

## **WORKING GROUP 7**

### **Research on Geometrical Thinking**

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## RESEARCH ON GEOMETRICAL THINKING

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The CERME-4-Working Group on Geometrical Thinking was a rather small group of about a dozen persons interested in Geometry and its teaching and learning from primary education to secondary education and teacher training. In all, seven papers were accepted before the conference and served as a basis for intensive discussions.

Continuing the work at the CERME-3-conference (for a report on this work see [http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7\\_introduction\\_cerme3.html](http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7_introduction_cerme3.html)), the Working Group started with the presentation of a special framework for Geometry and its teaching and learning by looking into geometrical paradigms. This framework was already presented at the CERME-3 conference (see the “geometrical approaches in the paper of Kuzniak and Rauscher). This description was meant to give some common ground for the discussions in the group. After this introductory session, the Working Group focussed on primary education. Papers from Italian colleagues gave an excellent opportunity to look deeply into geometrical concepts held by young students (see the papers by Medici et al. and Marchini et al.), but also into the role of specific tools for the teaching and learning of Geometry at that age level (see the paper by Vighi). Naturally, this debate also included Geometry teaching and learning at secondary level, what gradually brought us into issues more linked to the Geometry curriculum for grades 5 and above. Hamiti and Xhevdet presented ideas for the implementation of a new Geometry curriculum in the recent new primary school curriculum in Kosovo, showing the way current theoretical frameworks and approaches for teaching Geometry have influenced the curriculum development and replaced the traditional Geometry curriculum in Kosovo. The paper by Jones et al. gave a description of geometrical reasoning in Chinese and Japanese classroom (mainly from the perspective of teachers), but also trying to be specific about students’ geometrical thinking. The analysis of students’ reasoning went nicely together with the paper by Markopoulos and Potari. In addition to this, the paper from Greece also opened a window on spatial Geometry by analysing dynamic transformations of solids.

The second part of the paper by Kuzniak and Rauscher rounded off the travel through Geometry and its teaching and learning by analysing problems and potentials of Geometry in in-service teacher training. At the end of the seven sessions, the Working Group even had time for a general closing debate and the preparation of ideas for the report of the group in the final plenary at CERME-4 and this summary. For detailed information on the individual papers the reader is nevertheless directed to these papers.

Looking back on the discussions of the Working Group during the last session, we came up with four major issues: For research, it was obvious that existing

frameworks (like the ones from Piaget or the famous van-Hiele-levels) are helpful to analyse only some aspects of the variety of data in research on geometry teaching and learning. Additional research categories are needed and new local theories are needed to better analyse and understand the data collected in recent empirical studies. At present, we are not in a position to offer a unified theory to completely cover the richness and diversity of the data on the teaching and learning of Geometry. Theoretical innovations visible in the work of Kuzniak and Rauscher as well as Markopoulos and Potari are only indications of this trend, while the paper of Vighi (by looking into a simple artefact like squared paper) reminds us of the importance of the tools (and their use) for the teaching and learning of Geometry.

If one wants **change in teaching Geometry**, for instance because of the necessity of defining an adequate curriculum, one faces a dilemma closely linked to the epistemology of the knowledge to be taught. By its very "nature", Geometry is organised around wide conceptual networks with far-reaching relations inside the area, but also implying links to other mathematical and extra-mathematical areas, especially cultural ones. In contrast to this, school teaching usually oversimplifies such wide networks, particularly in the Geometry lessons. This seems to be at least one reason for the poor learning often occurring in our classrooms. Some of the papers linked to this Working Group can also be read as examples of more open approaches to teaching and learning Geometry.

Implications for the educational **policy** are most obvious, but the Working Group wants to especially mention one issue here: Textbooks are crucial instruments of teaching and learning. According to the research results available now, they are the most important teaching and learning tool even in the age of new technologies like computers and software, especially Dynamic Geometry Software (DGS). Nevertheless, textbooks available at present seldom meet the expectations of the members of the WG - both on choice of content and variety of teaching approaches.

Finally, the Working Group looked into **teacher training**. The participants took for granted that there is an urgent need for training future and practicing teachers with respect to Geometry taught at school. The situation for Geometry seems to be particularly difficult because of the poor knowledge of the teaching force within this mathematical area - and additional training should include both content, i.e. Geometry as a sub-domain of Mathematics, as well as the "Didactics" of Geometry, for instance the theoretical background supporting the organisation of Geometry curricula and specific suggestions and innovations why, how and what to teach in Geometry lessons.

# DEVELOPING GEOMETRICAL REASONING IN THE CLASSROOM: LEARNING FROM HIGHLY EXPERIENCED TEACHERS FROM CHINA AND JAPAN

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**Abstract:** *Mathematics education has been the subject of considerable international comparative research, mostly focussed on pupil achievement but also examining teaching methods, curricula, and so on. In all this, and perhaps unsurprisingly, the role of teachers has emerged as a key influence on pupil learning. Given that the development of pupils' capability in geometrical reasoning continues to be an issue of considerable international concern, this paper reports an analysis of lower secondary school lesson suggestions prepared by highly experienced "expert teachers" from China and Japan, countries selected because they represent some interesting similarities and contrasts. The paper also gives background to these lesson suggestions in terms of the educational context in which they are presented.*

## Introduction

The (recently renamed) *Trends in International Mathematics and Science Study* (TIMSS) is continuing to investigate pupil achievement, the mathematics curricula, teaching methods, and so on, across almost 50 countries around the world (see, for example, Mullis *et al*, 2000). Overall, the results to date of TIMSS suggest that there are significant similarities between the mathematics curricula across countries, especially in terms of topics specified, if not in overall curricular design (Schmidt *et al*, 1997; Valverde *et al*, 2002). Yet these broad correspondences of grade level and content become differences if examined more closely; both in the range of content addressed at a particular grade level and in particular developmental sequences where common content is addressed over several grade levels.

In terms of geometry teaching, while analysis of TIMSS data continues, a detailed comparative study of geometry specifications (Hoyles, Foxman and Küchemann, 2002), though covering fewer countries than TIMSS, found considerable variation in current approaches to the design of the school geometry curriculum. Thus, for example, the study found, a 'realistic' or practical approach apparent in Holland, while a theoretical approach is most evident in France and Japan. The study concludes by noting "there is evidence of a state of flux in the geometry curriculum, with most countries looking to change" (*op cit* p. 121).

