Our approach attempts to interleave the theoretical framework of communities of practice with epistemological observations arising from the specific knowledge domain of number sequences. As a case study, we have chosen one of our experiments involving a game called *Guess my Robot*. Our analysis suggests several factors which contribute to the extent and to the quality of interactions: facilitation, reciprocation and audience-awareness. Supporting these elements has guided our design of the webreports system. Nevertheless, along with its potentials the technology raises challenges – which need to be addressed by adjusting patterns of behaviour as well as social norms. The fundamental elements of a community of practice are reflected both in our analysis and in the design of the tools, the rules and the roles in our activities.

Mutual engagement, in the sense of sharing and discussing artefacts, is afforded by the features built into the WebReports system; its support of joint and individual authoring of documents, the ease of commenting on others' document, and most importantly – the ability to include models of ideas as manipulable objects in these documents. The notions of facilitation and reciprocation elaborate on the idea of mutual engagement. Implicit rules of engagement emerge by which, for example, harder challenges are more esteemed and provoke richer responses.

A sense of joint enterprise is valuable in motivating students to engage in the activity. This motivation is related to participants' audience-awareness; a factor that is easy to neglect in traditional environments, but takes prominence in a web-based

environment, where the communication channels are thin. As the accepted value of the enterprise rises, in terms of its mathematical richness, so does the level of collaboration.

The concept of shared repertoire is related to that of sociomathematical norms, but also the domain-specific questions, such as the implicit agreement on what constitutes a hard challenge and the positive value of one. Using programming (specifically ToonTalk) as a taken-as-shared resource enriches the repertoire with a language that is both rigorous and expressive. As students master the multiple facets of their repertoire, the boundaries between the verbal and computational languages they use are blurred. Their argumentation is shaped by the tools, while at the same time they shape the tools to express their arguments.

Synergising distinct paradigms is always a challenging task. In our case, we still see more questions than answers before us, but these questions are enough to make the effort worthwhile.

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# MOTIVATION: THE CONTRIBUTION OF INTERACTIVE WHITEBOARDS TO TEACHING AND LEARNING IN MATHEMATICS

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**Abstract:** *A research team from Keele University has worked with 12 mathematics departments in partner school to evaluate the motivational effects of using interactive whiteboards in mathematics classrooms. Although at times it is not easy to separate presentational and motivational effects a number of factors are considered by teachers and pupils to impact upon pupil motivation. Interest and enjoyment were most evident in lessons where the interactive whiteboard, not the teacher, was the focus of the lesson. However, the interactive whiteboard in itself is not sufficient to ensure that pupils are motivated, it is instead the pedagogical stance and the quality of the teaching that enhance motivation.*

**Keywords:** Interaction, Interactive Whiteboard, Mathematics, Motivation, Pedagogy, Secondary School, Technology.

## Background

Between April 2002 and March 2004 members of the Keele University Department of Education Interactive Whiteboard (IAW) group took part in research funded by the Nuffield Foundation to ascertain the rationale, practicalities, pedagogic implications and outcomes of the use of interactive whiteboards in secondary school mathematics departments within the Keele University Partnership of initial teacher education schools. From September 2003 to March 2004 the same research team were involved in British Educational Communications and Technology Agency (Becta) funded research looking into 'best practice' in mathematics and modern foreign language teaching using IAWs.

There has been considerable research into the way in which pupils are attracted by teaching or learning involving the IAW. Carr (1999) considers whole class use of the IAW whilst Blane (2003) deals with motivation in the primary classroom; Clemens et al (2001) describes the gains from the IAW when used in learning enhancement for slower learners, and Bell (2000) and Blanton and Helms-Breazeale (2000) describe attempts to enhance motivation through the use of technology to help those with special needs and literacy learning problems. Miller et al. (2003) report on the perception of teachers in training on the impact of using an IAW on pupil motivation.

Miller et al. (2004) suggest that there is a developmental process where teachers might progress, or not, through three stages: 'supported didactic' where the teacher makes some use of the IAW but only as a visual support to the lesson and not as integral to conceptual development; 'interactive' where the teacher makes some use of the potential of the IAW to stimulate pupil responses from time to time in the lesson and to demonstrate some concepts; and 'enhanced interactive' which is characterised by the development of teaching and learning strategies to shift the focus from the teacher to the IAW and pupil centred learning.

### **Methodology**

The Nuffield Foundation research involved 11 meetings with the 12 mathematics teachers working with the research team, classroom observation of teaching and semi-structured interviews with teachers as well as group interviews of pupils.

At the initial meetings, where most of the teachers were relatively inexperienced IAW users, time was set aside for the teachers to consider the gains that might be made from IAW use, and this led to the development of a structure for the observation of lessons. So that the lessons might be analysed in more detail it was agreed that they would be video-recorded, and the teachers agreed to taking responsibility for ensuring that appropriate permissions were granted. The research team, following consultation with the group of teachers, drew up the semi-structured interview schedule; this was piloted, adapted and then used for the teacher interviews. At five of the later meetings time was set aside to discuss the summaries of classroom observation and interview evidence in order that it might allow for a 'grounded' analysis.

This research team built the Becta research on the early Nuffield Foundation work and used the same classroom observation and semi-structured interview schedules. The teachers included in this part of the research were 'identified' as likely to be working at a 'best practice' level.

This paper is concerned with 30 mathematics lessons that were observed and video-recorded. The majority of the lessons lasted for between 45 minutes and one hour and all were a single age group of pupils with pupil ages varying from 11-12 year olds to 15-16 year olds. Most observations were made of a variety of groups classified by ability in terms of 'upper', 'middle' or 'lower'. Virtually all groups had 20-30 pupils in the class and included both males and females.

In total 22 mathematics specialist teachers were interviewed, though not all were video-recorded. The interviews looked to probe, amongst other things, the perceived motivational aspects of the IAW and how it made a difference to pupil engagement and learning. Two groups of ten pupils each were interviewed in two schools to gain some triangulation with teacher opinion.

The video-recorded lessons were analysed according to a set format with observation of: the timeline and activity sequence in each lesson; classroom management issues; the nature of IAW techniques used within the lesson and their perception by pupils;

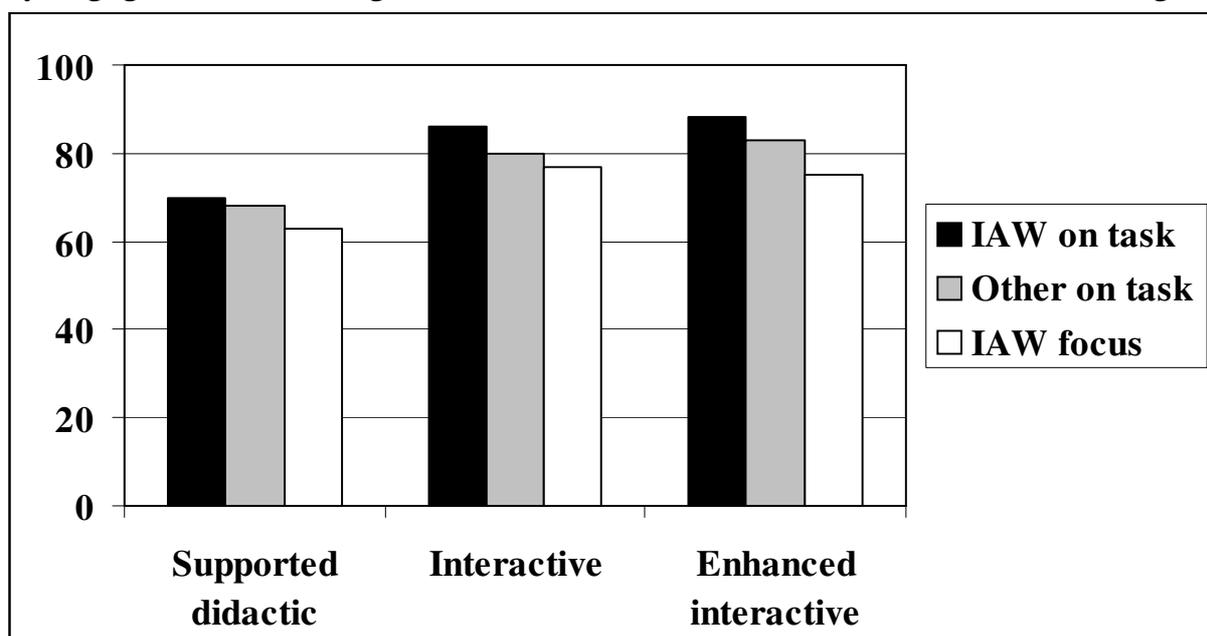
an assessment of the teaching style used in the lesson; teacher and pupil technological fluency; identification of practical and pedagogic issues; enhancement resulting from IAW use within a framework of pedagogic elements; the extent of ‘on task’ work when the IAW was the focus of attention and when it was not, judged by observation of a single pupil; the percentage of the lesson when the IAW was the focus of teaching and learning; the contribution IAW use made to conceptual development; and the contribution IAW use made to cognitive development.

At the conclusion of each observation the lessons and the teachers were classified according to the teaching style observed in that lesson, using the categories of ‘supported didactic’, ‘interactive’ and ‘enhanced interactive’. This gave a measure of the extent to which the teacher had incorporated pedagogic change into the lessons through enhanced activity.

### Findings

In the report that follows the analysis necessarily addresses subject specific issues but we believe that many of our observations and comments may well be generic and these could prompt further understanding of gains from IAW use.

Of the 30 lessons observed 8 of them were classified as having a supported didactic teaching approach, 10 an interactive approach and 12 an enhanced interactive approach. This suggests that in just under three quarters of the lessons (22 out of 30) the teachers demonstrated fluency in the use of IAW techniques and had access to a range of techniques and material that allowed them to work at the interactive or enhanced interactive stage. This was not completely unexpected since many of the observations were made of teachers ‘identified’ as likely to be working at a ‘best practice’ level. A small number of teachers worked at two different levels in different lessons and this appeared to be determined by the materials available as much as by the way in which they were used. This appears to show that these teachers had not fully engaged with working at either the interactive or enhanced interactive stage.



**Figure one: The relationship between teaching approach and IAW use**

The observed lessons all showed that the teacher was not the focus of the lesson in the way that they might previously have been. Figure one shows both pupil 'on task' time as a percentage of the time that the IAW was in use, a similar percentage for when the IAW was not in use, and the percentage of the lesson when the IAW was in use. These are all shown plotted against the type of teaching approach observed. There is a subjective element in these observations because not all 'targeted' pupils were visible for the whole lesson. The proportion of the lesson where the IAW was the focus is generally a more reliable indicator but even so activities sometimes continued whilst pupils were working in pairs or with exercise books.

We believe that the impact of enhanced motivation can be seen in the attitudes to learning as shown in Figure one, with the highest time for 'on task' activity, whether the IAW was in use or not, in the observed lessons where the teacher was working at the enhanced interactive stage and the lowest times when the teacher was working at the supported didactic stage. In those lessons where the IAW was used 'only' as a support, categorised as 'supportive didactic', there were clear changes of pupil attention and attitude. In some of these lessons, when the teacher replaced the IAW as the focus of activity, pupils' interest waned and, at times, there were behavioural management issues that were not evident during the IAW based activity.

Initially there were concerns that there could be a novelty value associated with the use of the new technology and that any motivational gains might disappear with time, particularly if pupils had all lessons in all subjects with teachers using IAWs. But there were also worries about not using IAWs, expressed by one teacher who commented '*there is now danger that if we don't use the technology we will be seen as lacking in some way*'. To address these concerns teachers had developed strategies to ensure that there would be a continuing upward progression in learning and attainment. For example, in one mathematics lesson the teacher started with the aims of the lesson on the IAW, used these as the 'pegs' upon which activities were to be developed and then used different methods of assessment at the conclusion of each learning stage so that '*pupils get a continuing spur to go further, a check that they have understood what they have done, and a set of targets towards which they are working*'.

This recognition that the IAW in itself was a motivating factor was moderated by the way in that the teachers intuitively recognised that the motivation of pupils stemmed from the way in which teachers exploited a '*different type of contact with the lesson in the pupils hands*'. Good practice obviously builds upon knowledge of particular groups and of individuals within the groups and a realistic assessment was that '*the IAW still doesn't mean that we shall have a lesson where all the pupils are paying attention all the time*'.

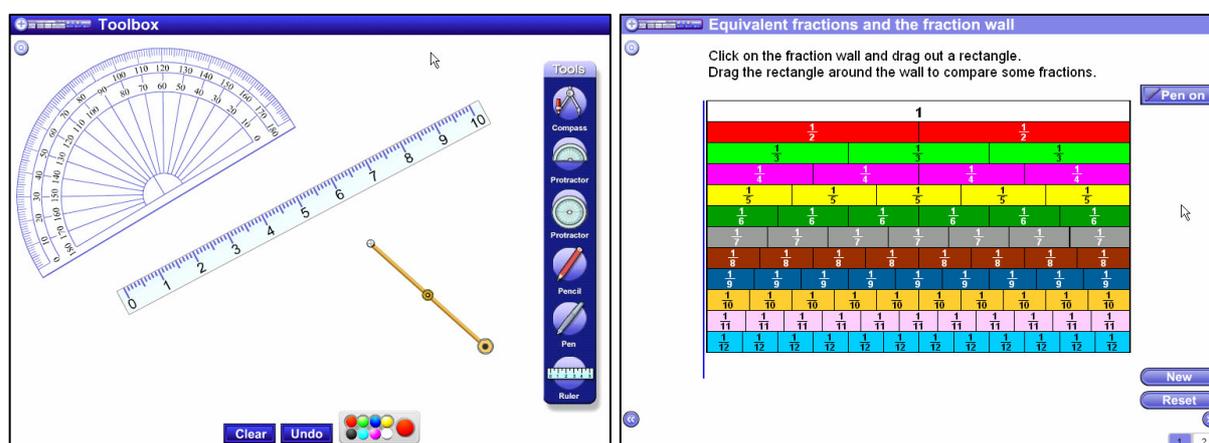
All the teachers were enthusiastic about the technology and argued that the nature of their teaching had changed since the introduction of IAW technology into their

classrooms, suggesting that major changes had occurred in their classrooms. A number also commented that the IAW had been a motivating factor for them and had renewed their enthusiasm for teaching mathematics. However two of those interviewed had reservations about the way in which the IAW was prompting them into a certain form of teaching.

In discussion with participant teachers it was at times difficult to differentiate the motivational factors from the presentational or pedagogic in the successful use of the IAW. Broadly, the evidence showed that the perceived major features that encourage pupil motivation can be classified in three ways: first the *intrinsic stimulation* provided by the combination of the visual, kinaesthetic and auditory paths to learning; second the *sustained focus* maintained throughout the lesson by the teacher's management and 'orchestration' skills; and third *stepped learning* through constant challenges with frequent assessment of achievement as a stimulant to further involvement. The second and third of these three classifications are features of *effective management* that can be seen typically in IAW lessons where the teaching approach is classified as enhanced interactive.

### Intrinsic stimulation

In all lessons observed, teachers were able to capitalise on the intrinsic stimulation offered by the IAW. The use of 'colour, highlighting and shading' was extensively used in work on fractions, angles and algebra to engage and enable pupils to see clearly what was being discussed, to describe parts of the diagrams in explanations and to clarify, for example, equal angles. Similarly, the dynamic features of the IAW, such as 'drag and drop', i.e. moving an on-screen object from one place to another, allowed the use of 'virtual manipulatives', significantly in work on geometrical construction (the virtual manipulatives of on-screen pair of compasses, ruler and protractor), and in demonstrating equivalence of fractions using the virtual manipulative of an on-screen fraction wall. Pupils were also motivated by the opportunities to use virtual manipulatives, seeing it as '*fun arising from the use of tools*'. Examples of these virtual manipulatives, taken from one commercial software package (EXP Maths 7, (2003) Miller and Sherran, Nelson Thornes) are shown in Figure two.



### **Figure two: virtual manipulatives - tools and a fraction wall**

The use of ‘hide and reveal’, hiding an on-screen object so that it might be ‘revealed’ at an appropriate point, enabled teachers to promote conjecture and discussion before answers were shown.

Teachers in interview were clear that interaction based on these features made explanation easier and sustained pupil motivation. However the only auditory stimuli used in the lessons were the voices of the teachers and the pupils, however in the enhanced interactive lessons it was suggested that pupils’ voices were heard more often than in non-IAW lessons.

Evidence from the pupil groups mirrored that of the teachers. When they were asked to identify why lessons were of greater interest than in traditional teaching they also identified the intrinsic stimulation of ‘colour, highlighting and shading’, ‘drag and drop’ and ‘hide and reveal’.

It was clear that where lessons had dynamism and attraction they appeared to offer interest and challenge. Typically such lessons supported both revision of earlier work and enhanced understanding of new work. Teachers were conscious however of the time demands for preparation even when using commercial materials, and three referred to the problems of technology that could inhibit use of the IAW. In the observed lessons there were problems with the technology in 10% of the recorded lessons. When such demands hindered the progress of lessons the motivational advantages to pupils and staff were lost.