As part of TIMSS, or related to it, a number of projects have examined the teaching methods that teachers (typically) use in various countries and, related to this, how teachers structure their lessons (see, for example, Shimizu, 2002; Stigler and Hiebert, 1999). To date there has been little comparative work specifically on how teachers structure mathematics lessons to develop geometrical reasoning. This is despite the issue of geometry teaching being of considerable international concern, especially its role in developing students' powers of reasoning (Mammana and Villani, 1998; Royal Society, 2001).

The analysis presented in this paper compares suggestions from highly experienced "expert teachers" for geometry lessons for lower secondary school classes in China and Japan, countries taken in alphabetic order and selected as they represent an interesting comparison (see methodology section for more on the choice of countries). The paper also analyses the range of influences that impinge on the way lessons are likely to be structured in the selected countries.

### **Comparative research on geometry teaching**

Internationally, on average, it seems that the Grade 8 (UK Year 9) curriculum specifies greatest coverage of topics in fractions and measurement (see Mullis *et al*, 2000, chapter 5). Very few students internationally are given a major emphasis in geometry (three percent, on average), with, it seems, Tunisia the only country where 20 percent or more of the students are in classes that emphasise geometry over other areas of the mathematics curriculum. In terms of what is actually taught, teachers in the TIMSS survey report a range of instructional coverage across topics in geometry. For example, the topic "Simple two dimensional geometry – angles on a straight line, parallel lines, triangles and quadrilaterals" is reportedly taught to 95 percent of students (on average), while "visualization of three-dimensional shapes" is taught to only 57 percent, on average (with a variation across countries from 7 - 99%). Another geometrical topic that shows a large variation across countries is "symmetry and transformations", varying from being taught to 11% to 98% of Grade 8 students. According to their teachers, most students in Grade 8 receive moderate emphasis on geometry. On average internationally, by the end of their eighth grade, it seems that 22 percent of students are yet to be taught 50 percent or more of the geometry topics listed in the TIMSS survey (the list being generated by comparing curricula across countries).

Overall, and perhaps unsurprisingly, the role of the teacher emerges as a key influence on pupil learning. The latest TIMSS research related to the way teachers structure their lessons, the *TIMSS 1999 Video Study* (Hiebert *et al*, 2003), covered seven countries, including a number where students scored highly on the TIMSS achievement tests. This study found that some general features of Grade 8 mathematics lessons (including geometry lessons) were shared across the seven countries studied. For example, lessons were generally organised to include some public whole-class work and some private student work, the latter being mostly individual but with some involving small groups. Most lessons included some review

of previous content as well as some attention to new content and, in the majority of cases, made use of a textbook or worksheet of some kind.

Notwithstanding these shared general features, the study reports discernible variation across the countries studies. Distinctions included how new content was introduced, the coherence across mathematical problems and within their presentation (ie the interrelation, both implicit and explicit, of the mathematical components of the lesson), the number and form of topics covered, the procedural complexity of the mathematical problems tackled, and classroom practices regarding individual student work and homework in class (although the report is not detailed enough to say anything specific about geometry lessons).

Overall, as Hiebert *et al* (2003, p149-50) emphasise, the video study found that the countries that show high levels of student achievement in the TIMSS achievement tests do not all employ teaching methods that combine and emphasise features in the same way. As they conclude:

“The results of this study make it clear that an international comparison of teaching, even among mostly high-achieving countries, cannot, by itself, yield a clear answer to the question of which method of mathematics teaching may be best to implement in a given country”.

Hiebert *et al* (2003, p150)

This confirms that further research is needed to shed light on how teachers might best structure their lessons to develop geometrical reasoning.

## **Aims and theoretical framework**

The principal aims of the research project, an initial analysis from which is reported in this paper, are two-fold:

- To determine the influences on the way geometry lessons might be taught in the selected countries;
- To analyse selected suggestions from highly experienced “expert teachers” in these selected countries – suggestions that regular teachers might use as a guide to structuring geometry lessons for lower secondary school students.

At the time of writing the authors are considering a range of theoretical notions with a view to determining which may be appropriate. For the purposes of the analysis presented below, the approach to analysing the lessons is derived in part from the study of textbook ‘lessons’ by Valverde *et al* (2002) – see next section for more on this.

## **Research methodology**

The countries selected for study are China and Japan, chosen because they represent some interesting similarities and contrasts. Both countries have National Curricula for mathematics that covers geometry, amongst other mathematical topics. Yet, for

teachers in the two countries there are different traditions and different ways in which they have responded to international developments over the years.

In terms of the influences on teaching, the sources of primary data selected for analysis in this research include:

- Government guidelines and other official documents
- Guidance documents and /or books for teachers

The specific sources of data providing suggested lessons are as follows:

- China: the data are mainly from the national teaching references (*The Compulsory Education Nine-Year Secondary School Mathematical References, 1995-1996*) and a popular teaching reference, *Master teachers' lessons records (Lower secondary school mathematics), 1992*. Such items are currently in widespread use by secondary school mathematics teachers throughout China.
- Japan: the data are suggested lesson plans by experienced teachers and university researchers (each with more than 10 years experiences, in general). The plans include information on the aims of lessons, problems for students, suggested activities for both teachers and students, time allocations, etc.

The analysis of the lesson suggestions is framed by the following procedure, derived in part from the study of textbook 'lessons' by Valverde *et al* (2002, Appendix A):

- Division of the suggested lesson into 'blocks' in terms of content, focus, and purpose;
- Identification of key features of geometry teaching, especially that focusing on the development of geometrical reasoning.

The analysis of the range of influences on lesson structure is based on a review of the literature.

## Analysis

**China:** As a country with an extensive teaching tradition, teaching practices in China continue to be influenced by the ideas of Confucius (551-479 BCE) and by texts written in subsequent centuries. For example, the distinctive character of Confucianism in respect of learning is to ask questions constantly and to review previous knowledge frequently. In terms of mathematics teaching, the *Arithmetic of Nine Chapters*, a classic Chinese mathematics work of the Tang dynasty (618-907 CE), has greatly affected mathematics teaching and learning in China over centuries. This text lays down rules for solving problems and a sequencing of questions, answers and principles that continue to play an important role in the instructional model of teaching in China (An *et al.*, 2002, p 106). Traditionally, therefore, questioning is a key part in mathematics learning and teachers are likely to use good questions in motivating students to explore new problems. In addition, as Ashmore

and Zhen (1997) demonstrate, review and conclusion are indispensable in classroom lessons in China

As is common in education, National Standard Examinations plays a critical role in school mathematics curriculum (Chongqing [China] Conference, 2002). Thus, according to Li (2002), mathematics teachers are likely to carefully select a considerable quality of exercises as one of their main teaching strategies. Consequently, completing exercises is a major feature of mathematics lessons. In addition, national textbooks are an essential teaching and learning resource. Teachers usually plan lessons by referring to such textbooks. The current textbooks in Shanghai, for instance, are arranged as a “spiral” curriculum, with new theorems, rules and formulae appearing in each unit. Consequently, mathematical terms and methods, which have already been taught, have to be frequently repeated through review, conclusion and exercises made by teachers in the lessons. Subsequently, new knowledge often follows introduction or experiment and this usually requires students to review previous knowledge. Given the above, mathematics lessons in China are likely to comprise the following segments:

1. Introduction/review/experiment (about 5 minutes)
2. The teaching of new content (about 25 minutes)
3. Exercises on the content introduced (about 10 minutes)
4. Homework assignment (about 5 minutes)

The case study below is a lesson record of a lesson from what, in China, is referred to as a “master teacher” (the teacher has more than 30 years teaching experience).

Lesson on ‘Corresponding Angles, Alternate Angles, Interior Angles at the same side of a line’; grade 7, students aged 13-14, school in SiChuan Province, in south-west of China (Li, 1992, translated by Ding, 2004).

Objectives of teaching and learning of this lesson:

1. To clearly understand the concepts of corresponding angles, alternative angles and interior angles at the same side of a line.
2. To correctly recognise these angles in complex figures;
3. To be fully prepared for further studying about the properties of parallel lines

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### Introduction (+/- 5 minutes):

Discuss the location relationship of three lines on a plane

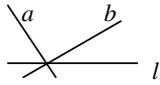


Figure 1

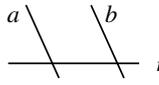


Figure 2

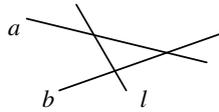


Figure 3

Focus on a figure in which two unparallel lines are crossed by the third line and review the concepts of vertically opposite angles and neighbour complementary angles;



### Teaching new knowledge (+/- 20 minutes):

- 1) Teach the concepts of 'Corresponding Angles, Alternate Angles, Interior Angles at the same side of a line' through observing figures:

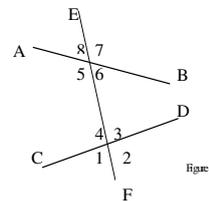


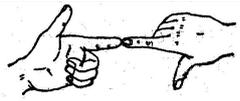
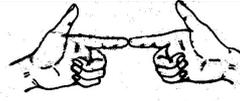
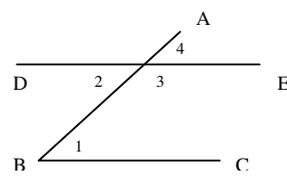
Figure 4

- 2) Complete the diagram as follows:

The name of angles	Basic figures	The characters of location	One side of the angles on the same cross line	The other side of the angles (which side of the cross line are they?)
Corresponding Angles			The same direction	The same side
Alternate Angles			The opposite direction	The different side
Interior Angles at the same side of a line			The opposite direction	The same side

### Conclusion (+/- 5 minutes):

- 1) Review the concepts of the three types of angles learned in this lesson;
- 2) Use hands to present the different angles (See pictures below).

 <p>Picture 1</p>	 <p>Picture 2</p>
<p><i>Exercises (+/- 10 minutes):</i></p> <p>a) To recognise corresponding angles, alternate angles and interior angles at the same side of a line in figure 7;</p> <p>b) To discuss whether a pair of alternate angles is equal and the sum of degree of a pair of interior angles at the same side of a line is <math>180^\circ</math>, when a pair of corresponding angles is equal? Why?</p>	
 <p>Figure 7</p>	

**Japan:** The way teachers structure their lessons in Japan is influenced by the specification of the mathematics curriculum, the design of textbooks, the occurrence of ‘Lesson Studies’, and research into the learning and teaching of mathematics. ‘Lesson study’, practiced in Japan for the last several decades, is one of the most common forms of professional development for Japanese teachers and involves teachers working in small teams collaboratively crafting lesson plans through a cycle of planning, teaching and reviewing (Yoshida, 1999). Through this process, Japanese teachers have collaboratively developed a view about ‘good lessons of mathematics’.

Research in the learning and teaching of mathematics that has influenced how teachers structure lessons includes the “Open-ended approach” in which ‘the teacher gives the students a problem situation in which the solutions or answers are not necessary determined in only one way’ (Sawada, 1997, p. 23). Considering the influences described above, in summary, Japanese teachers tend to structure mathematics lessons as follows (as also described in other research, including, for example, Stigler and Hiebert, 1999, pp.79-80):

1. Presenting the problem(s) for the day:

- a) The problem(s) selected is/are designed to make students engage in mathematical activity in a challenging (or sometimes open-ended) situation
- b) Reviews of the previous lessons are sometimes included before the problem(s)

2. Development of the problem(s):

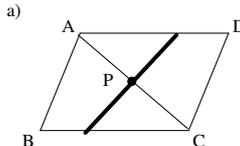
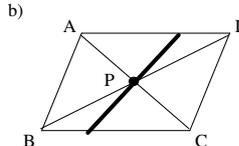
- a) Students work the problem(s) individually or in groups
- b) Discussion and presentations of solutions are often included
- c) Teachers clarify and/or extend the mathematical thinking of the students
- d) New problems, usually related to the problems for the day, are sometimes introduced

3. Highlighting and summarising the main point(s):

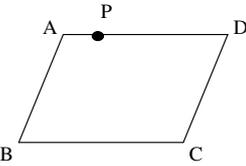
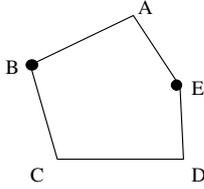
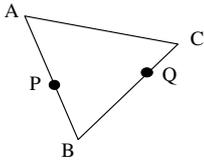
- a) Students' ideas are often used, and sometimes students are asked to explain their solutions
- b) The solutions of the problem(s) are summarised by the teacher
- c) By the end of the lesson, students would grasp mathematical concepts and deepen their mathematical thinking (often main goals of the lesson)

The case study presented below is a lesson record taken from Haneda (2002):

Perpendicular bisectors of segments; students aged 12-13 (Haneda, 2002, p. 38, translated by Fujita, 2004).