The impact of enhanced motivation can be seen in the attitudes to learning prompted by the IAW. One pupil commented of their teacher *‘she has become a bit of an expert since she had the IAW’* but it was noticeable that in the same class the pupils had also gained, according to the teacher, and this was shown in neater exercise books, greater use of colour and presentational techniques and a higher standard of homework completion – the IAW appeared to offer a standard not previously seen with conventional boards.

One final contribution to motivation offered by the intrinsic stimulation of the IAW and highlighted by the research is the relative ease with which it is possible to show the same concept in different ways to ensure understanding and retention. Being able to represent and consider fractions in their many forms, such as on a fraction wall, as fractions of a whole and as a numerator over a denominator, means for the teacher that fewer pupils are likely to be excluded from the lesson. In this respect 20% of the teachers commented upon the particular advantages for slower learning pupils or those who need reinforcement through the presentation of data or processes using more than one learning style. One comment is significant in that it may have highlighted a particular feature of the slower learning group concerned. *‘You have to remember that the lower groups are rather small – in this school averaging only 16 and often with a classroom assistant – and this allows a much greater level of pupil participation. As a result they achieve and feel wanted.’*

## **Effective management**

The extent to which motivation is developed and maintained by what the IAW offers in terms of effective management opportunities is discussed under two headings: the first, how the IAW is used as a sustained focus for the lesson and the second, how the board is central to stepped learning.

### **Sustained focus**

When teachers used an enhanced interactive teaching style or, to a lesser extent, an interactive teaching style the focus of the lesson shifted from themselves to the IAW. This allowed them to sustain interest and engagement as discussion and activity were focused on the IAW. One teacher spoke of himself as: *'an orchestrator whilst the pupils explain, illustrate and direct from the IAW and this has changed the way in which I can involve them all in the lesson'*. The use of the IAW in this way was regarded as a key factor in enhancing motivation.

Effective teaching with the IAW appeared to motivate through the way in which it stimulated learning through participation and understanding. In so far as it affected motivation just under three quarters of the interviewed teachers commented upon aspects of involvement (i.e. the sustained focus of pupils), and 60% noted that the progression of the lesson fostered understanding and achievement as the basis of enhanced self-esteem. In some classrooms this was demonstrated by more movement by the teacher, and by pupils working in groups or at the IAW, than in conventional teaching. Such collaborative work appeared to increase participation and self-esteem, central to maintaining motivation. Pupils' responses supported this notion of sustained focus with responses typified by *'lessons had less wasted time'*, and that *'they moved with more pace so that they didn't want them to come to an end'* a view supported by classroom observation evidence. Three teachers noted that the constant progression in an interactive situation maintained the pace of the lesson and as such absorbed those who might otherwise go 'off task' in a traditional classroom, with the result that the pupils were less 'nagged' by the teacher during the lesson, thereby increasing enjoyment and supporting motivation. Motivational influences thus appeared to become integrated with the pedagogic aims and teaching strategy of IAW use.

### **Stepped learning**

Once the board was established as the focus of the lesson, the teacher was able to sustain pace and develop a teaching strategy by using *stepped learning*. This was a particular feature of effective lessons that were classified as enhanced interactive teaching style.

Perceptions of *stepped learning* were suggested by teachers in comments about the sequential development of ideas, constant challenges and constant feedback with exemplars resulting from pre-prepared and commercial software, such as EXP Maths 7 (Miller and Sherran, 2003, Nelson Thornes). They also mentioned that the opportunity to revisit earlier concepts and examples allowed them to underpin

understanding. A particular motivational gain highlighted by teacher interview and lesson observation concerned the impact of visual recall from lesson to lesson (i.e. stepped learning across lessons), often stimulated through IAW specific software as a means of sustaining pupil understanding and achievement. As one teacher commented: *'recall from lesson to lesson is helped by the use of previous screens... emendations and amendments are all recalled quickly and personally I gain because PowerPoint files are available from home using the Internet gateway'*.

Additionally teachers mentioned demonstration using 'movement and animation', in which the IAW's features were used to 'run through' or 'animate' routines (operating with fractions) or exemplify what was being discussed (the angle sum of a triangle).

If there is one single motivational factor for pupils during lessons that ensures maintained interest it appears to be the immediacy of response. Although not referred to by teachers, pupils consider the availability of games that support learning to be a key motivational factor. These were usually features of commercial software or Internet sites. Such games required responses that can be immediately assessed and then linked to a scoring system. Such competition, properly managed, between individuals and/or groups promoted engagement and the drive to succeed – a key stimulus in the cognitive interactionists' model of motivation (Bigge and Shermis, 1999).

### **Conclusion**

Whilst it would be easy to claim great advantages for the IAW in motivating pupils at all ages it is evident that it is the pedagogical stance and the quality of the teaching that ensures progress.

In an assessment of two lessons, both using professionally developed fraction wall materials to enhance learning of fraction equivalence the less successful began with problems of vision of the screen, continued with three longer periods of activity during which pupils lost interest, and degenerated into a conversation between the teacher and those who were nearest to the board and most interested in the lesson. By contrast in the other lesson the teacher used groups of pupils to demonstrate equivalence, and then worked with the whole class to establish rules of process as a *'genuine voyage of discovery for them – they saw that they were doing the learning, I was merely opening the gate of understanding for them'*.

Perhaps one comment from a pupil sums up the motivational impact of the IAW. After a lesson in which the stages of equation solving were developed in three different ways, one girl said *'Oh, my God, it is so easy when you put it like that – and I won't forget again'*.

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# DEVELOPING NEW REPRESENTATIONS AND MATHEMATICAL MODELS IN A COMPUTATIONAL LEARNING ENVIRONMENT

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**Abstract:** *In this study, we explore the relationship between the learners' visualisations, actions and the environment in which these are articulated. ToonTalk is a visual programming microworld, designed to help students construct mathematical meanings by linking their actions with multiple representations and with the mediation of programming to build structures and relations. Through a case study of two students interacting with ToonTalk and WebReports, we illustrate a view of mathematics learning which places at its core the medium of expression, and the building of connections between different modes of representations to discover the related mathematical concepts.*

**Keywords:** Representation, model, learning environment, ToonTalk, microworld.

## INTRODUCTION

In the past two decades the teaching of mathematics has undergone many significant changes. The importance and emphasis placed on the development of a deep understanding of mathematical concepts and processes via active learning, inquiry and problem solving have been well documented (NCTM, 2000). The central themes in mathematics learning include strategies for generating and solving problems, looking for patterns, formulating and testing hypotheses, making conjectures, evaluating constraints, predicting outcomes, justifying and verifying (NCTM, 2000). Along this line, Noss, Healy and Hoyles (1997) have pointed that it is crucial to explore the potential of environments that help students to build connections and derive mathematical understandings using tools that may assist them in the process.

The present study originates in *WebLabs*, a three-year European research project on the use of programming and web-based collaboration in mathematics and science education. The focus of this project is on communities of young learners (10-14 years), engaged in collaborative modelling of mathematical and scientific phenomena, across six European countries. In this framework students interact within ToonTalk, a programming microworld and WebReports, a web-based system, designed to allow students and researchers to share and discuss models of mathematical objects, processes and concepts.

In an attempt to inform the development of better pedagogical models, this paper reports some of the findings from a study of the integration of a microworld based environment in exploring number sequences. The tools, definitions, exploration techniques, and visual representations associated with the microworld contribute to a learning environment fundamentally removed from traditional teaching (Healy & Hoyles, 1999). Specifically, the purpose of this study is to provide some indications on how this environment can promote the construction of new representations and models for conceptual understanding in number sequences and especially in Fibonacci Number Sequences.

### **THEORETICAL FRAMEWORK**

Recent mathematics curriculum documents, such as the NCTM *Standards* (2000), as well as researchers in mathematics education (Hoyles, Noss & Pozzi, 2001; Kaput, 1999; diSessa, 2000), value mathematical investigation based on the pedagogical belief that students learn best when they are given the opportunity to actively construct personal understandings of mathematical concepts and relationships. In the area of number sequences in most cases, the mathematical structures, which underpin the sequences, are invisible; there was no access to the process so students are constraint to searching for patterns in the output data rather than within the processes by which they were constructed (Arzarello & Domingo, 2003). As a result, students are diverted by surface relationships between numbers and ignore the structural, mathematical reasons why they may or may not be related (Healy & Hoyles, 1999).

There are good reasons to assume that microworlds have the potential to help students enhance their problem solving abilities in various contexts and provide students with a powerful means for conceptual understanding by applying different heuristic approaches (Hoyles et al., 2002). Papert (1998) defines microworld as a place where certain kinds of mathematical thinking could hatch and grow with particular ease, while Edwards (1995) stresses knowledge as a central element in the microworld idea. The microworld as a concrete embodiment of a mathematical structure is extensible (so tools and objects can be combined to build new ones), but also transparent (so its workings are visible) and rich in various representations (Edwards, 1995).

In exploring number sequences and investigating properties of these sequences, microworlds could be considered as powerful environments for learning, since they provide the opportunity for pupils to make sense of ideas in a context where true and false, right and wrong are not the only decisive criteria. In line with that, Noss et al., (1997) have suggested that microworlds include almost any exploratory learning environment; a microworld was “simultaneously rich and simple enough to study learning behavior and promoted the notion that both of these components should be considered in any conceptualization today”. Within this framework, in ToonTalk, the environment used in the present study, the learner is able to actively explore concepts

in number sequences in new and dynamic ways, which would not have been possible without the technology, and in doing so, the learner through explorations of number sequences and investigations or their properties, constructs knowledge that has meaning for her. Among the general aims of the Weblabs project, was to provide a successful microworld, which combines all these features in an attractive and enjoyable environment, allowing students to explore and understand its structures and relationships and use programming to actively interact with these structures (Noss, 1998). It should further provide students opportunities to work individually and/or collaboratively, being creative and having fun (Piaget, 1951).

As Edwards (1995) argued, a well-designed microworld does not necessarily involve programming; that is even a Logo-based microworld may not contain modification of code or interaction with Logo at all (p. 134). In that case, students only manipulate ready-made tools, without any chance to build, modify or debug the tools. Programming can be fun and empowering activity, which can promote learning (Noss et al., 1997). It is the prototypical tool for the constructionist vision, and a microworld without programming runs the risk of avoiding the feature that gives a microworld its power (Hoyles et al., 2002). If children cannot program at all, how can they build the tools that they need to model, make the necessary connections and finally come to understand a mathematical concept (Papert, 1996)? A good microworld example is expected to provide students with opportunities to observe, understand and modify existing tools and structures, and gradually build their own tools. Involvement with programming will allow students to explore, discover and construct actively the mathematical ideas (Edwards, 1995; Healy & Hoyles, 1999).

### **The ToonTalk Environment**

ToonTalk is an animated programming environment, based on a metaphor of a game in the computer environment (Kahn, 1999). The fundamental idea behind animated programming is to replace computational abstractions by concrete familiar objects, which behavior captures the essence of the corresponding computational concept (Kahn, 1999). Therefore, students, as ToonTalk programmers can build, run and debug programs while understanding only ToonTalk's concretizations. The animated source code appears in the form of animated cartoon robots (ToonTalk is so named because the user is **talking to cartoons**). Animation is central and operational, since it formulates the communication between users and software and between the different tools and structures of the software.

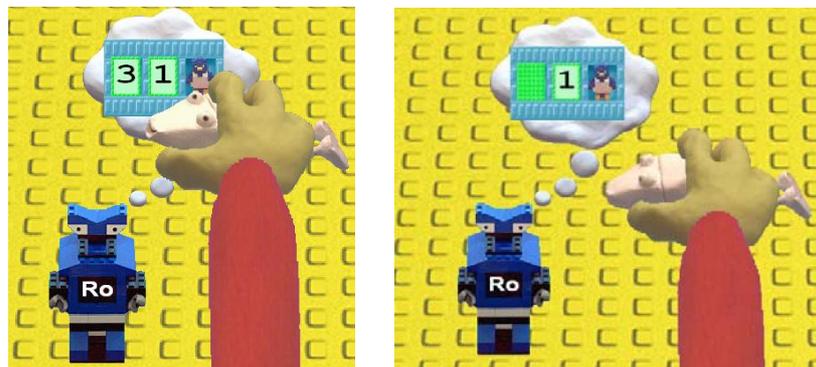
Programs are created by directly manipulating animated characters, like boxes, numbers and text pads, trucks and robots. In ToonTalk, programming is by example, that is the user can construct programming code by training a robot who is given an example input (in its thought bubble) to work on and show what the user can do (Kahn, 1999). The important point is that the process is made concrete in a robot,

which can be pointed at, named, picked up and moved around. Figure 1 shows a robot trained to count through the natural numbers.



*Figure 1: Training a robot to count*

ToonTalk microworld has many common features with Logo programming language and Logo based microworlds, though there are differences, like the one related with the variable concept. In Logo environment the user needs to know a priori what a variable is or what it represents; in ToonTalk, the introduction to the concept of variable follows a completely different and visual metaphor, since the user can generate a variable (using a small hover machine) by erasing the value (number, text or image) of the input from the robot's memory. Figure 2 shows how a student erases number 3 from the box a robot is working with, so the robot will continue to work with any number. This metaphor is an immediate and central characteristic of the ToonTalk environment (Hoyle & Noss, 1996).



*Figure 2: Visual metaphor of the variable concept*

## THE STUDY

The purpose of the study is to provide some indications through a learning snapshot on how the environment can enhance the construction of new representations and models for conceptual understanding in number sequences. The environment consists of ToonTalk microworld and a web-based system, named WebReports, which is designed to allow students to share and discuss ideas on tools and models of

mathematical concepts and processes. The students publish reports on their programs, conjectures and ideas in ToonTalk, comment on and annotate on others' reports. The use of these reports allows students to share what they have done, debug and comment on others' attempts to model mathematical concepts and procedures.

Within this framework, students explore and construct number sequences, making conjectures, suggesting solutions and finding patterns. An extension of number sequences is the activity on Fibonacci number sequences. In this activity, after a short introduction, the students were expected to make programs to model the Fibonacci sequences in ToonTalk environment and explore properties (mostly divisibility properties) of the sequences. The general purpose of this activity was to provide students with opportunities not only to apply the sequence's form, but also to model the sequence, so to get insights into its generation and therefore to collaborate with other students to construct and compare different models, analyze the sequences and discover properties and identities of the sequence. These will allow students to understand cause and dependence concepts in a mathematical concepts framework (Papert, 1998).

## Setting

Twenty students aged 13 years old participate in hourly meetings twice a week, in an established club, named "Using New Technologies in Mathematics". The Club meetings take place in Mathematics Computer Lab, which is equipped with the ToonTalk microworld and Internet access. A video recording of the sessions was decided as the means of recording the meetings since we wished to capture not only the discussions but also the actions occurring on the computer screen as interviewees talked about their work. For the purpose of the present study, we analyse the work of two students participated in an activity on Fibonacci number sequences. The whole setting was informal with students being able to analyze and build Fibonacci number sequences and find properties of the sequence. The analysis of the data followed interpretative techniques (Miles & Huberman, 1994). Video records helped us identify the unique ways the microworld facilitated the students to construct and model the number sequences. Detailed analysis of all the data posted on the Web helped us to gain insights into students' ideas and models.

## RESULTS

Students worked on two activities in Fibonacci sequence. The first one was on analyzing, modelling and generating the sequence in ToonTalk microworld and the second part of the activity on exploring properties and identities of the sequence. Due to space limitations only the results of the first part are presented here. After a short introduction to Fibonacci number sequences, students worked with paper and pencil to find the following and the preceding numbers in the sequence ... 8, 13, 21, 34 ... In the next part of the activity students worked in ToonTalk's microworld, trying to

### Working Group 9

generate the Fibonacci sequence. Students could train their own robots or use a set of ready-made tools. This set of tools consists of three tools; a tool adding 1 to a given number, a tool adding up two numbers and a tool for generating a constant number. Both students started by training a new robot. Chris decided to model the sequence using paper and pencil, while Alex started working immediately in the microworld.

*Chris:* Can I use paper and pencil?

*Researcher:* Sure. If you prefer working first with paper and pencil is fine. Do you find it easier to work like this?

*Chris:* I want to draw the figures on paper. Then I will make a sketch to show how I will train the robot.

*Alex:* I think it is not necessary. I only need to use a box with two numbers and train a robot to add them.

Alex managed to build the sequence, but instead of adding new numbers in the sequence, the robot kept replacing the second number of the box with the new number of the sequence. On the other hand, Chris deployed the rule of the sequence on the paper, while making an accurate model of the sequence. He then moved to microworld's environment trying to model the sequence. Alex told him that he managed to generate the numbers of the sequence, but not the sequence. They decided to work together to build the sequence.

*Alex:* I did not add a new box each time. This is what we have to do. It might be better to write the steps down on paper first.

*Chris:* Good idea. You can see my sketch (presents his model). I think it is fine, although I did not write the necessary steps.

*Alex:* First we give the robot a box with two 1s. Then train the robot to add the two numbers. Now be careful!

*Chris:* Take a box from the toolbox and put it next to the other two. Place the number inside. I think we are done.

*Alex:* Yes. The robot is going to generate the sequence.

The robot performed the operation once and then stopped, since students had forgotten to erase the numbers from robot's memory, so it could work with all numbers. It was quite easy for them to debug the robot and they finally managed to generate the sequence.