<b>Year 7 (students 12~13)</b>	The lesson on perpendicular bisectors of segments
<b>Aim of the lesson</b>	By the end of the lesson, students will be able to a) grasp the meaning of perpendicular bisectors of segments, and b) grasp the method of the construction, and be able to construct perpendicular bisectors of segments
<b>Segment</b>	<b>Description</b>
<b>1 : Introduction</b>	<p><b>Introducing problem 1</b></p> <p>Problem 1: Let us fold a parallelogram ABCD so that C will fall on A, and consider how to draw the folded line.</p> <p>a) </p> <p>b) </p> <p>a) Solution: drawing the perpendicular bisector of AC  b) Solution: taking the intersection P of AC and BD, and drawing a perpendicular line to AC</p> <p><b>Undertaking the construction by students</b></p>
	<p>Notes for teachers</p> <ul style="list-style-type: none"> <li>- Give paper parallelograms and worksheet</li> <li>- Encourage students to try various ways of solutions</li> <li>- It is expected that students would notice the solutions a) or b) by looking at the facts that APC, 180 degree, is bisected when they actually fold paper parallelograms</li> <li>- In addition to the solutions a) and b), it is expected that students would use congruent quadrilaterals or angle bisectors which they have learnt to draw the line.</li> </ul>

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<p><b>2: Development</b></p>	<p><b>Introducing similar problems</b></p> <p>Problem 2: Also consider how to draw folded lines in the following case</p> <p>1. Fold the shape so that C falls on P</p>  <p>2. Fold the shape so that B falls on E</p>  <p>3. Fold the shape so that P falls on Q</p>  <p>Undertaking the constructions by students</p> <p>Notes for teachers</p> <ul style="list-style-type: none"> <li>- Give worksheet for students</li> <li>- Give further tasks to students who finished the three problems</li> <li>- It is expected that students would use the construction of angle bisectors</li> </ul>
<p><b>3: Summary</b></p>	<p><b>Summary</b></p> <p>Knowing the lines which students drew are perpendicular bisectors of the segments</p> <p>Clarifying how to draw perpendicular bisectors of segments</p> <p>Notes for teachers</p> <ul style="list-style-type: none"> <li>- Explain clearly and precisely the words such as the mid-point or perpendicular bisectors</li> <li>- Clarify the simplest methods of the construction</li> </ul>

## Discussion

In each of the countries, the lesson structure followed the pattern expected for that country, something not altogether surprising given the evidence from existing research. Thus, in the lesson from China, new content is introduced and a considerable number of short tasks and questions are included in each segment of the lesson. In the lesson from Japan, the three-part structure is followed with a problem introduced in the first part and developed in the second before the main teacher explanation is given in the third.

As was found in the *TIMSS video studies* (Stigler and Hiebert, 1999; Hiebert *et al*, 2003), notwithstanding these shared general features, there is variation across the countries studied. For example, there is some variation in how new content is introduced – in the Chinese lesson through the teacher asking many questions, in the Japanese lesson through the teacher posing fewer, but perhaps more substantial, problems. Variation occurred, as in the *TIMSS video studies*, in the coherence of the lesson (ie the interrelation, both implicit and explicit, of the mathematical components of the lesson) and the procedural complexity of the mathematical

problems tackled. There was also variation in the type of individual student work and the sort of homework set (if any).

### Concluding comment

What this study has not been able to ascertain as yet are what the implications might be for student achievement in geometry in the countries under consideration. This is as an area for future research. Further research also needs to focus on what teachers actually do in lessons and whether, if, or how, they may make use of the advice that is available on how they might structure their geometry lessons.

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# ON THE GEOMETRICAL THINKING OF PRE-SERVICE SCHOOL TEACHERS

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**Abstract:** *We present a classification of geometrical approaches used by pre-service schoolteachers. The analysis is based on the notion of geometrical paradigms and levels of argumentation. Even if it is focused on a particular population, this study can be used to evaluate the long-term effects of education in geometry.*

To assess the long-term effects of mathematics' education is an important issue in didactics, which is often left aside, though its social impact is fundamental. Difficulties in measuring such effects probably explain the lack of studies. In a way, primary school teachers' training can give, as we shall see, information for assessing the long-term impact of education. The future teachers have to be ready to teach a subject that they often stopped using for several years. It becomes justifiable to test them and work with them on mathematical subjects that they will have to present to their pupils.

We focus our study on the case of elementary geometry and we examine only two particular questions here. How do students with initial studies on various subjects react when they are faced with elementary geometrical exercises? What can we learn from their reactions about the knowledge and conceptions of geometry they retain from their schooling?

## **What this study is not**

Since the early 80ies and in different countries, numerous studies have evaluated the teachers' mathematics level based upon the belief that the higher the level, the better the teaching. With respect to geometry, content knowledge among elementary school teachers appears low (Hershkowitz and Vinner, 1984, Carayol, 1983) due to various reasons (Niss, 1998), such as syllabus content variations and teachers' initial studies. Most researches use Van Hiele levels to assess reasoning abilities in geometry (Swaford and Ali, 1997). This use of the Van Hiele theory can surprise, it describes geometrical thinking development among pupils and not for adults like teachers who have achieved their development. But again, these studies show that schoolteachers seem to master only low levels (Hershkowitz, 1984 and Mayberry quoted by Swaford, 1997). Our aim is not to add another study to this subject but to build teachers' training devices based upon this assessment.

## **What this study tries to be**

Beside the content knowledge, another way of thinking about teachers' training is to define the kind of knowledge of mathematics which is necessary for teaching (Ball and Ali, 2001). In this area we encountered studies which aim to change belief and

practice of mathematics: for example, Leikin (2003) presents a research-development based upon different ways of solving problems, Houdement and Kuzniak (2001) use short situations to *provoke* students' beliefs. Steinbring (1998) insists on the necessity of epistemological knowledge for teachers and in France the accent is put on didactical knowledge with a lot of homological situations as described in the book *Concertum* (Copirelem, 2003), which presents 10 years of pre-service schoolteacher training in this country.

We situate our approach in this trend of research. But to be effective, we need to go deeper in the understanding and the interpretation of pre-service teachers' difficulties. In France, graduate students from any university (three years of study) are accepted in the teachers training institutes (IUFM) after a first selection. During one year, these students prepare for a competitive examination. The mathematical examination part is composed of classical mathematical questions and also of questions about the teaching of mathematics in primary school (study of pupils' errors, comparative analysis of textbooks). The successful candidates receive a theoretical and practical education during one year (the "second year") in all the subjects of the primary school; they receive a salary and are almost sure to become effective primary school teachers the following year.

### **A training device that offers elements of answers**

Within the setting of the "first year" teachers' training, an original training device (Kuzniak A. and Rauscher JC, 2003) gives us the basis for answers to the former questions. This device tries to make students sensitive to the variety of approaches to geometry and to the difficulties that this variety creates for their future teaching.

First, the students have to solve geometrical exercises and write the doubts and difficulties they encountered during the resolution. Then, they look at solutions and opinions that their peers wrote. They then review their initial answers to the exercises. We worked with eight groups of students during these four last years, but in this paper we speak only of two groups: 57 students, 19 with degree in science, 23 in Literature (French or foreign languages) and 15 in Art or Physical Training

The training takes into account the actual student's personal Geometrical Working Space, then it aims at changing it through activities. The choice of exercises and the analysis of students' productions is based on a theoretical framework, which we have presented at the Cerme-3-conference (Kuzniak and Houdement, 2003). This framework contains an epistemological dimension which is based on geometrical paradigms: the hypothesis is that there exist various meanings of the word "geometry" which cover different geometrical approaches. This variety creates an obstacle and a source of didactical misunderstanding. The framework is completed by a cognitive point of view that allows us to describe students' level of argumentation.

### **The variety of geometrical approaches**

School, and more generally compulsory education, offers the pupils several "mathematical worlds". Among these, the "geometrical world" has the basic characteristic of making an abstraction close to reality. So, the geometrical figure,

totally determined by its definition, is confronted by a drawing, which in turn, is the basis for the definition. This partly explains why pupils and students have so many difficulties understanding geometry. This also creates very ambiguous situations where the problem of coexistence or play between two possible paradigms appears.

We distinguished:

-Natural Geometry (Geometry I), which has reality and the sensitive world for source of validation. In this Geometry, an assertion is justifiable by using arguments based upon experiment and deduction. The confusion between model and reality is great and all arguments are allowed to justify an assertion and convince.

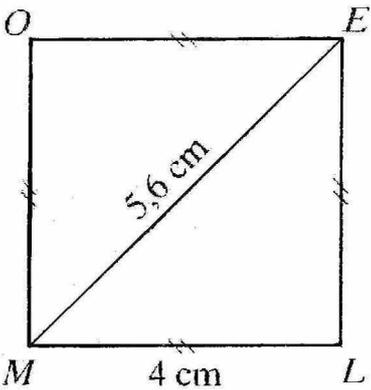
-Natural Axiomatic Geometry, whose archetype is classic Euclidean Geometry. This Geometry (Geometry II) is built on one model close to reality. But once the axioms are fixed, demonstrations have to be inside the system of axioms to be valid.

To these two approaches, it is necessary to add Formal Axiomatic Geometry (Geometry III) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers. Usually, they are mathematicians who have studied mathematics in university, which is very influenced by this formal and logical approach.

These various approaches (and this is one originality of our point of view) are not ranked: their horizons are different and so the nature and the handling of problems are changing.

### The chosen exercise<sup>1</sup>.

To make the students react, we chose problems where play exists between both Geometries. In all the proposed exercises, a drawing is given but its role is ambiguous, which raises the question of the existence of an appropriate working space to solve the problem. Let us detail this point on the problem of «Charlotte and Marie».

<p>1. Why can we assert that the quadrilateral OELM is a rhombus?</p> <p>2. Marie maintains that OELM is a square. Charlotte is sure that it is not true.</p> <p>Who is right?</p>	
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The drawing, proposed for the problem, looks like a square but its status in the problem is not clear. It looks like a sketch with dimension: codes are on the sides of

<sup>1</sup> On top of the « first year » preservice schoolteachers, we gave this exercise to pupils from 14 to 18, to inservice schoolteachers, to preservice and inservice highschoolteachers. In Chile, we gave the problem to students wishing to become teacher.

the quadrangle and indicate their equality, measurements appear on the drawing. But what is the origin of these dimensions? Are they made on a pre-existent figure or are they, especially for the diagonal, the fruit of calculation? The length of the diagonal [ME] is given in the nearest tenth of a cm (5,6 cm), which can lead us to interpret it as a real measurement. But, the problem comes from a textbook for the last year of secondary school and that leads us to see it as theoretical measurement which better corresponds to the usual didactic contract in this kind of class.

Is the drawing a first datum, a real object, which the problem suggests studying or does it result from a construction whose conditions are given in a text? In this case, is the practical realization important or is it only a support to help reasoning?

The text of a problem usually allows us to answer these questions and to determine the status of the represented object: this in turn orients us towards a precise geometrical paradigm. But, the wording in this problem gives no such indications as a student points out: "there are no texts for the wording, only a drawing that can deceive". "The quadrangle is a rhombus" seems to be the only fact taken for granted. To know if the quadrangle is a square or not is left to the pupil, who can situate and solve the problem in Geometry I or in Geometry II.

Finally, who is right? Charlotte or Marie? A classic way to handle this kind of exercise is to use Pythagoras' theorem, which doesn't require the real measurement of the angle. But even there, the ambiguity of the choice of the working space reappears. For our purpose, we shall introduce two forms of Pythagoras' theorem, the classic one, an abstracted form, with real numbers and equalities:

*If the triangle ABC is right in B then  $AB^2 + BC^2 = AC^2$*

and the other one, a practical form, which uses approximate numbers and, in a less common way, approximate figures

*If the triangle ABC is « almost » right in B then  $AB^2 + BC^2 \approx AC^2$*

The first form leads to a work in Geometry, which deviates from data of experiment by arguing in the numeric setting. The second formulation appears rather as an advanced form of Geometry I.

If we situate ourselves in Geometry II by using the abstracted form of Pythagoras' theorem, then we can argue, as one student suggests, giving reason to Charlotte:

*We know that if OEM is right in O then we have  $OE^2 + OM^2 = ME^2$*

*We verify  $4^2 + 4^2 = 5,6^2$  and  $32 \neq 31,26$ . Thus, OEM is not a right triangle.*

If we use the practical Pythagoras' theorem in the measured setting then we shall rather follow the reasoning proposed by another student who concludes:

*Marie is right: OELM is a square because  $32 \approx 5,6^2$ .*

In fact, it would be necessary to conclude that OELM is "almost" a square. But, for lack of an adapted language, students cannot play on these various distinctions. They

are at the same time faced with an epistemological and didactical misunderstanding. To us, the interplay between Geometry I and Geometry II can explain and work on this problem.