Students have next packaged the tool and posted it on the web, while writing a web report on their work. A few days later both students came to the next session. They were very excited to have received comments on their report, particularly from students on the other side of Europe! A student from Bulgaria commented that their model was quite effective and suggested that they should try to build the sequence

using the ready made robot AddUp. Students were very fascinated, since a participant from another country commented on their report. In the following meeting they decided to deploy the AddUp robot and try to generate the Fibonacci sequence in a neat way!

*Alex:* This robot can add two numbers and sends the result out using a bird. What we need to do is to make it add up the two preceding numbers to generate the following one.

*Chris:* We have to use the output. Actually add it to the previous numbers.

*Alex:* What do you mean?

*Chris:* Find the result of the addition and then use it to express the next number. We can add two robots. This one (the AddUp) to find the sum and another one to generate the sequence.

*Researcher:* You could do that. But the challenge is to build the sequence, using only one robot, the AddUp one.

*Chris:* We have to bring the output back as an input. Right?

*Alex:* We have to copy the nest, back to the robot.

*Chris:* Cool! We use the magic wand and copy the nest back to robot's box.

*Researcher:* Where do you have to put the copy of the nest? Does it matter?

*Chris:* We need to erase the first of the two initial numbers and then add up the number in the nest-copy with the second number.

*Alex:* So the robot will continue working with the numbers, while the nest will contain the numbers of the Fibonacci sequence.

The two students constructed the robot successfully, while in the meantime they debugged it many times, trying to build the best one they could. They posted their robot in the website of the project, asking from other countries to send comments, or different ideas on constructing the Fibonacci sequence. Figure 3 presents the modified AddUp robot, which presents a new solution for generating the Fibonacci sequence.



*Figure 3: A robot trained to generate the Fibonacci sequence*

## Working Group 9

The London team congratulated the two students for their nice solution to the problem and posted a question, asking from students to predict whether these two robots were equivalent that is if they were going to generate the same sequence for ever.

*Alex:* We can place each robot in a different house and find out their outputs. If the two outputs are the same, that means they are equivalent.

*Chris:* We could also use a third robot ... Which will compare the output from the two robots and return a “Yes” if the two outputs are the same or a “No” if they are different.

*Researcher:* Well done! Your ideas are really productive. Can you think of something else?

*Alex:* I think Chris’ solution is quite difficult. It might be easier to find the differences. The new robot will subtracts the output numbers from the two Fibonacci robots and generates a sequence with the differences.

*Chris:* The new sequence will contain only 0s.

The previous extract shows how the students effectively employed the provided tools to create alternative models, which helped them organize their ideas and provide solutions for a quite difficult problem.

## DISCUSSION

In this study we tried to show some of the ways in which a computational learning environment can provide students not only data to confirm or reject a conjecture, but ideas and representations to model and understand mathematical concepts. To this end, the results of the study were presented in the light of how the environment (both the microworld as well as the Webreports) mediated students’ understandings. The environment not only facilitates mental processes, but also fundamentally shapes and transforms them, while offering opportunities for students to act and react. The example was provided to show how the computing, modelling and debugging capabilities of the microworld in conjunction with the webreports mechanism can enable students to explore and make mathematical conjectures, model sequences and solve problems. Moreover, the interactions both within students in the same site as well as across the different countries participate in Weblabs project enhanced students’ work and promoted conceptual understanding in number sequences.

However, the main issue is whether the involvement of students in a learning environment like ToonTalk may result in understandings that would not be possible to reach through traditional instruction. Another issue is whether instructional devices are actually used and transformed by students in exploring, modelling and discovering the mathematical concepts, using the programming facilities of the microworld (Doerr, 1996; Matos, 1995). The students’ processes in the activity showed that they were engaging in making conjectures, exploring and modifying

solutions at some level, something which might not happen if the same problems were assigned to students without the use of technology (Noss et al., 1997).

The environment was used to give students evidence that their conjectures were not always valid. Cognitive conflict and/or surprise, as it appears, make students eager to understand *why*. The example provided showed that the microworld facilitated students' understanding of mathematical ideas through the observing contradictions arising in the exploration of generating the number sequences (Noss, 1998). Moreover, the use of different modes of representations and the connections between them in the learning environment seems to help students formulate a coherent conceptual understanding of number sequences (Kaput, 2000). In addition, the opportunity students have to build their own representations may result in better connections and understandings (Noss, 1998) and can enhance students' meta-representational abilities and thus their meta-cognitive abilities (diSessa, 2000).

The purpose of the research project is to challenge the mathematics education community to focus in the dimension of applying new technological tools and designing appropriate activities which promote non static representational systems and the use of modelling in exploring mathematical ideas. Finally, on a more practical level, the present study of learning in a microworld based learning environment can benefit teachers, and curriculum developers. Teachers faced with limited time and crowded computer labs may use research results to identify fruitful ideas in the language and construction actions of their students. In addition, curriculum developers may find inspiration for new activities aimed at the needs of mathematics learners.

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# INTEGRATING TECHNOLOGY IN A MATHEMATICS COGNITIVE INTERVENTION PROGRAM

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**Abstract:** *This paper presents a project which aimed at providing a framework for cognitive acceleration in mathematics education. The innovative aspect of this project was the integration of contemporary technological tools (e.g. Dynamic Geometry Software and spreadsheets). Two groups of 6<sup>th</sup> grade students (11 and 12 year olds) were examined: An experimental (133 students) and a control group (89 students). The results obtained by a post-test which was administered, after the completion of the program, to both groups of students, suggest that the experimental group, which participated in this acceleration program, performed significantly better in the Qualitative-Analytic, Spatial-Imaginal and Causal-Experimental Specialized Capacity Systems, than the control group who only used the school textbooks.*

**Keywords:** cognitive acceleration, educational technology, enrichment program, ICT.

## Introduction

The term *cognitive acceleration* has been created under the theoretical premise that students' thinking and ability to learn can be enhanced and developed through systematic training (Demetriou, Efklides, & Gustafson, 1992). For the past two decades, there has been a growing development of intervention programs that aimed to accelerate cognitive development in primary and secondary education (Shayer & Adey, 2002). The major goal of these programs was the fostering of students' ability to think effectively and thus to increase their general problem solving ability and academic achievement (Adey & Shayer, 1994).

In this paper we present a project which aims at providing a framework for cognitive acceleration through mathematics education. The innovative aspect of this project is the integration of contemporary technological tools in mathematics teaching (e.g. Dynamic Geometry Software and spreadsheets). The background of the project is premised on the experiential structuralism theory, as it was presented in CERME 3 conference (Christou, Demetriou & Pitta-Pantazi, 2003). The first part of this paper summarizes the general premises of cognitive intervention programs, the core concepts of experiential structuralism theory and the necessity of the integration of technology. The second part presents the design and the implementation, while the last part presents the results of the intervention and discusses some of the findings of the project.

## Theoretical framework

Adey (1999) defines cognitive acceleration program as the systematic training that targets to improve children's thinking processes by accelerating progress towards higher-order thinking skills. The idea of cognitive intervention has been around since the 1950's (Feuerstein, Rand, Hoffman, & Miller, 1980). Two kinds of intervention programs have been developed, context-free and context-based (Hamers & Csapó, 1999). The context-free programs adopt the development of cognitive intervention programs based on general cognitive skill tasks, whilst, the context-based integrate the teaching of cognitive skills in the curriculum of domain-specific subjects. The best-known example of context-free cognitive intervention program is the "*Feuerstein's Instrumental Enrichment*" (Feuerstein, Rand, Hoffman, & Miller, 1980) which was developed with the purpose to enhance learning in low-performing Israeli adolescents. One of the most successful context-specific cognitive acceleration programs is CASE (Cognitive Acceleration in Science Education) (Adey & Shayer 1993, 1994). Students who participated in CASE outperformed students in the control schools even two years after the completion of the program (Shayer & Adey, 2002). The success of this project motivated the development of CAME (Cognitive Acceleration in Mathematics Education), which focused on teaching mathematics as a mean for a rapid intellectual development (Adhami, Johnson, & Shayer, 1998). The first phase of this project was concluded in 1997, showing that CAME students' achievement effect sizes, compared to control students were significant in mathematics and English (Shayer & Adey, 2002).

### Psychological Premise

According to experiential structuralism theory (Demetriou, Christou, Spanoudis, & Platsidou, 2002), the human mind is organized into three levels. The first involves a set of environment-oriented Specialized Capacity Systems (SCS), each of them including a characteristic set of operations and processes which are appropriate for thinking and problem solving within its domain of application (Demetriou et al, 2002). The input to this level is information coming from the environment and its output are actions, overt or covert, directed to the environment. The second level involves a set of higher-order control structures governing self-understanding, self-monitoring, and self-regulation (hypercognitive system). The third level of the mind involves processes and functions underlying the processing of information. This is regarded as the dynamic field where information is presented and processed by the thinker for the necessary time-span, in order to make sense of the information and accomplish the problem-solving tasks.

Empirical research in laboratory led to the identification of five SCSs: (1) *the qualitative-analytic*, specializing on the representation and processing of similarity and difference relations (Demetriou et al., 2002). Its functioning is based on the specification of the properties that may co-define the mathematical objects. The abilities required in the qualitative-analytic SCS contribute to the understanding of mathematical concepts that are characterized by the inclusion relations connecting the

elements of a hierarchy. (2) *The quantitative-relational*, involves abilities and skills of quantitative specification, for example counting, pointing, bringing in and removing, and sharing. Internalization of these skills into coordinated mental actions results in the four basic arithmetic operations, which provide understanding of the basic quantitative functions of increase, decrease, redistribution and so forth. This system also involves rules and operations for identification of various types of quantitative relations such as fractions. These processes constitute the basis of complex mathematical thinking, such as proportional or algebraic reasoning. (3) *The causal-experimental* includes the following abilities: (i) combinatorial abilities which form the cornerstone of this SCS, (ii) the hypothesis formation abilities that enable the individual to induce predictions about possible causal connections on the basis of data patterns, and (iii) the experimentation abilities that enable the individual to “materialize” hypotheses in the form of experiments. (4) *The spatial-imaginal* system is directed to those aspects of reality which can be visualized by the “mind’s eye” as integral wholes and processed as such. This system involves abilities such as mental rotation, image integration, and image reconstruction. (5) *The verbal-propositional* is concerned with the formal relations between mental elements. The main characteristic of this SCS is the ability to differentiate the contextual from the formal elements of a series of statements and operate on the latter.

### Cognitive Technology

Although CASE and CAME projects have been extremely successful, they have not taken into account the dramatic growth of computer-based technologies and the immense potentials of the integration of these technologies into mathematics; new technologies can enrich and give totally new potential to the development of cognitive intervention programs. Evidence is mounting to support that 21<sup>st</sup> century information and communication tools, which can positively influence student learning processes and outcomes. A review of studies conducted by the CEO Forum (2001) emphasizes that technology can have the greatest impact when integrated into the curriculum to achieve clear, measurable educational objectives. Many research findings support the argument that the integration of new technologies in the teaching of mathematics enables students to self-construct their mathematical knowledge (Mok & Johnson, 2000). The effective integration of technology into mathematics teaching can and will result in higher levels of achievement; the addition of technology in mathematics classroom can help students master fundamental skills and more importantly motivates them to higher levels of achievement by promoting and developing higher-order skills. Mathematics software can present a mathematical concept in various representational systems (symbolic equations, tabular form, graphs). Research findings support that many mathematical concepts can be more efficiently taught with the aid of contemporary software (Tinsley & Johnson, 1998). For example, Laborde (1998) argues that dynamic geometry software develop higher order thinking skills such as synthesizing, analyzing, conjecturing, experimenting, generalizing and reasoning. Jones (2000) asserts that dynamic geometry software develops both deductive and inductive reasoning. Mok and Johnson (2000) report

that various mathematical software facilitates algebra's conceptual understanding because of the simultaneous use of a number of variables and the visualization of algebra's properties.

### **The present study**

The purpose of the present study was twofold. First, to present the program, the structure, the design, the content, the philosophy and the ways of integrating the appropriate mathematics software. Second, to examine whether the students who participated in the program accelerated their ability in the five SCSs, as they are defined in experiential structuralism theory (Demetriou et al., 2002).

### The Program

The program developed consisted of twenty 80-minute lessons. The teaching scenario of each lesson had the following main rationales: (i) to develop technology based mathematical activities that foster students' mathematical thinking and their ability in the five SCSs, and (ii) to integrate in the lessons self-planning, self-monitoring and self-evaluating skills which can affect the functioning of the SCSs. The design of the student materials is based on the assumption that the integration of new technologies in mathematics teaching can create a powerful learning environment that can give students the opportunity to build on their own the mathematical knowledge and by doing that to develop their cognitive abilities, as they are defined by the experiential structuralism theory. To achieve this, we developed the twenty lessons of the programs based on real-world problems that can be solved by using appropriate software.

The core idea of the 20 lessons lies on the existence of a fantastic hero, Jason, who is traveling around the world and helping local people to overcome everyday mathematical problems. The program is called 'Jason's 19+1 feats' and every 'feat' constitute a problem to be solved. Students are challenged and encouraged to help Jason to solve his problem with the use of specific software according to the content of the problem. The twenty lessons of the program were divided into four groups: (i) the first one involved the following: intuitive development of the variable concept, algebraic relations, function concept, graphic representations of linear relations, analogical and inductive thinking. Students had to use 'Ms-Excel' to solve the problems of these lessons. The role of the software in this group of lessons was to help students model the problem, handle and represent algebraic relations and variables in a symbolic form, represent graphically linear relations, and interpret graphic representations of linear relations (slope, point of intersection with the y-axis). The use of the software helped also the students to be involved in self-planning processes. For example, students used the software to organize their data, to select the necessary information, to implement problem solving strategies, such as trial-error, to make conjectures and to inductively extract rules and relations. The software gave feedback to the students, and so they could re-plan their problem solving procedure. The activities involved in these lessons lie, in general terms, in the domains of the

quantitative-relational and the qualitative-analytic SCSs. (ii) The second group involved geometrical thinking. Students had to explore the sum of the angles of a triangle, relations between the sides of a triangle, to explore and discover the area of a parallelogram, a triangle and other polygons based on the conservation of area principle, to explore  $\pi$  and circle's area, to discover the Pythagorean theorem, to study tessellation properties, to investigate polygons' properties and some fundamental geometric transformations. In these lessons students used the Geometer's Sketchpad (DGS). The DGS-based activities were designed to involve students in the processes of modeling, conjecturing, experimenting, generalizing and developing their inductive and deductive reasoning. Modeling through the visual representation of the solution-strategy, reasoning through the interpretation of dragging and measuring facilities of the software, using perception strategies for estimating area, developing inductive and deductive reasoning through the numerous examples that the software gives the opportunity to the students to examine and check the correctness of their conjectures and conclusions. The activities included in these lessons involved mainly thinking skills belonging to the Spatial-Imaginal and Qualitative-Analytic SCSs. (iii) The third group included activities involving fundamental statistical concepts and probabilistic thought. Students had to handle data, represent graphically data in various ways, to calculate basic statistical concepts, use propositional conjunctions, design and execute probability experiments. The statistical software 'Table-Top' was used for the statistics lessons and the 'Probability Explorer' in lessons dealing with probability. Table-Top helped students to organize, analyze and interpret their data and to solve decision-taking problems. The availability of different diagrams in the software, and especially the scatter diagram was used to identify and explore relations between variables. 'Probability Explorer' was used as a mean to design and execute numerous experiments and by doing so to conceptually understand the relation between experimental and theoretical probability. The activities of this group activated mainly thinking skills belonging to the Qualitative-Analytic and the Causal-Experimental SCSs. (iv) The fourth group included mainly problem-solving activities. Students were administered two-step word problems and complex procedure ones. For the word problems students used software developed by the researchers and was based on Marshall's schema theory (1995). Students had to solve and pose problems using the diagrams provided by the software. For the complex procedure problems students chose on their own the software that they thought it was the most appropriate one. During the solution procedure, teachers tried to engage students in self-planning, self-monitoring and self-evaluating strategies.

The design of the students' material adopted the five-pillar scheme of cognitive acceleration, proposed by Adey (1999): (i) *concrete preparation*. There was an introductory phase of preparation in which the language of the problem was introduced, along with examples to be used and a context in which the problem were set. (ii) *cognitive conflict*. After the discussion in the *concrete preparation* phase students encountered a problem that they cannot solve or have never encountered

before. By working collaboratively, students had to find ways to plan a solution strategy taking advantage of the facilities of the software used. (iii) *construction*. Students got involved in cognitively stimulating experiences. Not only did the integration of the software aimed at engendering cognitive conflict and/or surprise, but also, the software acted as a mediation tool and encouraged students to use in problem solving higher order thinking skills and processes that could no have been achieved without the use of technology. (iv) *metacognition*. The philosophy of the activities developed tried to “force” students to be engaged in self-planning, self-monitoring, self-evaluating, reasoning and reflecting processes during the ‘construction zone’ individually or in pair work by solving appropriate tasks. Most of these tasks integrated the software presented above. These self-regulating activities continued in the (v) *bridging phase*, where the linking and extension of ways of thinking developed in the context of the activities of the program to other contexts within mathematics or other parts of the curriculum and to experiences in real life (Adey & Shayer, 1994). Table 1 presents the description of one feat of this project.

TABLE 1: Description of the Feat “Jason in Iran”

Concrete Preparation	<p>Jason is visiting a carpet industry and tries to advise the owners of the industry about their financial management. The owners gave the following information to Jason. “The fixed costs of the industry are 18000 rials (Iran currency). The production of each extra carpet costs 45 rials”. Jason has to answer the following questions: (i) what is the cost of the production of 500 and 600 carpets, (ii) how much will the profit or loss be of the production of 700 carpets if each carpet is sold for 75 rials, (iii) how many carpets should the industry sell to make no profit or loss?</p> <p>Students are introduced into the requirements of the problem. The terms of “fixed costs” and “cost per extra carpet” are discussed.</p>
Cognitive Conflict	<p>Students have to deal with a problem that have never encountered before. The teacher asks students what is the cost of the industry when it does not produce any carpets. Cognitive conflict may arise when students discover, realize that the industry has costs even if it does not produce goods. Students in schools are used to deal with relations of the form <math>y=ax</math>, thus to conceptually understand and use a relation of the form <math>y=ax+b</math> is a cognitively demanding task.</p>
Construction	<p>Students are prompted to self-plan their solution strategy. Teachers ask students to think how they can take advantage of the Ms-Excel. Students may model the problem, identify the necessary variables and represent them by using the columns of the software. The concept of variable is introduced; the teacher discusses with the students how they can take advantage of the names of the cells in order to answer the first and second questions. Students handle the variables “fixed costs”, “cost of extra carpet”, “number of carpets”, “sell-price”, “income” and “profit/loss”. They must plan their solution strategy to answer the third question. Students are motivated to compare and evaluate their solution plans.</p>
Metacognition	<p>During the lesson students are asked to explain their reasoning, to evaluate their solution plan, and reflect on how they used the software to reach their solutions</p>

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Bridging	Students are asked to pose their own problems for their butterfly-hatch involving the concepts they learned
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## Method

The participants of the study were 222 6<sup>th</sup> grade (11 and 12 years old) students. One hundred thirty three individuals participated in the experimental group and eighty-nine students in the control group (five experimental and four control classes in four urban primary schools in Nicosia, Cyprus). The teachers of the experimental classes participated on a voluntary basis and attended a 6 hour seminar on the philosophy, the development and teaching strategies of the program. They also attended a three hours seminar on the integration of computers in elementary education, emphasizing the integration of dynamic geometry and spreadsheets. The experimental and control classes were selected from the same schools. The participants were tested with a cognitive development test two weeks before the beginning of the lessons (pre-test). Every experimental class attended the twenty 80 minute lessons of the project during a five months period (in an average of one lesson per week). The lessons were conducted in each school's computer lab, which were equipped with computers loaded with the Greek version of the Geometer's Sketchpad, Ms Excel, TableTop, Probability Explorer and the Problem Solving software. The control classes used only the school textbooks and never visited the computer lab. One week after the completion of the program the same cognitive development test was assigned to the students of both groups (post-test). The task batteries of the cognitive development test are not presented here due to space limitations but have been extensively presented in previous studies (Christou et al, 2003; Demetriou et al, 2002). However, it can be clearly said that none of the test items had any relation with the teaching material of the two groups.

## Results

The question of the study was to examine whether this project accelerated 6<sup>th</sup> grade student's ability in the five SCSs, as they are defined in the experiential structuralism theory (Demetriou et al, 2002). Therefore, we examined if experimental group student's attainment in the five SCSs tasks was significantly better compared to the attainment of the control group students in the post-test measurement, right after the completion of the program, taking student's initial attainment as a covariate. Initially, we conducted a multiple analysis of variance test (MANOVA) having student's pre-test attainment in the five SCSs as dependent variables to examine whether there were statistically significant differences between the two groups. The results revealed there were no significant differences between the two groups (Pillai's  $F_{(5, 193)}=.032$ ,  $p>0.05$ ), meaning that the two groups were equivalent, thereby, we could proceed to further analysis. In Table 2 we present both groups' pre and post-tests attainment in the five SCSs and the mathematical ability.

Then, we conducted a multiple analysis of covariance test (MANCOVA) having student's post-test attainment in the five SCSs as dependent variables and the corresponding pre-test scores as covariates. The results showed that there were statistically significant differences in students' post-test attainment between the two groups, Pillai's  $F_{(5, 193)}=.057$ ,  $p<0.05$ . Table 3 presents the results of the MANCOVA test, showing that there were significant differences between the two groups in the Qualitative-Analytic SCS ( $F_{(1, 193)}=8.794$ ,  $p<0.05$ ), the Causal-Experimental ( $F_{(1, 193)}=4.109$ ,  $p<0.05$ ) and the Spatial-Imaginal SCS ( $F_{(1, 193)}=4.299$ ,  $p<0.05$ ) and the Causal-Experimental SCS ( $F_{(1, 193)}=4.109$ ,  $p<0.05$ ). Taking also into consideration the mean values of both groups in the pre and post-tests (presented in Table 2), we can conclude that the students of the experimental group performed significantly better than the students of the control group in three of the five SCSs, that altogether constitute the cognitive ability. Specifically, they performed better in the Qualitative-Analytic, the Spatial-Imaginal and the Causal-Experimental SCSs.

TABLE 2: Means of SCSs by the Experimental and the Control groups in the two measurements

	<u>Pre-test</u>				<u>Post-test</u>			
	Experim. Group		Control Group		Experim. Group		Control Group	
	<u>Mean</u>	<u>S.D.</u>	<u>Mean</u>	<u>S.D.</u>	<u>Mean</u>	<u>S.D.</u>	<u>Mean</u>	<u>S.D.</u>
Quantitative-Relational SCS	0.49	0.22	0.46	0.19	0.60	0.25	0.54	0.24
Qualitative-Analytic SCS	0.57	0.18	0.51	0.17	0.67	0.22	0.60	0.17
Verbal-Propositional SCS	0.39	0.18	0.39	0.17	0.52	0.19	0.46	0.15
Causal-Experimental SCS	0.50	0.22	0.45	0.24	0.60	0.28	0.47	0.24
Spatial-Imaginal SCS	0.64	0.25	0.67	0.23	0.78	0.22	0.70	0.22

TABLE 3: Differences between Experimental and Control groups' means in the five SCSs

Source of variance	Dependent Variable	Sum of Squares	D. F	<u>F</u>	p-value
Group	Quantitative-Relational SCS	.116	1	3.142	>.05
	Qualitative-Analytic SCS	.232	1	8.794	<.05
	Verbal-Propositional SCS	.001	1	0.057	>.05
	Spatial-Imaginal SCS	.143	1	4.299	<.05
	Causal-Experimental SCS	.112	1	4.109	<.05

## Discussion

In this study we tried to provide a framework for the development of a cognitive acceleration program through mathematics with the aid of technological tools. We presented the design, the structure, the content and the psychological premise of the project developed and focused in the ways in which we integrated contemporary technological tools in the program.

The results of the implementation of the project showed that students accelerated their ability in the domains of the Qualitative-Analytic, Spatial-Imaginal and Causal-Experimental SCSs. These results may support the argument that the project achieved a general cognitive change effect. Experimental group students performed better than the control ones in three out of the five SCSs. This study showed that the project had a major positive effect in the Qualitative-Analytic and the Spatial-Imaginal SCSs, finding which might implies that the lessons that incorporated Ms-Excel and Geometer's Sketchpad software implemented successfully the aims of the project. We may conclude that the integration of technology in mathematics teaching not only develops mathematical thinking (Tinsley & Johnson, 1998), but can also enhance general thinking abilities, such as qualitative-analytic, spatial-imaginal and causal-experimental thinking, and promote higher-order thinking skills by developing appropriate teaching scenarios. However, it should be said that the results of this study can only give us an indication of the effectiveness of the integration of technology in cognitive acceleration intervention programs in mathematics because many other factors may cause the differences in the two groups' attainment. A delayed post-test a year after the completion of the program may give us useful information about the duration of the cognitive change.

A surprisingly result is the fact that experimental group students did not show a significant effect in the Quantitative-Relational SCS, which is assumed to include the main mathematical abilities (Demetriou et al, 2002). This may be due to the fact that the control group which followed the traditional instruction focused mainly on this SCS abilities, such as the four operations, mathematic relations and numeric patterns.

In this study we tried to identify, grasp and take advantage of core concepts that research and practice in various domain fields have accumulated as a framework for enhancing cognitive development. The success of this project shows the necessity of the integration of technology in mathematics curriculum and the necessity of the development of appropriate activities to achieve thinking curricula, thinking classrooms and thinking schools. At a practical level, the present study can benefit teachers, who are faced with limited time and crowded classrooms. They may use research results to identify fruitful ideas in the language and construction actions of their students to learn mathematics and further develop their general cognitive ability.

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# MATHEMATICS LABORATORY ACTIVITIES WITH *DERIVE*: EXAMPLES OF APPROACHES TO ALGEBRA

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**Abstract:** *The aim of this paper is to analyse some attempts to implement mathematics laboratory activities with a computer algebra system (Derive), examining the role played in the considered activities by the various actors (the teacher, pupils, the instrument, the discipline), their mutual interactions and the consequent construction or non-construction of concepts and meanings.*

*Chosen examples concern particular situations of approach to algebra: hence, significant features are pupils' age (young if compared to more frequent uses of CAS), their competence/incompetence in relation to mathematical contents dealt with, importance of the involved linguistic aspects (languages, formalism, abstraction, use of variables).*

**Key words:** Algebra, technology, Derive, Mathematics laboratory, Secondary School

## 1. Introduction

In the last few years studies on the educational use of ICT have been focusing on the problem of integrating computer technologies in teaching and the consequent transformation of teaching practices with relation to the complex interactions between mathematical objects, tools, modalities of use, students, teachers, interpreted in different theoretical ways by researchers (see for instance Lagrange, 2000; Artigue, 2001; Bottino & Chiappini, 2002; Hoyles & Noss, 2003 etc.).

Within this debate various studies carried out by Italian researchers contributed to elaborating the idea of mathematics laboratory (e.g. Mariotti, 2002; Chiappini & Reggiani, 2003; Bonotto et al., 2002). The idea is synthesised in a document concerning new mathematics curricula, elaborated by Italian Mathematics Union (UMI):

*'A mathematics laboratory is not intended as opposed to a classroom, but rather as a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people (students and teachers), structures (classrooms, tools, organisation and management), ideas (projects, didactical planning and experiments). We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practising. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together. It is important to bear in mind that a tool is always the result of a*

*cultural evolution, and that it has been made for specific aims, and insofar, that it embodies ideas. This has a great significance for the teaching practices, because the meaning can not be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used, in the schemes of use of the tool itself. The construction of meaning, moreover, requires also to think individually of mathematical objects and activities'*  
(<http://www.dm.unibo.it/umi/italiano/Didattica/2003/secondaria.pdf>).

In particular, I will refer to the idea of mathematics laboratory mediated by ICT, as proposed and discussed in Chiappini & Reggiani, 2003, Chiappini et al., 2003, and to the related theoretical framework. In these studies mathematics laboratory is intended as a teaching and learning activity in which the aim is to bring about an integrated use of technical tools and psychological tools (Vygotskij, 1978) oriented to the construction of the experiential basis which is needed to appropriate mathematical concepts.

In this view, the role of the teacher who builds the teaching and learning activity, chooses the technological tool and tries to create the conditions (utilization schemes) which allow the concepts construction, is central (Mariotti, 2002).

## **2. Research problem**

In the context of the theoretical framework recalled above, this paper is meant to present and discuss two teaching and learning activities carried out with the *Derive* for Windows CAS, and addressed to middle school and first two years of secondary school students, not expert in algebraic manipulation. The two units belong to different research projects and have their own specific objectives.

The first activity (Giuliani & Tagliabue, 2002; Reggiani, 2002) is an approach to algebra and aims to favour the use of algebraic language as an instrument for solving problems, as well as to introduce pupils to an aware use of symbolic calculus in grade 8<sup>th</sup>. The main research aim was to verify whether the presence of CAS favours these objectives and study students' "spontaneous" use of the software. The activity was part of a project called SeT (Science and technology), aimed at the elaboration and spreading through the web of teaching materials for scientific and technological education.

The other activity (Baldrighi et alii, 2004), carried out transversally between middle school and secondary school, proposes an itinerary leading to both the construction and the study of the equation of a line, starting from direct proportionality problems. Pupils are required to use different software functions and commands, in particular the "vector" function which permits point by point constructions and parametric representations. This work was carried out within a teacher training project and focused on construction and management of the teaching unit in the classroom.

The paper is meant to analyze and compare some features of the two teaching and learning activities, meaningful in terms of the outlined framework, and precisely

- pupils' age and competencies
- different modalities of use of the software
- different roles played by the teacher
- mathematical objects and linguistic aspects

The aim of analysis is distinguishing different modalities of use of C.A.S. and *Derive* in particular, which characterize the role of these instruments in concepts construction (Artigue, 2001). Topic of this paper, as already said, is algebra approach in inferior school, underlining in particular positive effects and risks connected to software different ways of use.

### 3. Methodology

Exposed work consists in extrapolating, from analyzed teaching activities, suggestions about the questions of the previous paragraph, so it is necessary to clarify which are the methods used in classroom activities discussed here and for their observation.

#### a. Classroom work

In both working units *Derive*, version 4.09 for Windows, was used. The choice of this version is due to the fact that although not being one of the latest, is available in many schools and has commands in Italian.

In the first teaching unit the use of the software is limited to few commands, and exclusively to the “algebra window”. In particular, the software is used to allow pupils to verify properties both using very large numbers and exploiting its potential for factorisation or generalisation through “writing a formula”. In this working unit the software is substantially viewed as support to the elaboration and verification of conjectures. User-friendly features of the system makes students' acquisition of necessary competencies easier.

In the second teaching unit both algebra and 2D graphical pages are used. The “vector” function is used, as already mentioned, together with commands that make its management possible, although their syntax may be not trivial for pupils at the considered level of age and competencies.

Both experiences were carried out with pupils aged 13, 14 and 15 and developed through moments in the classroom (paper and pencil activities, discussion moments) and moments in the computer laboratory where two or three pupils could share one computer. A worksheet for each activities was proposed to pupils, on which the problem situation and some guidelines were written. In the second teaching unit, worksheets proposed a guided path, through a sequence of questions, because of the higher difficulty from both formal and conceptual level. In the two activities pupils were given worksheets individually and one could decide whether he/she wanted to complete it individually or sharing it with his/her companion at the machine. Students were anyway invited to collaborate and discuss with companion(s).

During the activities the teacher tried not to give hints: in a later moment a discussion was carried out and it was a fundamental moment for both acquisition and sharing of knowledge. In this phase all the different answers were taken into consideration, compared and commented upon: the teacher tried to make everybody participate in the discussion, acting as “moderator” and proposing the emerged ideas so that they could be clarified or possibly corrected and shared.

At the beginning of both teaching units pupils were able at least to operate with natural numbers, in particular they knew operations, their properties, including powers and relation of divisibility; they had also dealt with cases in which a situation expressed in verbal language must be translated into symbolic language (the most common case, proposed to pupils of every age, is the translation of the situation described by a problem’s text into an expression) and had also come across situations in which letters are used to represent numbers.

A particularly interesting problem was the *management of time* in activities which involve working partly in the classroom and partly in the laboratory. Often pupils struggled to link work carried out in the laboratory with that carried out in the classroom, also due to time constraints imposed by the weekly timetable and the availability of the computer laboratory. Reflection on this issue required a deeper analysis of how the teacher can manage computer activities and discussion in the classroom; moreover, in the case of secondary school, an issue emerged of analysing ways of better integrating these activities in the usual curricular work.

#### **b. Observation**

A qualitative analysis of the activity was carried out through the use of both completed worksheets and observation protocols collected by two university students writing their dissertation, who participated in each of the activities as observers in laboratory and discussion moments. Synthesising discussions were recorded whereas difficulties were found in observing both interaction with the software and interaction of pupils working in small groups at the same computer. The actual layout of the computer laboratory, which could not be changed for a lack of space, influenced observation. We acknowledge that a completed worksheet or a possible printed version of the work carried out are only products and provide partial information. In the case of activities analysed here, the presence of external observers allowed us to follow small groups’ work. However this remains a major problem for teachers managing the activity in normal classroom situations.

### **4. Synthetic description of examples**

#### **a. Approach to algebra**

This teaching unit involves three different types of activities:

A first type of problems is meant to make pupils reflect upon conventional aspects of both arithmetical and algebraic language, with particular attention to the use of

parentheses and to priorities in operations. This reflection is solicited through translation of some situations from natural language to algebraic language and subsequent writing of the obtained expression in *Derive*'s "algebra" window. We aim to make pupils get to master conventions of algebraic writing in a line, necessary for inserting algebraic expressions when using the *Derive* software, and consolidate their knowledge of conventions of arithmetical-algebraic language through a transition from usual symbolism to the one required by the software. Pupils are solicited to think about operations involved in an expression, their priorities and how they must be codified in order to make *Derive* transcribe them according to the usual conventions. Pupils usually know and easily accept conventional priorities in operations and follow them when they need to make written calculations. A higher degree of awareness is required though when an expression must be translated from one code to another one, as is the case of software or programming languages, which are characterised by conventions that partially differ from and are more rigid than those adopted in algebra. In particular it is necessary to use parentheses correctly in order to get the desired expression on the computer screen.