### **Cognitive dimension and reasoning structure**

A first analysis of students' answers and reactions to the exercises which we gave them shows that besides paradigms it is also necessary to consider levels of argumentation to describe more precisely the students' geometrical thinking. Students give lines of arguments in very variable structures. We based our analysis on the Van Hiele theory. As noted above, this theory is interested in the development of geometrical thinking. But, students are adults who use sophisticated reasoning outside mathematics: before their entrance in the Institute, numerous students studied abstract subjects and are able to argue in their domain.

On the other hand, authors like Duval have criticized "the naiveté" of the Van Hiele Theory from a cognitive point of view. We don't enter in this discussion but we freely use Van Hiele levels outside his theory to give us good benchmarks about the levels of the mathematical thinking of the students. In fact, it gives us a different view, maybe more easily recognisable, on intuition, experiment and deduction. We speak rather of levels of argumentation. The observation of students' levels of observation is getting interesting: They can argue about the nature of a figure by "accumulating" reasons, they can also develop a demonstration based on a necessary and sufficient condition.

So, all the students' productions are analyzed thanks to a double approach, which incorporates geometrical paradigms and levels of argumentation. The last contact our students had with geometry before entering the training institute took place in secondary school or at university where they learnt Geometry II or III. In the primary school, Geometry I is predominant.

### **Towards a classification of students' answers**

From the answers given by the students, we can sketch a classification which takes into account the nature of the geometrical paradigm, which is favored in the resolution. We have also identified four kinds of answers to the "Charlotte and Marie" problem. This allows us to bring out four main types of approaches.

We represent these four types by GII, GIprop, GIperc, and GIexp. We shall clarify the meaning of these abbreviations farther. Every time, we shall give a typical answer of the studied population.

First, answers using theorems are common among two groups of students, GII and GIprop.

**GII** In this case, the answers are close to this one [**Et A**]

1) *OELM* is a rhombus because its successive sides are equal.  
 2) If *OELM* is a square, then *MEL* is a right-angled triangle *L*. According to the Pythagoras' theorem we would have  $ME^2 = ML^2 + LE^2$   $ML^2 + LE^2 = 16 + 16 = 32$   
 $ME^2 = 5,6^2 = 31,36$   
 Thus, angle *ELM* is not a right angle.  
 Consequently, *OELM* is not a square and it is Charlotte who is right.

The classic Pythagoras' theorem is applied inside the world of abstract figures and numbers without considering the real appearance of the object. Only information which is given by words and signals (code of segments, indications on the dimension of the lengths), is used, and Pythagoras' theorem is applied in its entire formal rigor. To prove that the quadrangle is a rhombus (four sides of the same length) and to show that it is not a square (contraposition of Pythagoras' theorem), students use minimal and sufficient properties. We shall consider this population as being inside GeometryII. For some students, the lack of a look, even retrospective, at the drawing already indicates a geometrical conception of type Geometry III.

**GIprop.** This population groups together students who apply the practical Pythagoras' Theorem, in fact, to be rigorous, the converse. They give an answer similar to this one [**Et B**]:

1°) *OELM* is a rhombus, for  $OE=OM=ML=LE$  and a rhombus has four sides of the same length.  
 2°) Marie is right because all the sides of the quadrangle have the same length and there is at least a right angle. We can verify it by Pythagoras' theorem.  $ML^2 + LE^2 = ME^2$   
 $4^2 + 4^2 = 16+16=32$   
 $ME = \sqrt{32} = 4\sqrt{2} \approx 5,6$  thus  $MLE=90^\circ$

This student did not forget properties. She has the necessary knowledge to justify her answer. She applies the practical Pythagoras' Theorem in a form which we almost never see in schoolpupils (according to a similar study with pupils) but which appears several years later. Properties are used as tools to produce new information about the geometrical objects, which are seen as real objects.

In that case, the students recognize the importance of the drawing and of the measurements' approximation. The practical Pythagoras' theorem appears as a tool of Geometry I. We have designated this population as GIprop to insist on the fact that individuals of this group use properties to argue. The question whether these students can play with the differences between Geometry I and Geometry II or if their horizon remains only technological.

An addition to these answers, here are those of the students who did not use Pythagoras' theorem and that we place a priori in Geometry I.

**GIexp.** We group together students who use their measuring and drawing tools to arrive at an answer. They are situated in the experimental world of Geometry I.

Generally, this type of students concludes that Marie is right. But, it is not always the case: a student, using his compass, verifies that the vertices of the quadrangle are not cocyclic and he can assert that OELM is not a square.

Here is a response, which is based on the findings with instruments. [Et C]

*1°) OELM is a rhombus, for its diagonals cut themselves in their middle (measuring) by forming right angles (using a set square).*

*Remark: the student built the second diagonal on the figure.*

*2°) Marie is right. It is a square, for besides being a rhombus, OELM has its angles right (set square).*

**GIper.** In this last category, we group together students whose answers are based on perception: Their interpretation of the drawing is the basis for their answer. They do not give us any information about their tools of investigation. [Et D]

*1°) Four sides of the quadrangle are parallel between them and of the same length  $OE=ML$  and  $OM=EL$ . According to the definition of a rhombus, we can say that diagonals have the same middle point and are perpendicular between each other.*

*2°) Marie is right; OELM is also square because its sides form a right angle.*

### **A look at reasoning difficulties**

The previous productions are logically rather coherent and do not contain too many reasoning errors and formulation problems. But naturally, it is not always the case and as noted above, we proceeded to an analysis of the proofs and reasoning structure based on levels of argumentation inspired by Van Hiele levels.

We classify in level 1 productions, which enumerate a non-minimal list of quadrangle properties to justify assertions. In level 2, we place productions, which evoke a correct relation of inclusion between square set and rhombus set. In level 3, we set productions that use minimal and sufficient information to justify assertions.

This analysis allows us to separate two categories of students. In the first one, widely illustrated by our previous examples, students have solid knowledge concerning the figures' properties and use level 3 reasoning. The students of the second category argue with an accumulation of properties and show not very sound knowledge of the geometrical properties. Here are two examples illustrating this second group. [Et E]

*1°) The quadrilateral OELM is a rhombus. This one has the characteristics of such a figure: 4 sides are equal; the diagonals cut themselves in their midpoint and form a right angle.*

*2°) Both girls are right; OELM is a square, for it has 4 equal sides and 4 right angles. It is also a rhombus, even if this figure, which is a rhombus, was not necessarily constructed using right angles.*

This student justifies her first answer by enumerating a list of properties of rhombuses. Thus, we classify her production at level 1. The properties employed are partially justified through visual or instrumented indications. This student considers

the figure in its material reality and her approach of the problem comes within Geometry I.