The second group of activities aims to lead pupils to use algebraic language as an instrument for generalisation and verification, so that they can acknowledge symbolic language as an effective means of expression. Proposed activities concern properties of even and odd numbers and other divisibility issues. The aim is to provoke pupils' thinking around properties of natural numbers, to introduce them to generalisation through a study of many cases, to make them use letters for generalising properties. Situations are purposefully chosen so that the simplest ones can be solved almost without calculations and generalised through verbal language, and the increasing complexity may suggest the use of many numerical trials and the translation into a formula. These problems are often tackled with paper and pencil: here a fundamental mediational role is played by the available software, which enables pupils to perform many numerical trials and to write and transform expressions, thus focusing on meanings rather than on calculations.

Again the last group worksheets concern properties of natural numbers, but tackle more complex situations and are centred on the transformation of formulae.

Let us provide an example of a proposed problem

*Take a natural number, make it to the power of 3 and subtract the number itself from the result. What do you get? What properties does the obtained number have?*

Some pupils try to provide the answer after carrying out several numerical trials with *Derive*. Others represent the proposed situation in formal language, at very different levels:

- Some pupils simply write the expression  $n^3 - n$  without transforming it
- Others formalise in  $n^3 - n$  and then try to factorise without the software's help, getting either to wrong expressions or to the correct expression  $n(n^2 - 1)$ , but do not further elaborate the latter.
- Others get to the expression  $n^3 - n = n(n+1)(n-1)$  in which factorisation is obtained using *Derive*.

The difficult thing is for pupils to use the obtained factorisation to “read” the number’s properties.

The teacher’s work is clearly essential at this stage: he proposed comparison of strategies, results and possible interpretations and also the analysis of algebraic transformations carried out by *Derive*.

### **b. From proportionality to the equation of the line**

The teaching and learning path is long and complex and developed through several steps:

- Analysis of situations of direct proportionality
- Comparison between equivalent algebraic symbolic expressions
- Transition to graphical representation
- Reflection on the definition set of a function: difference between discrete and continuous
- Transition from representation of a straight line to analysis of its features: gradient, intercept.

Here our analysis will be limited to the description of some activities in which the use of the *Vector* function was introduced in view of a later use in the production of tables representing direct proportionality situations, a first step toward representation on the Cartesian plane.

At later stages the *Vector* function was also used to vary either gradient or intercept and observe the systems of lines thus obtained.

The first task proposed to pupils is to explore how the *Vector* function (which allows the production of ordered n-tuples of elements) works and which is its potential.

They are suggested to insert:

*Vector* ( $a, a, 1, 10, 1$ )

*Vector* ( $a, a, 0, 15, 2$ )

in the Algebra Page and observe what they produce and try to interpret them. As it is well known, the former produces the natural numbers from 1 to 10 step 1 and the latter the natural numbers between 0 and 15 step 2.

Many pupils met difficulties in understanding the relationship between what is produced by *Derive* and the related symbolic expression, in particular the role of the third numerical value which is in the instruction (the step): for some it is about a term to add, for others about a multiplicative factor.

At this stage it was useful to make pupils examine other examples they had freely constructed. In order to understand the meaning of the step and exclude a possible misunderstanding related to multiples it is enough to modify the starting point as in *Vector*( $a, a, 3, 10, 2$ ).

After making explicit the meaning of previous symbolic expressions and their syntactic structure: *Vector* (*expression, variable, initial value, final value, step*), the expression *Vector* ( $2x, x, 1, 10, 1$ ), where the novelty is the element  $2x$ , was examined again. The aim of this example is to make pupils distinguish between the role of

expressions that appear in the first two entries of the vector and to highlight the distinction between independent variable and function value.

Later students are asked to produce even numbers between 1 and 11 using the *Vector* function.

Among the proposed solutions: *Vector* (2x, x, 1, 5, 1), *Vector* (x, x, 2, 11, 2), *Vector*(x+1, x, 1, 9, 2). In this case pupils often used trial and error strategies, using the software as a validation instrument.

Later, after producing with paper and pencil a 2-dimensional table to study the relation  $x \rightarrow 2x$ , giving the variable  $x$  natural values between 0 and 5, pupils learn how to construct it with *Derive*.

To reach this goal pupils learn how to create a 2-dimensional Vector having  $x$  as its first element and  $2x$  as its second element (the two headings of the table), and which is visualised on the computer screen as  $[x, 2x]$ .

Next step, guided by the teacher, is the writing of *Vector* ( $[x, 2x]$ , x, 0, 5, 1), which, once simplified, generates the required table.

Pupils are asked to describe the procedure and interpret both the used expression and the result. Possibly this step is too difficult for these pupils' age and competencies. In particular  $x$  plays the role of independent variable both in the *Vector* function and in the function which is parametrically represented in the vector  $[x, 2x]$  and this may raise difficulties.

Protocols highlight that pupils, although not grasping clearly the link between the writing inside square parenthesis and the variable in the round parentheses, did not find any difficulty in following the procedure needed to make *Derive* produce the 2-dimensional table. At the level of terminology they use terms such as step, initial and final value and, in some cases, the term "variable" rather correctly.

The next step is the transition from representation of the proportionality relationship through the table constructed in the *Algebra page* to representation in the *graphical page* of the points corresponding to pairs of values in the table, and their possible link through a command of the *Options* menu.

Protocols' analysis does not show particular difficulties in the management of both *Algebra* and *Graphical* pages.

In the final phase the *Vector* function is used again, as mentioned earlier, to produce variations in the parameters of the equation of the line, that is gradient and intercept. This can certainly be a further cause of difficulty, since a parameter is used as a variable and not the independent variable of the represented function.

In this case the comparison between algebraic representation and graphical representation is particularly meaningful, since it makes possible for pupils to visualise the effects of parameters' variation also through the aid of different colours.

Let us point out that successive versions of the same software make management of parameters' variation more immediate and probably more effective for a perceptive-motory approach to knowledge.

## 5. Sharing and discussion

Teaching units presented in these pages propose the use of a symbolic manipulator in the phase of approaching algebra with the double objective of making pupils reflect upon conventions and properties of algebraic formalism and make available for them the software's potential in terms of arithmetical and algebraic calculations: this potential allows pupils to tackle complex situations without the distraction of calculations and avoiding that mistakes in algebraic transformations lead to wrong conclusions. Moreover the second activity is meant to make graphical potential of the software available to pupils and to use the possibility of "perceiving" visually some operations carried out at algebraic level.

The points mentioned in the research problem constitute the outline of the following discussion.

### – Pupils' age and competencies

The choice of using a software like *Derive* in the approach to algebra, that is with young pupils, novices in algebraic manipulation, requires that the teacher be very careful and supervising the situation. On the one hand the request for a higher attention in writing expressions and using parentheses, needed in order to use the software, and the possibility/need to compare different ways of writing the same algebraic expression are certainly useful opportunities to improve pupils' mastery of algebraic language. On the other hand, the possibility of carrying out not completely controlled algebraic manipulations or even manipulations that one would not be able to carry out without the software brings up a situation that might deviate from teacher's intentions. Therefore, sometimes the teacher might usefully ask pupils to verify with paper and pencil what they found, thus suggesting the importance of checking results and an integration of the use of software with competencies in algebraic calculations they already have.

### – Different modalities of use of the software

The two experiences show significant differences in the use of the software proposed to pupils, especially according to the different phases.

In both experiences, mostly in the initial phase, pupils are asked to use the software to write expressions that solve problems expressed in natural language (in the first case) or to write sequences of numbers (in the second case).

In this phase using the software and understanding how it works is their task. We might define this as a "forced" use, since pupils do not find it useful in terms of the proposed problem, although they generally find it "fun".

The aim of these activities is on the one hand to allow pupils to get familiar with the software, with the rules required to insert expressions in the algebra page and with the syntax of some of its functions, and on the other hand to make them think about formal rules characterising the languages involved.

Moreover a "spontaneous" use of the software has been observed, especially in the first activity, to make numerical trials, to manipulate and transform formulae, to elaborate and verify conjectures. This use is not always corresponding to what was expected a priori, i.e. in this case the using schemes employed by pupils do not

necessarily coincide with those proposed and foreseen by the teacher. In particular, we noticed that, within the constraints of the proposed problems, pupils made few numerical trials before elaborating their conjectures, whereas they made extensive use of the software for algebraic elaboration of expressions, thus facing the already mentioned risks of lack of control.

In the second activity there is a frequent “guided” use of the software for writing tables and plotting graphs, also in relation to the already underlined difficulty of the proposed situation. In this case we notice that the phase of interpretation of the meaning of written expressions and their results becomes particularly important. This phase makes pupils engaged both at perceptual level, in observing, and at the level of conceptual elaboration, necessary for a meaningful verbalisation. In this phase there is a significant possibility of constructing other expressions, analogous to the proposed ones, and to use the software to validate one’s conjectures. Examples are provided by the different solutions proposed by pupils to the problem of producing even numbers between 1 and 11, using the vector function. Here, maybe, we can see an example of integration of *technical tools* and *psychological tools* (Vigotskji).

Finally, Derive plays an important role of “visualising tool” in the second activity we proposed, to favour the comparison of analytical expression and graphical representation, as parameters vary, and enact, for instance, the construction of a meaning for parameter.

– Different roles played by the teacher

The role of the teacher, analogous in the setting of the two activities, which are based on the same scheme described in the methodology section, is different in the two cases, because of the differences in the worksheets’ outline. The first ones are based on presentation of a problem to be solved, with the software available, the second ones are directed and often finalised to the use of the software.

– Mathematical objects and linguistic aspects

It is important to point out that in the second activity mathematical objects such as relations, vectors, variables, parameters are involved, and that they raise several difficulties for pupils at the age and competence level we considered. Representation produced by the software is not always immediate at perceptual level and in order to operate with it, at least in the considered activities, a pupil is supposed to be able to decode symbolic language (see, in particular, the use of the *vector* function). However the mediational role of the software allows an approach to symbolic representations, that would not be proposed otherwise, with a positive follow up on competencies in algebraic language.

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## PUTTING THE *LEARNING* BACK INTO E-LEARNING

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**Abstract:** *The design of web-based learning environments is primarily focused on the production and delivery of content to a learner. The principles of constructionism are intended to guide the development of learning environments where the learner has more control. In this paper, we describe characteristics of constructionist and learning environments that can foster the learning of mathematics. Our experiences are drawn from the development of microworlds for an e-museum. Reflecting on this process turns out to provide some fresh insights into how e-learning environments might be re-conceptualised in the future.*

**Keywords:** mathematics; learning; microworld; constructionism; design.

### 1. INTRODUCTION

In this paper, we reflect on our experiences of developing microworlds as part of an e-museum to draw inferences about issues related to using web-based environments for the teaching and learning of mathematics. The broad aim of the e-Muse project<sup>1</sup> was to investigate the concept of developing an Internet museum. A museum consists primarily of exhibits, supplementary explanatory material related to the exhibits together with hands-on activities to engage visitors. The e-Muse website is in essence a large collection of assets related to the ancient Olympic Games that comprises text, images, videos, interactive areas for participating in discussions and facilities for uploading work and downloading other children's work.

When we began this project, we were interested in two tensions. In order to develop a virtual museum that bridged museum and school environments, it was apparent that there was likely to be a cultural conflict. Perhaps museologists would be concerned primarily with accuracy and appropriate presentation, whereas classroom practitioners' foremost concern was likely to be about interaction and engagement. Of course this is a characterisation in so far as both cultures would have concerns about accuracy and engagement but we felt that the priorities might be distinctive.

The second tension is an extension of the first. In a sense, museologists might be characterised as most interested in the efficient *delivery* of accurate materials, and we perceive this to be an aspiration shared by designers of so-called e-

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<sup>1</sup> E-Muse: e-learning for museum and schools environments, <http://emuse.cti.gr>  
EC e-learning initiative: 2002-4084/001-001 EDU-ELEARN

learning environments. In contrast, our own approach is heavily influenced by the constructionist literature (Harel & Papert, 1991), which places emphasis on ownership of ideas by the learner. In that respect we would tend to align ourselves more closely with classroom practitioners who place the accent on learning rather than delivery.

To provide an interactive experience for e-museum visitors, we have developed two microworlds that are intended to engage and stimulate exploration of the e-museum. These microworlds, based on the throwing events of the Olympics, are targeted at children of 10 years old and upwards. Our aim in this paper is to describe our experiences of developing these microworlds in order to explore the larger question: How do we invest constructionist principles into web-based situations? In section 2, we will describe related literature before describing the development of the two microworlds in section 3. In section 4, we discuss the characteristics of these microworlds, and then return in section 5 to consider the above question in light of our research.

## 2. THE PEDAGOGIC CONTRIBUTION OF MICROWORLDS

Examples of the careful design of microworlds began to emerge in the 1960's when a team, headed by Papert and Feurzeig, was developing the computer language, Logo, at MIT. This early work was primarily concerned with programming and problem-solving (see Papert, Watt, diSessa, & Weir, 1979; Watt, 1979). In particular, they advanced the radical notion that children need to play with and use mathematical concepts within a supportive computer-based environment before being introduced to formal work with those concepts (Papert, 1972).

When mathematizing familiar processes is a fluent, natural and enjoyable activity, then is the time to talk about mathematizing mathematical structures, as in a good pure course on modern algebra. (p.18)

These initial ideas reached a climax (Papert, 1980) in which a radical vision of education was proposed. Since then, the work has been elaborated to the point where a new paradigm for the teaching and learning of mathematics, the *constructionist* approach, was put forward (Harel & Papert, 1991). We believe that this paradigm has much to teach developers of e-learning platforms and that reflection on the design of our microworlds can help to crystallize what those lessons are. First, let us distil six constructionist criteria from the literature.

### i) Quasi-Concrete Objects

Turkle and Papert (1991) have referred to the way that the computer offers access to formal ideas in a concrete way, since abstract mathematical ideas, represented in iconic form on the screen, can be manipulated directly by the user.

### ii) Using Before Knowing

In our everyday lives, we typically use artefacts for particular purposes. Through that use, we learn about the effectiveness of the tool, its limitations, how well it serves that purpose and sometimes we may gain some understanding of how it works. In schools, mathematics is a subject where you learn how to generate the object before you use it. In practice, more often than not, the former task proves too difficult, especially when disconnected from purpose. The computer offers the possibility of turning the learning of mathematics round so that use precedes generation (see the *Power Principle* Papert, 1996).

iii) Integrating the Informal and the Formal

diSessa has suggested that we incorporate versions of the formal representations of the mathematical objects in such a way that the child may be able to make connections between the various formalisations and their informal use (diSessa, 1988).

iv) Dynamic Expression

When Papert proposed the turtle as a tool for constructing a dynamic notion of angle (and of course much else), he acknowledged that the computer offers a medium which unlike paper and pencil can incorporate dynamic representations of the world. He suggests that the use of systems which are expressive of dynamic and interactive aspects of the world are more engaging to learn than static and abstract formalisms.

v) Building

Constructionists base their approach on a tenet that encouraging the building of artefacts is a particularly felicitous way of teaching mathematics. Pratt (2000) has demonstrated how this approach can be modified into related approaches such as mending.

vi) Purpose and Utility

The microworld approach can encourage purposeful activity through the building and modification of artefacts. In so doing, emergent knowledge is imbued with *utility* (Ainley, Pratt & Hansen, in press), in which the abstractions are seen as useful and the limitations of those abstractions are gradually discriminated.

In the next section, we move on to describe the microworlds themselves.

### 3. THE MICROWORLDS

As described above, the primary motivation for the development of the two throwing microworlds was to provide context and motivation for engaging with the museum content. We adopted the methodology of design experiments (Cobb

etc, 2003). Using this approach, we cycled between design and testing phases. As the design stabilised, we used increasing numbers of children, allowing us to be more systematic in our study of their activity. Each design in effect encapsulated emergent conjectures about the relationship between the tools and the children's learning. We describe below the objectives of the microworlds and discuss their final designs.

### 3.1 Shotput

The shotput microworld was intended as a multidisciplinary environment, bringing together physics, maths and physical education. Its primary objective was to explore factors involved in projectile motion, situated in the challenge of maximising how far a child might throw a shotput. Children were given the opportunity to throw the shotput, after which the distance thrown, the time of flight and their release height were measured. These values were entered into the computer microworld, which could replay the actual throw. In Figure 1, the flight path of an example throw can be seen.

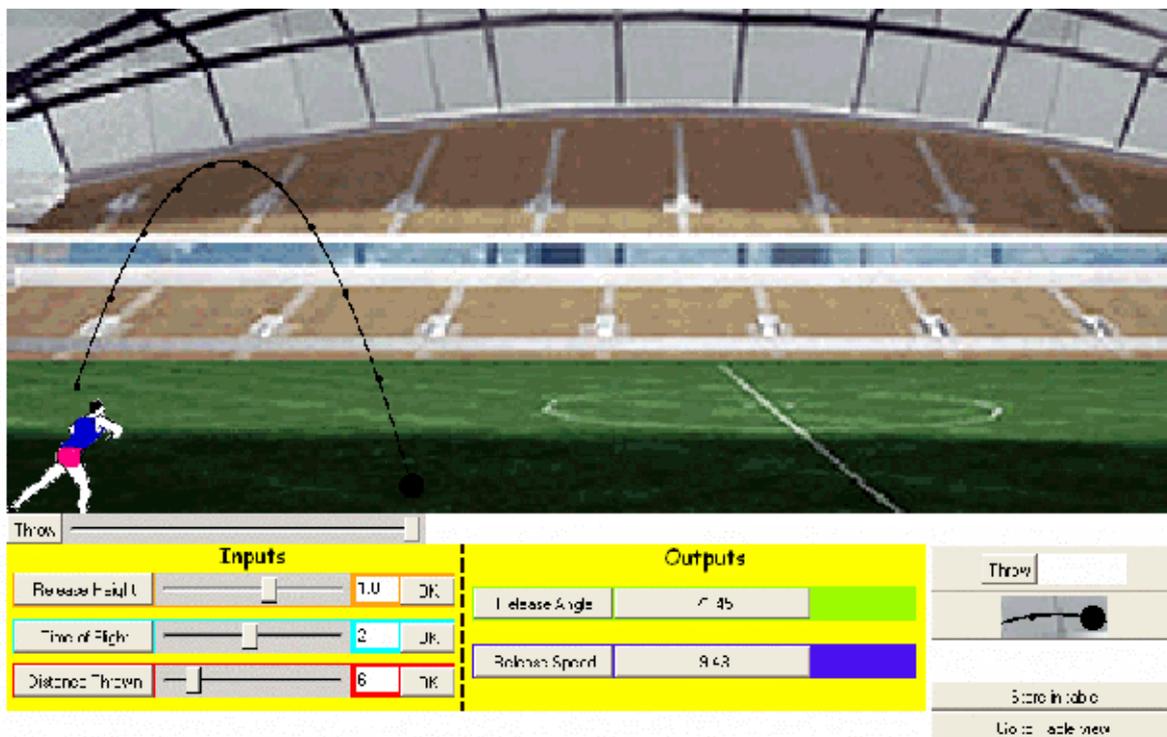


Figure 1: The shotput microworld

Children were then able to experiment with the different parameters in the model to try and improve their throw, aiming to establish the optimal release angle for a given release height and release speed. The main challenge was to understand the distinction between inputs and outputs, knowing which variables were sensible to change and how they might be changed. The microworld also contained facilities for children to tabulate interesting results, compare multiple flight paths in parallel, and produce graphs of the table of results.

### 3.2 Discus

The discus microworld shared common interface structures with the shotput microworld, enabling prior experience to be leveraged. Children threw a discus, made relevant measurements and then entered that data into the microworld to produce a simulation of their throw<sup>2</sup>. Children could then explore how to improve their throw and how to design a good discus. Experimentation with the input variables (the release height, release angle, release speed, discus tilt and the wind speed) could establish the optimal flight path for each individual. As with the shotput microworld, there were facilities for storing interesting throws, comparing multiple flight paths in parallel and producing graphs of the tabulated results. The discus microworld also contained a design view where children could experiment with discus design to explore how diameter, weight and colour affect the distance thrown (see Figure 2).

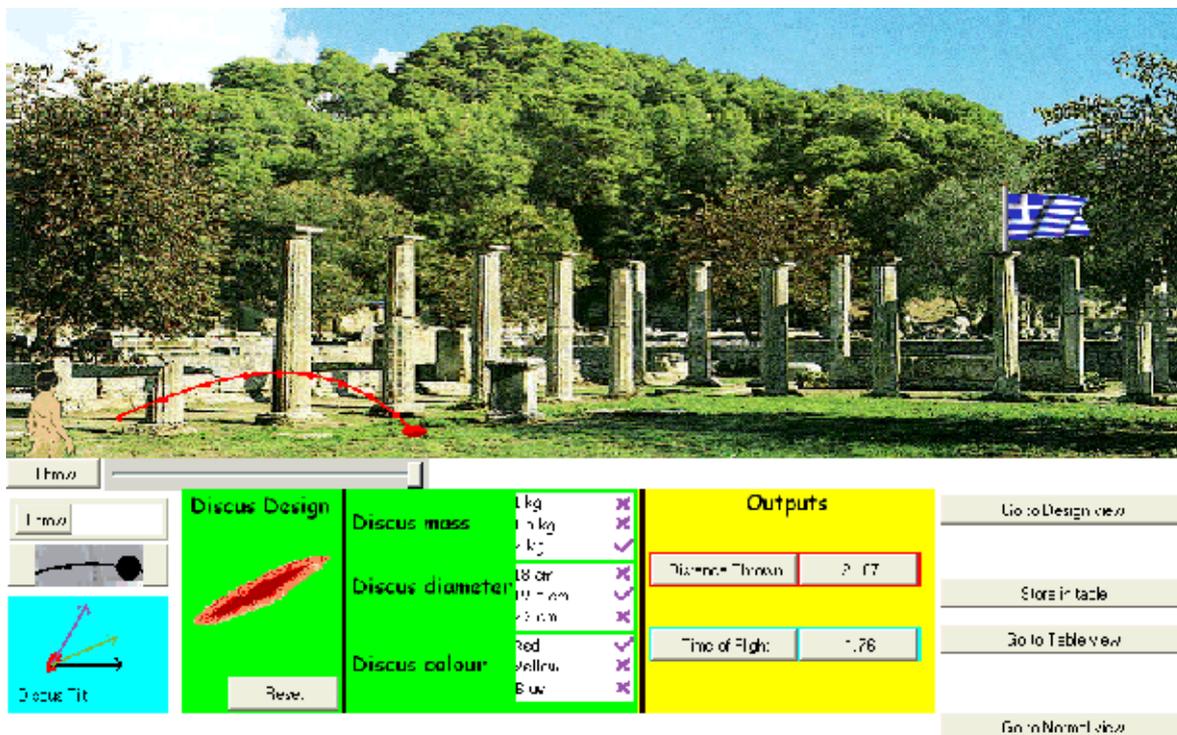


Figure 2: The discus microworld in design view

## 4. TOOL CHARACTERISTICS

Having described the microworlds, we now wish to reflect on some of the tensions that we faced during the design process, expecting that such deliberation should yield useful insights into the process of designing web-based resources. In particular, we wish to articulate how our struggle with those tensions distributed across the six constructionist principles outlined above.

<sup>2</sup> Completely accurate determination of a discus flight path is exceedingly complex. Our model is based on the work of Frohlich, 1981, and Hubbard & Hummel, 2000.

#### 4.1 Plug-and-play versus programming

Since the earliest days of Logo, programming has been an integral part of the constructionist paradigm. Yet modern languages have become increasingly high-level, and direct manipulation tools have become so available, that it is increasingly difficult to distinguish programming from related activities. Our microworlds were written in Imagine<sup>3</sup>, an extraordinarily powerful version of Logo. The designer (or indeed user) has available a vast array of direct manipulation devices such as buttons, switches, text boxes, sliders and so on. These features afford the quasi-concrete representation of mathematical or physics concepts. The sliders for release angle and speed, for example, gave the children direct control over complex ideas and, through exerting this control, they began to appreciate projectile motion, a demonstration, we would say, of Papert's Power Principle.

However, the same features that allow direct manipulation also make it relatively easy for a designer to design conventional programming *out of* the microworld. In our context of integrating the two microworlds into an e-museum, we exploited a facet of Imagine to create web-based projects, in which the user can run the project from a web browser without requiring Imagine itself. However this facility does not permit programming by the child. Compared to the creativity afforded by more conventional microworlds, we felt this was a loss. The plug-and-play nature of web-based resources seems to constrain the integration of the formal and the informal.

#### 4.2 Open/Closed microworlds

Designers of educational software have to consider just how open or closed they should make their software. The constructionist principles of Papert assert that children will learn best if they are left to their own devices to explore and construct in line with their own interests (Papert, 1980). As such, the design of educational software would be as open as possible – children would be free to follow their own interests within an environment where a particular theme could be investigated. For instance, in Logo, children free to explore projectile motion in idiosyncratic ways might develop a mediaeval project involving catapults or they may instead find the optimum way of throwing a cricket ball. In an educational system where accountability is important, the constructionist approach is hazardous since the teacher has relatively little control over the material, making assessment more difficult.

An alternative approach is closed software where a program is designed to support restricted interaction related to solving a particular task. Within such software, a child is shielded from making mistakes and exploring their own hypotheses, both of which are important elements of the learning process

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<sup>3</sup> Imagine is an object-oriented parallel-processing version of Logo that allows the programmer considerable interface design options. It is published by Logotron: <http://ns.logotron.co.uk/imagine/>

(Lewis, Brand, Cherry & Rader, 1998, include these ideas in a set of design principles emerging from work using *Agentsheets*, a graphical grid-based programming environment). For instance, a program for learning about projectile motion could simply allow the input of parameters for a throw (release angle, speed and height) to generate display of the flight path. In this type of environment, a child has little scope for either exploring a range of questions related to projectile motion or the ability to make and test personal hypotheses.

The perspectives of openness and closedness have impacts on the way that educational software can engage learners. In between the two extremes described above educational software can be partially open within a closed area.

For instance, the shotput microworld is closed within the domain of exploring projectile motion – yet it remains open to the possibility of exploring hypotheses, making mistakes or generating irrelevant results. In the shotput microworld, inputs are distinguished from outputs but in a way that may be unfamiliar to children. The children were comfortable with the notion that the inputs were those factors that they influenced during a physical throw (release angle, speed and height):

1. Researcher: As the person throwing the shot, what are the things that you can input?
2. L: What at the moment?
3. Researcher: If you were actually throwing it. What would you have control over?
4. J: The angle that you throw it.
5. L: Your release height... oh no you can't.
6. Researcher: I guess you could stand on a box, or something.
7. J: you can change your release speed.
8. Researcher: How?
9. J: You could throw it with more power.

Yet mathematically, any variable might be an input (as to a formula). Rather than protect them from this possible conflict, we felt this was an issue to be grappled with and hopefully understood:

10. Researcher: Are you happy with your inputs and outputs?
11. L: You can't really control the distance.
12. Researcher: What do you mean by that?
13. L: Well once you throw it you can't choose where it ends.

Without a programming language available to the children, there was an inevitable constraint on the creativity. We can repackage this issue as a lack of opportunity to build, one of the fundamental aspirations of constructionism. The children using the microworlds played with models but they did not construct their own versions.

### 4.3 Real-world familiarity / Design for purpose

Emergent understanding of projectile motion was of course contingent on feedback. Our microworlds exploited extensively the principle of dynamic expression. For example, the simultaneous throwing of several projectiles was designed to promote a 'feel' for the relative motion of one object against another.

Both of the throwing microworlds were designed to look and feel similar to their real-world counterparts. The microworlds exhibit both *surface familiarity* (objects look and behave like their real-world counterparts) and *cultural familiarity* (objects behave like their real-world counterparts) (Pratt, 1998). For instance, in the microworlds the animations of the throwers and the behaviour of the throwing implements exhibit the familiarity required to enable children to leverage prior experience of the activities into their understanding of the microworld. Indeed, by encouraging children to physically throw the shotput and discus, we reinforced that familiarity. This is not just of pedagogic advantage but also aids research into children's thinking since it provides a *window* on their thinking (Noss & Hoyles, 1996).

Familiarity supports the construction of purpose when sufficiently interesting tasks are created. Nevertheless, purpose does not guarantee the construction of utility by the child. According to constructionist principles, the child needs to be able to play with the pertinent concepts in order to take ownership of them. The more constrained the environment, the less likely it is that children will take this critical step.

## 5. CONCLUSIONS

The process of attempting to embed microworlds into an e-learning environment has illuminated what we see as particular problems with e-learning environments as they are currently designed. The development of e-learning environments has been driven by university needs where the lecture is the dominant teaching method. Lectures are essentially delivery and the Internet is an efficient mode of operationalising such delivery. In some situations, the delivery of factual information is entirely appropriate. On the other hand, educationalists recognise the importance of interaction and constructionists go further to propose a range of principles that facilitate learning.

We have shown that those principles are not easily embedded into a web-based resource. On the credit side, we have demonstrated that the range of direct-manipulation tools available in modern programming environments afford the forging of connections with complex scientific ideas through the use of quasi-concrete objects in dynamic settings. On the debit side, we would argue that integration of formal and informal representations was limited by the lack of facility to program, which would have allowed the children to build their own models. Similarly, the children were not able to test out idiosyncratic

conjectures about behaviour since they had limited facility to express their own ideas. The facility to recognise cognitive conflict and construct new meanings to resolve such tensions is an essential foundation of constructivist learning.

The predominant delivery model for e-learning exhibits this same failure, though perhaps to an even more marked extent. As Bannan-Ritland et al (2002) have indicated, designers of these environments structure content in a particular sequence for delivery to the learner. We agree that:

...there are alternative theoretical foundations other than a traditional instructional system design perspective that can be applied to learning object systems based on constructivist philosophy of learning. To the best of our knowledge, a learning object system based in theoretical approaches steeped in constructivism has not yet been developed. (p.12)

It is not of course self-evident that the level of interaction implied by constructivist philosophy is achievable. Indeed, Ehrmann (2000) has argued that the attainment of interactive courseware is a mirage. He claims that this mirage is due to the high human costs needed to achieve appropriate levels of interactivity. We maintain that the use of Constructionist principles offers the potential for achievement of far greater levels of such interactivity in e-learning environments.

We therefore exhort developers to re-consider design principles for such environments, in effect to put the learning back into e-learning. We are impressed by the approach of the WebLab project<sup>4</sup> where children are being encouraged by the design of the WebLab portal to share their projects, written in ToonTalk (Kahn 1996), with other, usually remote, children. Such sharing involves posting a project onto the website, commenting directly on other people's projects, running projects directly on the web, and downloading them to allow re-programming in ToonTalk. The Weblab project seems an important step forward in thinking about e-learning platform design, even if the download before programming style involves a certain degree of discontinuity in the constructionist process.

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<sup>4</sup> WebLabs is creating new ways of representing mathematical and scientific knowledge of young learners through collaboration, construction and interpretation of how things work. For more information, see the WebLabs project website: <http://www.weblabs.eu.com/>

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# PHYSICAL AND VIRTUAL WORLDS IN TEACHING MATHEMATICS: POSSIBILITIES FOR AN EFFECTIVE COOPERATION?

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**Abstract:** *Through a case study of a series of Galton Board activities, we try to offer some reflections about the possibility of integrating in the teaching model different tools, starting from completely physical activities and arriving - through gradual steps of abstraction - to the “virtualization” of the tools and to the corresponding abstraction of underlying mathematical concepts.*

**Key words:** *educational tools, teaching mathematics, educational technology, probabilistic thinking*

## 1 Several tools in teaching practice

In the debate about new possibilities offered by the computer presence in the school, we present several experiences we had in various different teaching contexts about the use of tools and technologies, both inside and outside classrooms, and offer some reflections about their features and usefulness.

We had the chance of making and using several kinds of tools. Also, in some cases, we added even “corporeal tools”, by organizing games to better introduce some mathematical concepts. Our proposals follow an ideal way from “completely physical” to “completely virtual”, with some intermediate steps, and, in several ways, all of these approaches imply a certain level of interactivity. The teachers, of course, can organize these activities, choosing each time the most proper one or suitably combining them, and develop different educational paths, depending upon their class and on the context. Of course these ways of introducing a new mathematical idea have different characteristic features that should be kept in mind when planning didactical activities; we now illustrate some of them.

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The first consideration regards the question of *time*: often the physical and instrumental approaches need more time to be organized and to be performed, while similar activities in a virtual level are easier to organize and faster to be performed (click-based environments). We think that this is not necessarily a negative aspect: the attention kept on the subject for a longer time can contribute to a more effective and deeper learning.

An other remark concerns students' *level of involvement*: we think that more an approach is corporeal (e.g. a game), more the actors are involved in the environment with all their senses and more they enter in the didactical experience, while in many virtual approaches (also in the so called "multimedial approaches") the actors often just interact with an external environment. Moreover, we think also that the physical/instrumental approaches can better foster the social and relational aspects and interactions inside the group of learners than virtual approaches (even though some kind of human interaction can appear also in the virtual level in the case of computer networks).

A further observation concerns the *accuracy*: the material tools are often lacking of exactness in contraposition with the precise results that can be obtained with a virtual simulation. We will discuss this aspect in more detail in the section regarding the material tools approach.

In order to better plug our arguments into reality, we will illustrate our experiments about a special argument, i. e. introductory activities to probabilistic and statistical concepts through the Galton board model, with some comments about several activities we propose.

The classical Galton board is a wooden table with some rows of pins stuck into it in a certain arrangement and a series of vertical compartments below. The game consists in letting some marbles fall down through the pins and observe how they bounce on the obstacles (one bouncing per row without horizontal shifting) and their final

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distribution in the compartments at the bottom; the apparatus was originally meant to illustrate the genesis of the Normal Curve of Frequency.

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<b>Mathematical concepts that can be introduced by Galton board like environments</b>	
1 <sup>st</sup> – 5 <sup>th</sup> grades	Randomness, uncertainty, qualitative assessment of probability, random variables
6 <sup>th</sup> – 8 <sup>th</sup> grades	Combinatorics, quantitative evaluation of probability
K12 / university students	Statistical concepts, empirical and theoretical distributions: the discrete case, extension from the discrete case to the continuous case.
Informal didactics	Idea of randomness, idea of the meaning of a random variable, qualitative evaluation and comparison of probabilities, analogies with the real world: examples of concrete distributions

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## 2 The Physical approach

In this first approach we suggest some corporeal activities that we call “*body in*” activities.