The answer to the second question "*both girls are right*" occurs frequently enough. Its justification shows that the statement "Charlotte is sure that it is not true" is wrongly interpreted as "Charlotte asserts that it is a rhombus "concealing the assertion" It is not a square". The student focuses on the question of the link between squares and rhombuses. It is a classic question (but not asked here) and the student knows how to answer. That shows that she has level 2 knowledge corresponding to the classification of figures.

With this student, we meet a rather frequent profile. [Et F]

*1° Four sides of the quadrangle are parallel between each other and of the same length  $OE=ML$  and  $OM=EL$*

*Definition of the rhombus: we can say that diagonals have the same midpoint and are perpendicular.*

*2° Marie is right, OELM is also a square because sides are all of the same length:  $OE=ML=EL=OM=4cm$ .*

*Let us remember that the square is also a rhombus but which has the peculiarity of having all sides with the same length (thus forming right angles) and having diagonals of the same length.*

The employed syntax could refer to level 3: some partially correct implications are evoked. But the body of knowledge is not very reliable. In particular, we find here a rather frequent "pupil's theorem": any quadrangle having four equal sides is a square. We are clearly within Geometry I where visual indications are used to support reasoning.

### Conclusion

What remains of geometrical learning when all else is forgotten? Our observations allow us to sketch a typology of geometrical approaches by the students some years after they have stopped studying elementary geometry.

First, we distinguish a set of students with sound knowledge on the figures' properties organized in an orderly and coherent set. These students are not necessarily situated in the same geometrical paradigm. Some have an approach referring clearly to Geometry II. Very often, these students have studied higher-level math and science. Others, contrary to the previous group, apply their knowledge to work in Geometry I, they are sensitive to the estimate of results within the measured setting.

In another group, we place students who do not give priority to deductive reasoning based on organized properties. We can here distinguish two approaches, the first one based only on visual indications while the second uses results obtained thanks to the instruments of construction and measuring. Students' levels of understanding and memorization of the bases of the elementary geometry differ greatly.

We find *again* students' levels described in former studies with respect to geometry, but our approach allows us to separate difficulties coming within reasoning or within a different geometrical belief. It seems that students keep the practical use of

Geometry: they forget the dimension geared towards reasoning about ideal forms which is yet favored in the French education.

Is it possible to change students within the setting of teachers' training? Are their positions fixed or on the contrary, malleable? The approach, which we have just developed, allows to tackle and revisit these questions in a finer way by taking into account the variety of students' geometrical thinking.

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### Annexe

#### Students' Answers in french

[Et A] 1) OELM est un losange car ses côtés successifs sont égaux deux à deux.  
2) Si OELM est un carré, alors MEL est un triangle rectangle en L. Selon le théorème de Pythagore on aurait alors,  $ME^2 = ML^2 + LE^2$   $ML^2 + LE^2 = 16 + 16 = 32$   
 $ME^2 = 5,6^2 = 31,36$   
L'angle ELM n'est donc pas un angle droit.  
Par conséquent, OELM n'est pas un carré et c'est Charlotte qui a raison.

[Et B] 1°) OELM est un losange car  $OE=OM=ML=LE$  et un losange a ses 4 côtés de même longueur.  
2°) Marie a raison car tous les côtés du quadrilatère ont la même longueur et il y a au moins un angle droit. On peut le vérifier par le théorème de Pythagore .  $ML^2+LE^2=ME^2$   
 $4^2+4^2=16+16=32$   
 $ME=\sqrt{32} = 4\sqrt{2} \approx 5,6$  donc  $MLE=90^\circ$

[Et C] 1°) OELM est un losange car ses diagonales se coupent en leur milieu (mesure) en formant des angles droits (avec l'équerre).  
Remarque : l'étudiant a construit la deuxième diagonale sur la figure.  
2°) Marie a raison. C'est un carré, puisque en plus d'être un losange, OELM a ses angles droits (équerre).

[Et D] 1°) Quatre côtés du quadrilatère sont parallèles entre eux et de la même longueur  $OE=ML$  et  $OM=EL$ . Selon la définition d'un losange, nous pouvons dire que les diagonales ont le même milieu et sont perpendiculaires entre elles.  
2°) Marie a raison; OELM est aussi carré parce que ses côtés forment un angle droit.

[Et E] 1°) Le quadrilatère OELM est un losange. Celui-ci répond aux caractéristiques d'une telle figure : les 4 côtés sont égaux ; les diagonales se coupent en leur milieu et forment un angle droit..  
2°) Les deux filles ont raison, OELM est un carré car il a 4 côtés égaux et 4 angles droits. Il est aussi un losange, même si cette figure qu'est le losange ne se construit pas forcément avec des angles droits.

[Et F] 1°) Les 4 côtés du quadrilatère sont parallèles entre eux et de même longueur  $OE=ML$  et  $OM=EL$   
Définition même du losange, de ce fait on peut dire que les diagonales ont même milieu et sont perpendiculaires entre elles.  
2°) Marie a raison, OELM est aussi un carré car les côtés sont tous de même longueur :  $OE=ML=EL=OM=4cm$ .  
Rappelons que le carré est aussi un losange mais qui a comme particularité d'avoir tous ses côtés de même longueur (donc forment des angles droits) et d'avoir ses diagonales de même longueur.

# GEOMETRICAL PRE-CONCEPTIONS OF 8 YEARS OLD PUPILS<sup>(\*)</sup>

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**Abstract:** *In an experimental study about isosceles triangles, we observed pupils' solution strategies revealing different naive approaches to the problem of measure in geometry. Our experiment discovered phenomena that should be taken into account in teaching Geometry.*

## The research

Our research group studied the influence of the drawing "orientation" (Cooper, 1998) in the perception of "isoscelity" of triangles, cf. (Marchini *et al.*, 2002), collaborating with Martin Cooper. The same test as used in Italy and Australia. In Italy the research was performed in 6 (third grade) classes of primary school, involving 105 pupils. The authors of this paper and L. Grugnetti elaborated the materials and the modalities of the experiment, in connection with M. Cooper.