There is actually an open debate on when and how it’s possible to introduce concept of uncertainty and thus a primary probabilistic thinking. Recently, J. Way (2003) elaborated a three stages model for the development of probabilistic reasoning and she launched an invite for further researches: “*The development, implementation and evaluation of sets of teaching/learning activities for each stage of probabilistic thinking could make an important contribution to mathematics education.*” (J. Way, 2003). Hence, we are planning an early introduction of probabilistic concepts for

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children of 3<sup>rd</sup> – 4<sup>th</sup> grade (8-9 years). According with Way's model, we will operate in the Stage of Emergent Probabilistic Thinking (and the foregoing Transition Phase), in which children become able to order likelihood through visual comparison and acquire the concept of equal likelihood. Our proposal is simple and funny, and tries to be “intellectually honest”<sup>1</sup> as Bruner claims. The idea is to organize the Galton game “physically” in the school.

We have elaborated two versions of the game, the first one to be played in a large room by about 30 children, and a smaller version, a table board to simulate the same game by counters, coins and tongue twisters to be played over the following cardboard scheme:

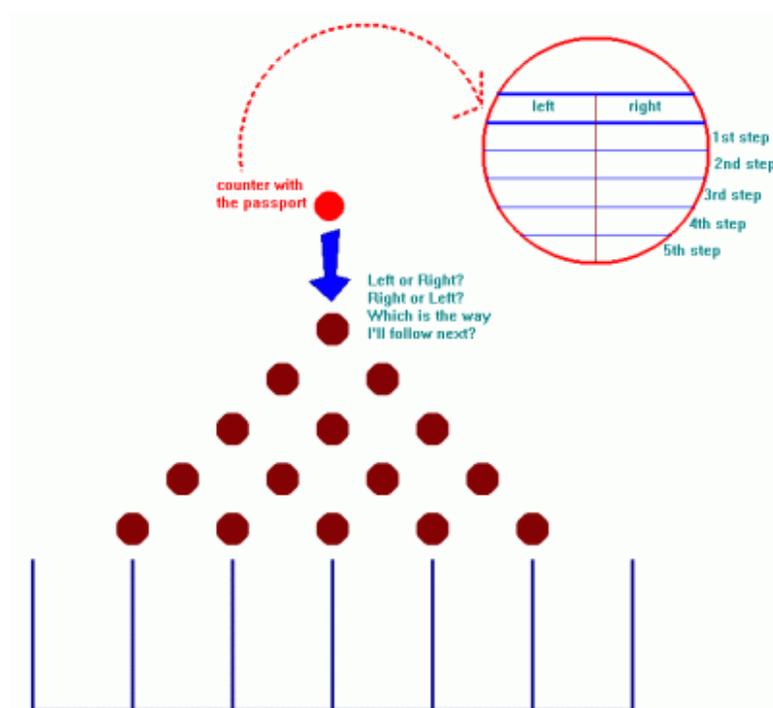


Figure 1 - The Galton Board game

The rules are those “inherited” by the real Galton apparatus: the children have some counters representing marbles, which have to “fall” from the upper part following a

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<sup>1</sup> “We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development.” (Bruner, 1962)

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simple rule: while falling the counter meets the pins, and *at each step* has to decide if going right or left. The simplest and most reasonable way of deciding is to sort “right” or “left” by a coin.

We propose to stress that the marble-counter *‘has to decide’* its way by suggesting the children to say a rhyme (created on purpose) while sorting:

*Left or right /Right or left Which is the way / I’ll follow next?*

Also, we ask the children to sign – directly over the counter or on a passport card – the way of the counter, simply writing the list of right-left turns they sorted. At the end, the counter will be stored in the lane, corresponding to Galton Board’s compartment, it reaches.

The analogous game can be played by children pretending to be a falling ball in a large room equipped with true human-size pins. At the end the children are physically “stored” in a lane marked on floor by coloured tapes and they form a living distribution diagram.

After several trials (not too few, in order to be able to show some evidence), it is possible to ask the children to observe the results, and then to study several questions about the combinatorial and probabilistic aspects of the board: in particular, they can, even in early ages, to recognize, at least roughly (in a qualitative manner), that

1. they were queued almost symmetrically with respect to the central lane of the living diagram,
2. there are always longer queues in the central lanes,
3. the passports of the children stored in a given compartment have the same number of left-turns (and so right-turns and vice versa).

We deeply prefer the human size-version of the game, in which it is easier to focus the attention over the behaviour of the “ball”, and its relationship with probability; we have made even a ball-shaped hat, to use during the “falling down of the child-ball”, to favour children’s motivation. Indeed, here the children are part of an enormous

Galton board; they are not only moved by random variables, but they impersonify random variables.

### 3 A further level: material tools

At this level we propose a series of activities, that can be placed within the classical sphere of “hands-on” activities, according to Piaget’s belief that “*mathematical understanding comes [...] from children’s reflection on the actions they perform on the objects.*” (as reported in Schliemann (2002)). Using Hogle’s words (1995), in this step we offer to pupils “*objects to think with*”, favouring them to build up their own knowledge.

We have at our disposal a “true”, professionally made, very expensive Galton board, owned by Cirdis<sup>2</sup>, but we have proposed a rougher version of it (entirely made by one of us with poor materials as wood, plexiglass, screws)<sup>3</sup>. It has six rows of pins, and works with a fistful of glass marbles. Children – and adults as well – could stay for long minutes to play, without the fear of damaging an expensive tool, hooked at the sound of the falling balls. After some trials, it is possible to ask them questions like these:

- What do you notice?
- Which compartment should you bet before the following trial?
- Why didn’t any marble go/did only few marbles go/ to this or that (first on the left or first on the right) compartment?
- Do you wonder about what happened? Do you wonder that balls arrived mostly in the central compartments?

It’s possible to offer a concise motivation based on the number of possible paths and thus probability of arriving in a given compartment and also to make some practical examples of normal statistical distributions (e.g. people’s distribution with respect to their height).

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<sup>2</sup> Cirdis is an Italian Interuniversity Centre of Research on Statistical Education involving four Italian Universities (Rome “La Sapienza”, Perugia, Padua and Palermo). See the website: <http://cirdis.stat.unipg.it>

<sup>3</sup> see photo at the end of the section.

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Somehow, it seems to us that this simpler exhibit has been more effective than the more perfect one, at least in the catching the people's attention for a longer time.

Also, we have to admit the truth: our exhibit is not working really well... but this fact is not a problem, but rather a true teaching possibility: people are naturally interested in exploring the exhibit, to understand the reasons of failure (mostly a not perfect horizontality), and hence they arrive to conceive '*the abstract idea*' of a Galton board. J.C.Maxwell perfectly describes the power of such 'imperfect' tools in his Introductory Lecture on Experimental Physics: "*The simpler the materials of an illustrative experiment, and the more familiar they are to the student, the more thoroughly is he likely to acquire the idea which it is meant to illustrate. The educational value of such experiments is often inversely proportional to the complexity of the apparatus. The student who uses home-made apparatus, which is always going wrong, often learns more than one who has the use of carefully adjusted instruments, to which he is apt to trust, and which he dares not take to pieces.*"

Also, on constructivist bases, we propose to children to reproduce an absolutely minimal version of the Galton Board at home, simply by using a very common game to plug pins in a holed board that can be their "*personally meaningful products*" (Willis, Tucker, 2001)<sup>4</sup>.

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<sup>4</sup> "learning is an active process, and learning is more effective when students are engaged in constructing personally meaningful products" (Willis, Tucker, 2001)



Figure 2 - Photo of our home-made Galton board



Figure 3 - Photo of the holed board with pins

#### 4 The virtual approach: “mind-on” activities

Simply browsing the internet, it is quite easy to find virtual “versions” of Galton board; one of us, with other authors, developed such a simulation, which has been embodied in a quite wide hypertextual environment. It can be found at the Cirdis website.

There are a lot of reasons to use virtual simulations:

- they allows to change inputs (number of rows, probability of bouncing to right),
- they “work” perfectly (the balls fall down truly pseudo-casually),
- they are more flexible (they can allow a larger number of rows of pins),
- the experiments can be made much more quickly.

We suggest anyway that these reasons can be better appreciate, and the virtual version better used, by students that *before* had the chance to develop their insight on the subject by one of our not virtual approaches. In this way computer technology becomes *just one* important tool among many as in the experience of Chaika: “*teachers taught concepts and then used technology to reinforce, enhance, and elaborate on that instruction*” (as reported in Poole, Axmann, 2002)

## **5 Some final considerations**

Several physical and virtual approaches to the same argument can be offered in different ways, adapted to the context by the teacher.

Such a multiform presentation of course requires a lot of time and energies to be prepared and organized. Teachers, at least in Italy, are sometimes reluctant to abandon a standard way of presentation, and to use their own body and their hands to make the materials needed in such an activity. So, to try this kind of approach requires a bit of “originality” and “courage”, but, in our experience, as a teacher tries this way, he/she realizes that the energy and the extra spent time turns back as other energy and motivation expressed by the students.

In conclusion we can say that there are some clear and definite advantages in such a proposal. First, starting by a game allows children to take the time they need to face the situation at ease, and usually each of them receives at least some small gratification and feels positive emotions. Indeed, even weak students are involved in the game and arrive at some final lane, or draw by segments the distribution diagram given by the counters on the cardboard, even if perhaps they do not immediately grasp the concept behind.

Moreover, it is possible to develop this teaching offer during the years, in a “spiral way”, exploring the same argument deeper and deeper, simply by adding less “concrete” approaches and more complex questions to examine.

Also, in this manner the students arrive to the computer version in a natural way, and understand the power of this instrument to fulfill some needs they have experimented themselves (precision, large number of balls, in our example), so that, paradoxically, we think that our body- and ‘hardtool’-approaches are really useful in destroying the “computer myth” (unfortunately so common), and helping in seeing it in its true light.

We are very worried indeed by the aptitude that children – and people in general – have towards the computer, since usually they do not have the feeling of having the

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intellectual power of understanding its possibilities; according to an Italian joke made by computer experts, a computer user is often an “utonto”<sup>5</sup>, i.e. a bit stupid user. We think that the school has the duty of fighting this situation, by educating “conscious users”, i.e. people that use the computer without hyperhoping from it or hyperextimating it.

Finally, we want to stress again that our approach is fun both for teachers and for students, and, as pointed out by several researchers (e.g. Quinn, 1997), it increases motivation and makes education more effective.

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<sup>5</sup> a play on words obtained by mixing “utente=user” and “tonto= a bit stupid”

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# CALCULATING WITH “RULE AND COMPASSES” AN EXAMPLE IN USING TECHNOLOGY FOR MATHEMATICS TEACHING

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**Abstract:** *This paper describes a complex experience carried out with a group of 43 future teachers, in order to encourage them in using technological tools in structural way in education. A didactic proposal, previously tested with 78 High School students, was presented to the trainees, involving them in an experience of meaningful use of technological tools for Mathematics teaching. Continuing a research trend in using technology for Geometry teaching with special attention for the contents, an unusual subject of Elementary Geometry is treated using DGS, aiming at exciting students' interest and at deepening geometrical concepts. The subject links geometrical and numerical aspects. The methodology involves the students in practical activities for carrying out a sort of 'analogic computer'.*

**Key words:** Dynamic Geometry Software, Geometry Teaching, Conceptions of Mathematics, Teachers' Education, High School, Geometrical Algebra.

## 1. Introduction

Nevertheless technological tools are at disposal of many schools from decades, for various reasons they are used in structural way in education only by a little part of Mathematics teachers, as observed in different times by several authors (e.g., see De Lange 1996 and the more recent Johns & Lagrange 2004). I think that one of the reasons of this fact, paraphrasing what is reported in (De Lange 1996, p. 91), is that many Mathematics teachers *do not see any substantial gain in conceptual understanding* in using technological tools.

Concerning the Dynamic Geometry Software (in the following, DGS), my opinion is supported by previous investigations about teachers' education in using technological tools, in which I observed how trainee teachers' conceptions<sup>1</sup> about Mathematics are reflected in their kind of use of technology for Mathematics teaching. As reported in (Zuccheri 2004), in the Specialisation School for Teachers' Education of Trieste University<sup>2</sup> I observed in fact that many future teachers having

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<sup>1</sup> We means 'implicit conceptions', not necessarily consciously and explicitly stated.

<sup>2</sup> Secondary School Teaching Specialisation School, a two-years post-graduated University course for teachers qualification. For information about the University training of Secondary School teachers in Italy see (Favilli and Tortora 2004).

good mathematical knowledge (graduates in Mathematics, Physics or Engineering, which can teach at High Schools) had a *formalist conception of Mathematics*<sup>3</sup>. Many others with lower mathematical knowledge had an *instrumental view of Mathematics*<sup>4</sup> (graduates in Life Science, Earth Science, Chemistry, as the majority of Mathematics teachers in Italian Middle Schools). This ‘classification’ was performed by considering data coming from individual interviews and from the observation of examinations for the periodic assessment. Successively, by analysing the written lesson projects performed by these trainees as home-work for the course on Didactical Technologies for Mathematics Teaching, I noted that the ‘*formalist*’ were inclined to use technological tools only for reproducing theory which they supposed to have explained before, in a preliminary traditional lesson without computer. Also the most part of ‘*instrumentalist*’ preferred to use technology only for illustrating concepts already explained in traditional way. In this kind of use, not structured and well-integrated in the education process, the technological tools become a support for the understanding which can be considered superfluous. Only a little part of the future teachers involved in that investigation had a *creative (or constructivist) view of Mathematics*<sup>5</sup>. In their lesson projects they applied a constructivist teaching methodology<sup>6</sup>, using DGS for stimulating the pupils to investigate problems: in this way DGS becomes a useful tool for giving a stronger conceptual understanding.

The importance, in the process of teachers’ education, of developing their beliefs and conceptions about Mathematics is stressed by several authors (for references about this research field see e.g. Thompson 1992 and the more recent Malara 2004). Indeed, more in general, a ‘*formalist*’ conception of Mathematics and, for different reasons, an ‘*instrumental*’ view, could be an obstacle for the learning of a constructivist teaching methodology. Aiming at developing a transition to a dynamic view of Mathematics as a process, in the Specialisation School of Trieste we base our work, as reported in (Bonotto and Zuccheri 2003), on the reflection about the fundamental mathematical concepts, their historical sources and development.

A deep analysis about the necessity of practical understanding of any mathematical theory is contained in (Sierpiska 2004<sup>7</sup>). According to it and

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<sup>3</sup> Here, ‘*formalist conception of Mathematics*’ means a descriptive, static, not constructive conception of Mathematics and rigour.

<sup>4</sup> Here, ‘*instrumental view of Mathematics*’ means to think that Mathematics is a not-connected set of rules to be applied.

<sup>5</sup> The term ‘constructivist’ has various meanings. Here, ‘*creative (or constructivist) view of Mathematics*’ means a dynamic view of Mathematics, as a continuous process carried out by investigation, posing and solving problem.

<sup>6</sup> It is well known that there are many versions of this important theory related to the education research field. Here we mean the basic idea of ‘constructivism’, referable for instance to Piaget work, i.e. that the knowledge must be actively constructed by the learner and cannot be simply transmitted by the teacher.

<sup>7</sup> For the distinction between *practical* and *theoretical* thinking, see in particular p.7.

considering that even Mathematics teaching theories must be experimented and acquired by previous practical understanding, I decided to present to the trainees some meaningful examples in order that they might experiment on themselves that technological tools can be really used for deepening the conceptual understanding. Taking in account what before expounded, the purpose was in fact to encourage the trainee teachers, especially them having good mathematical knowledge, in using DGS in structural way in their future teaching activity.

Continuing a research trend in using technology for Geometry teaching (see Gallopin and Zuccheri 2002), in which special attention is given to mathematical contents<sup>8</sup>, integrating them in the used didactical methodology, I carried out a didactic proposal which permits to introduce cultural, historical aspects of Mathematics, not only the technical ones. I chosen to treat by means of DGS an unusual subject of Elementary Geometry, in order to deepen some geometrical concepts and theorems exciting the discussion with and among the students about geometrical construction problems (as described in §2). Further, I chosen a subject linking geometrical and numerical aspects, for interesting also the future teachers having an instrumental view of Mathematics. In fact, as reported in (Zuccheri 2004), I noted that many '*instrumentalist*' used spontaneously, even inadequately from conceptual point of view, the tools of DGS related with numerical aspects, considering useful the verification by measuring.

I previously tested the didactic proposal with High School students, with satisfactory results, as described in §3. Then, I presented the subject to the trainees in the course of Didactical Technologies for Mathematics Teaching of the Specialisation School for teachers' education of the Trieste University. The results of this experience are described in §4.

## 2. The contents and methods of the didactic proposal

The subject is inspired to the so called '*geometrical Algebra*' of the Euclid Elements Second Book (see i.e. Heath 1956). The cultural importance and relevance of the subject can be understood by reading (Russo 1995), which collects results of author's original historical researches about the Greek Science in the third and second centuries B.C.. In particular, the author illustrates the importance of the tools 'rule and compasses' for the development of Greek Mathematics (Russo 1995, pp. 73-75) and stresses their use even for numerical calculation (Russo 1995, pp. 57-60).

The used teaching method is a constructivist method, based on problem solving activities, in which occurs the convergence of speech and practical activity, interactive communication and guided interaction, according to Vygotskij's theories

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<sup>8</sup> The relevance of the contents in Mathematics education was recently stressed by Anna Sierpinska, which affirmed that "*Didactic knowledge... requires the study of mathematical content of teaching*" (see Sierpinska 2004, p. 22).

(as expressed in particular in Vygotskij 1990).

The didactic proposal, divided into five steps, consists in involving actively the students in carrying out, using DGS<sup>9</sup>, a simulation of an '*analogic computer*' for performing some operations, as described in the following:

Step 1: the *sum* of any two positive numbers.

Step 2: the *division* of any positive number by a positive integer number.

Step 3: the *square root* of any positive number.

Step 4: the *product* of any two positive numbers.

Step 5: to *find the solutions of a 2<sup>nd</sup> degree problem* (i.e., to find two positive numbers of assigned sum and product).

To do it, the numbers are initially 'transformed' into segments of measure equal to them, by creating the initial segments and stretching them until they reach the given measure, or using the commands '*Numerical Edit*' and '*Measurement Transfer*' if we need more precision. Then, the students operate on the segments by means of 'rule and compasses' constructions realised by DGS.

The Cabri-measure of the segments obtained as '*Final Objects*' of these geometrical constructions produces numerical results, which are inserted in a table (without unit measure). In this context, the teacher stresses the conceptual difference among the following entities: the segment, its length and the measure of this length. Of course, dragging the initial segments or changing the numbers in input in '*Numerical Edit*' window, we can modify the initial data. Discussions among the students are stimulated about problems related to the numerical approximation, the choice of the convenient number of decimal digits to visualize and about the way in which the computer really works (in fact, the computer does not work analogically, contrarily to the real rule and compasses tools).

At any step, a main problem to solve is proposed to the students. Other problems linked to that are proposed. For instance, at Step 1, the students are asked to produce a '*Macro-Construction*' for doubling any segment and to use it for generating arithmetical and geometrical sequences.

During the first three steps, the teacher task is (if it is necessary) only to help the students, by means of appropriate questions, to put in evidence the data and the unknowns of the problem, until they suggest the solution. In fact, the theory to be applied is already known to them, but generally in different context and with different significance. For instance, at Step 2 we require to get a method for realising the division of any number by an integer number. This can be got by dividing a segment into an integer number of equal parts, by application of the well-known 'Thales Theorem' about a sheaf of parallel lines cut by straight lines. At Step 3 is proposed, as a 'black box', a 'machine' already realised for getting the square root (fig. 1.a).

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<sup>9</sup> These experiences were performed by means of Cabri II and Cabri II plus.

The students are asked to explore it, to discover which operation it is performing and to explain the geometrical construction, in which a well-known theorem connected with the Pythagorean Theorem is used. Then, they have to construct the ‘machine’ (the products are similar to fig.1.b) and to carry out a ‘*Macro-construction*’ for it.

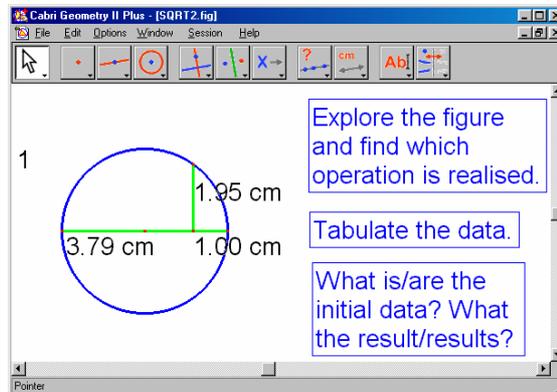


Figure 1.a

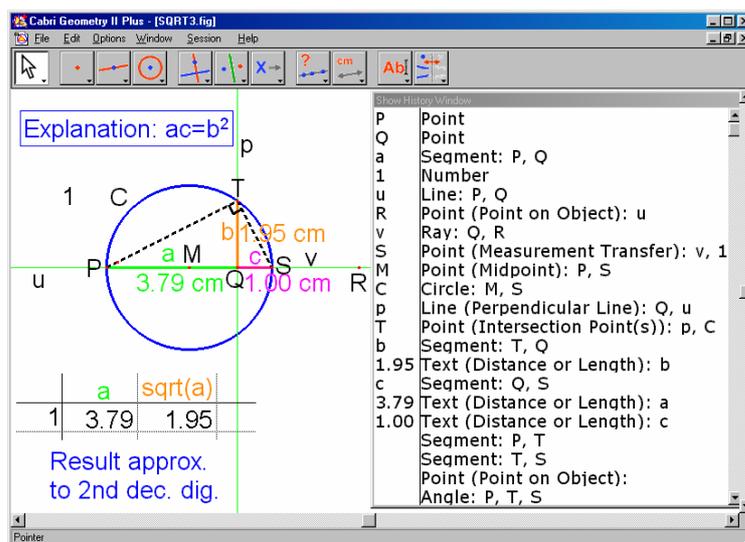


Figure 1.b

The last two steps require ‘new’ theory. Here (if it is necessary) the teacher may assume a stronger role of guide, introducing the problem, posing questions, exciting the discussion and giving suggestion to overcome the greater difficulties. At Step 4, for getting the product of any two numbers, the teacher leads the students to interpret the product of two numbers as the area of a rectangle. The problem becomes: “*How we can get a segment having as measure a number equal to this area?*”. Then, the students search a way for transforming this rectangle into another, with a side equal to 1 and having the same area (so, the measure of the other side will be equal to the required product). This problem is generally difficult to solve for the students. Then the teacher shows a ‘machine’ already realised, in which we use a theorem from the Euclid’s Elements First Book, which the students ignore; this is the Prop. 43-I, usually called the ‘Gnomon Theorem’: “*In any parallelogram, the complements of*

the parallelograms which stand around the diagonal are equal [that means: they have the same area]”. The students are required to analyse the ‘machine’ until they ‘discover’ in it this theorem, give its proof and apply it for realising themselves a ‘machine’ (a solution is similar to fig. 2).

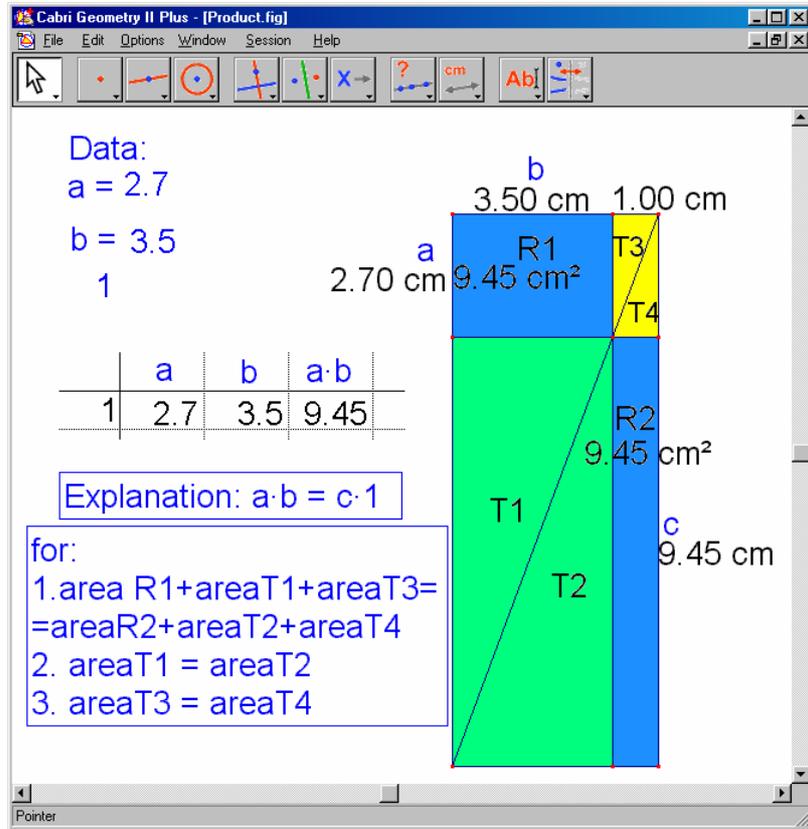


Figure 2

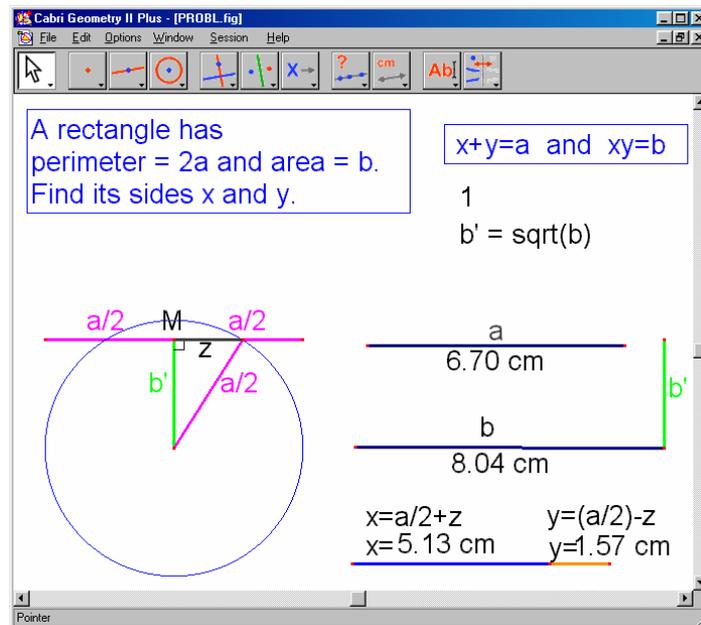


Figure 3

An analogous teaching method is used at Step 5. For the solution of the 2<sup>nd</sup> degree problem we use the Prop. 5-II from the Euclid's Elements 2<sup>nd</sup> Book, which states what we can translate as follows in algebraic symbols:  $(h+k)(h-k)=(h^2-k^2)$ . This theorem (see e.g. Franci & Toti Rigatelli 1979) and the Pythagorean Theorem suggest a graphical resolution of the problem (fig.3), in which we get the value of the square root by applying the '*Macro-Construction*' already realised for it. The teacher illustrates the graphical resolution, asking the students to solve little sub-problems, which conduce to the final result.

### 3. The experience with High School students

#### 3.a The context

I tested at first the didactic proposal in the academic years 2002/03 and 2003/04, during the workshop activities for Secondary School students of the Mathematics and Informatics Department of the Trieste University. These workshops are guided by University teachers and aim at approaching the secondary students to the University world, for orienting them in the future choice of scientific faculties. The subjects to be treated are chosen in order to stimulate students' interest, thus they are different from the usual curricular subjects and the presentation is self-contained.

#### 3.b The modalities

I repeated four times the experience, totally with 78 High School students, so divided into groups: Group A (16 students of Scientific Lyceum), Group B (4 students of Scientific Lyceum), Group C (27 students of Scientific Lyceum, 4 of Classical Lyceum), Group D (27 students of Pedagogical Lyceum). The students of Group B were 17-18 aged, the others were 15-16 aged. The workshop for any group consisted into 2 sessions, each of 3 hours. I used a computer and a video projector; 1 computer was at disposal for each student of Group A and B, whereas some students of Groups C and D worked in pairs. A co-operator recorded in writing the workshop development of Groups B and C. The teacher of Group D was present as observer.

The contents and the number of '*Macro-Constructions*' realised were arranged to school level and students' abilities. They were limited for Groups C and D, in which the work proceeded slowly, because of the numerousness of students. For this the solution of the 2<sup>nd</sup> degree problem (Step 5) was carried out only with Groups A and B. Each lesson included concise historical comments, which were deepened more accurately with Group B, which had already studied, as philosophers, Thales and Pythagoras. Few students of Groups A and C had some knowledge about the main features of the software, but nobody known the feature '*Macro-Construction*'.

#### 3.c Comment

The subject very interested the students. They reached a sufficiently good level of understanding, which I tested posing questions and checking the correctness of the constructions. The realisation of '*Macro-Constructions*' was useful even for this

purpose and for reinforcing the learning of DGS. A questionnaire was submitted to students of Group C after the workshop. From the answers to it and from the comments of students during the lessons it emerges that: a) the students were surprised to discover that some theorems which they learned before, without giving them any significance, have a practical utility; b) the totality of students never had seen Geometry under this aspect; c) the majority of them considered that in this way Geometry appears more understandable; d) the students appreciated the interactive working method. The teacher of Group D required materials for remaking the work.

#### **4. The experience with trainee teachers**

##### 4.a The context

In the academic year 2003/04, I presented the didactic proposal to the trainee teachers attending the course of Didactical Technologies for Mathematics Teaching of the Specialisation School for teachers' education of the Trieste University. Among them, I distinguish the Group TA with good mathematical knowledge (20 graduates in Mathematics, Physics, Engeneering) and the group TB with lower mathematical knowledge (23 graduates in Life Science, Earth Science, Chemistry).

##### 4.b The modalities

The total number of trainees was 43; among them, 27 was in the first year of the School (9 of Group TA, 18 of Group TB) and 16 in the second year (11 of Group TA, 5 of Group TB). The future teachers attending the second year of School had learned to use DGS in the previous course on Didactical Technologies.

The course consisted into 5 sessions, each of 3 hours. I used a computer and a video projector, whereas a computer was at disposal for any two trainees. The pairs of trainees were encouraged to discuss together for finding the solution of the problems and to technically help the colleague, if necessary. Each lesson included technical considerations about the software, historical deepening and didactical comments based on the previous experience carried out with the students. Each step of the didactic proposal was at first carried out by the trainees 'as learners' (I followed the methodology described in §2, as for the students) and successively analysed by means of discussions on technical and didactical questions. Additional contents were treated; i.e. the Theorem for the 'transport' of any segment.

##### 4.c Comment

All trainees showed a real interest for DGS and for the participation in the course. Especially the trainees of Group TA participated actively in the interactive work proposing their conjectures and ideas. It occurred some difficulties in realizing '*Macro-constructions*', which were useful for focusing technical aspects linked to the conceptual ones. Regarding the subject-matter, the trainees showed positive reactions as the students. In particular, many of them were struck by the practical significance of geometrical notions which they already studied only theoretically. In the past,

teaching to use technological tools, I observed often similar immediate positive reactions without effective long-term results, as stressed in (Zuccheri 2004). After this experience, conversely, I noted two long-term positive results confirming the permanence of a real interest in using technological tools in Mathematics teaching (especially for the group TA of trainee teachers with good mathematical knowledge) and a general tendency regarding the use of technological tools in education, different from that observed in previous experiences and described in §1, as follows:

4.c-1 I required the trainees, as usually, to produce as homework a short lesson project using DGS, compulsory only for Group TA. I received 19 projects from Group TA (8 from 1<sup>st</sup> year trainees, 10 from 2<sup>nd</sup> year trainees) and 15 from Group TB (10 from 1<sup>st</sup> year trainees, 5 from 2<sup>nd</sup> year trainees), three/four months after the end of the course. All projects were accurately realised. Differently from previous courses, it occurred that: a) for choosing the subjects, the trainees searched documentation in textbooks, specialised reviews and in the web; b) a high number of trainees of Group TA attending the second year of School (8 among 11) inserted in their classroom training activity a teaching experience with DGS and reported this in their homework [the 1<sup>st</sup> year trainees were not making the classroom training period]; c) only 1 trainee of Group TA and 1 of Group TB, both in the first year, described a project in which DGS was used exclusively for reproducing theory explained before in a lesson without computer, whereas the other projects used DGS in structural way, integrated with the lesson contents, even with some incorrectness. The integrated use of DGS in projects of Group TA consisted in a guided experimental observation of mathematical properties and concepts, with different levels of interactivity. The types of integrated use of DGS in projects of Group TB were: a) explanation of theory (2 trainees in the 2<sup>nd</sup> year); b) observation, guided by teacher, with interactive discussion (4 trainees in the 1<sup>st</sup> year and 4 in the 2<sup>nd</sup> year); c) exploration, conjecturing and problem solving activities (4 trainees in the 1<sup>st</sup> year and 4 in the 2<sup>nd</sup> year).

4.c-2 At the beginning of academic year 2004/2005, I submitted a questionnaire to the trainee of Group TA actually in the second year of School which attended the lessons the previous year, for testing if their opinion was unchanged. They remembered well the subject-matter. From the answers to the questionnaire it emerges that: a) they think that posing in relation in this way geometry with algebra make easier the understanding of both geometry and algebra (except for one, which affirm that it is more useful for the comprehension of algebra); b) all trainees want to make a classroom training activity with DGS.

Of course, teachers' professional education is a long and complex process (as the development of new conceptions and beliefs), in which the personal component is fundamental. Nevertheless, I think that what explained in this Section support the conviction expressed in §1, that is appealing to appropriate mathematical contents and exciting practical understanding of appropriate teaching methodologies, can be useful in order to encourage the teachers in using technological tools in structural way in education for improving Mathematics teaching.

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