The involved primary schools were from different places in Northern Italy (Viadana, Parma, Cattolica); in each of the three schools we chose a couple of parallel classes (with the same mathematics teacher) in which geometrical contents had never been treated before. This condition made us sure that we could observe some geometrical pupils' pre-conceptions, independent from schooling.

The choice of a couple of classes instead of a single class was necessary to reveal the influence of learning upon the establishment of pupils' mental images<sup>1</sup>: we introduced the notion of isosceles triangle in two different orientations, "roof" (A) and "flag" (B)<sup>2</sup>, cf. fig. 1. The tests were done at the same time in the two classes of the same couple to prevent possible exchange of information among pupils. The total pupils' numbers participating to the experiment were 49 for A training, and 56 for B training. The choice of introducing isoscelity for triangles as the condition of *equality of length of (at least) two sides* was due to an Italian school tradition in which measure of angles is introduced later than measure of lengths and to the etymology of the word.

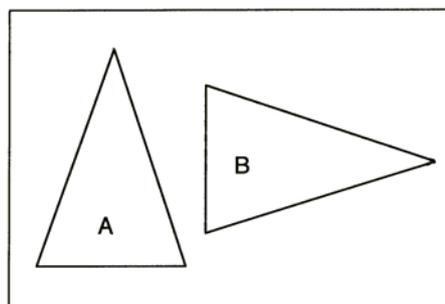


fig.1

<sup>(\*)</sup> Work done in the framework of activities of Parma Local Unity for Research on Mathematics Education.

<sup>1</sup> We use "mental image" as in (Fischbein, 1993): «a sensorial representation of an object»; in our opinion the establishment of a mental image is the first step in geometry since «the geometry deals with mental entities (the so-called geometrical figures), which possess simultaneously conceptual and figural characters». It is noteworthy that in *op.cit.*, the first example is an isosceles triangle drawn like triangle A in fig. 1.

<sup>2</sup> The words "roof" and "flag" were not used in the experiment; here they stand for references to the triangles A and B, respectively, in fig. 1.

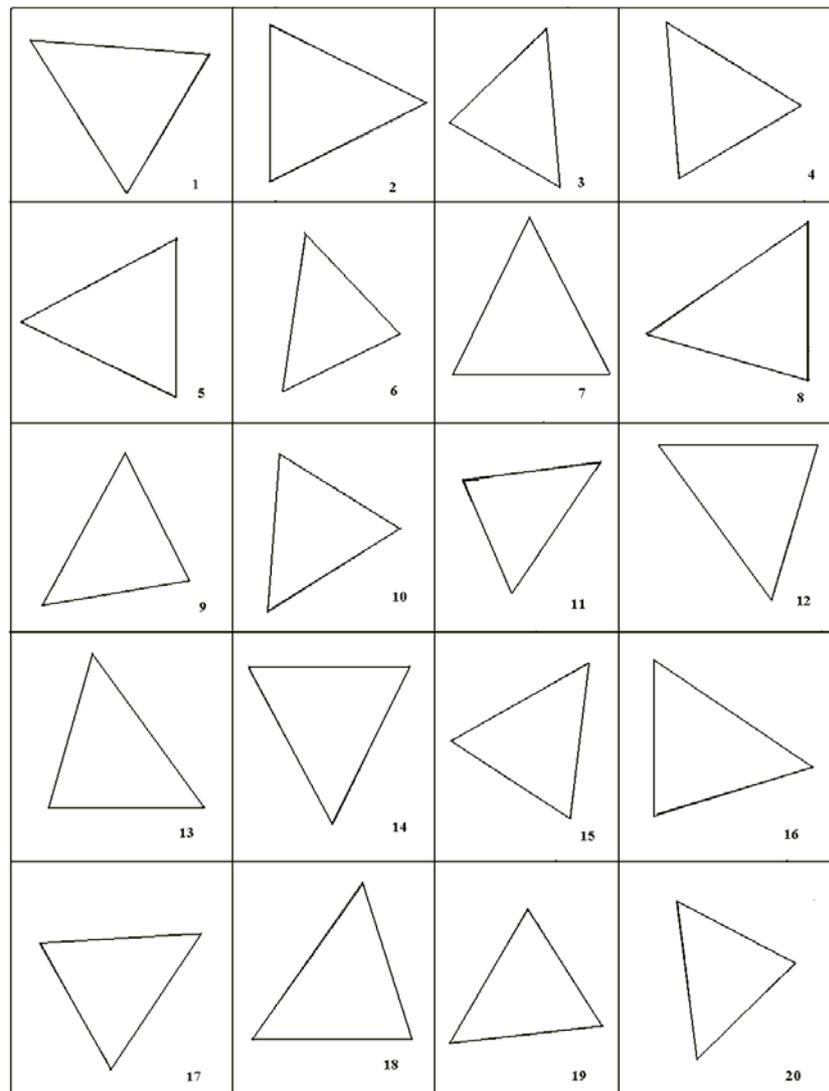


fig. 2

We decided to avoid the technical word *congruence*, because we thought that the meaning of this word requires a secondary intuition (Fischbein, 1987). The results of the Italian experiment can not be directly compared with those from Australia, where isoscelity is specified as a condition of equality of (at least) two angles. The experiment gave unexpected results and, as a by-product, we observed *de visu* the existence of strong pre-conceptions about the triangle and the measuring.

#### **The sessions of the experiment**

We did the experiment in the primary school of Viadana (1997 November 12th and November 21st), involving two classes: III C (15 pupils) and III D (20 pupils). The experiment was structured in two sessions: the first one composed of two steps for the learning and another for the testing; the second session divided in a first step for testing and a second one for written interviews.

We warned teachers to not discuss the test nor to present geometric topics between the two sessions. The first session lasted one hour, the second one approximately forty minutes.

In this paper we will focus on what we directly observed in Viadana. The same experiment was later performed in other schools (Parma 1997 December 12<sup>th</sup> and 19<sup>th</sup>; Cattolica 1998 January 23<sup>rd</sup> and 30<sup>th</sup>) in the same way, except for the written interview, since in the first experiment we had obtained uninteresting results. The researchers who performed the experiment in other schools confirmed the presence of the same relevant behaviour of pupils. In order to motivate pupils, we introduced the experiment as a contest among different classes of the third grade. We began the first step of session 1 by drawing triangles on the blackboard, in IIC "roof" triangles and in IID "flag" triangles, respectively, taking a few minutes only for this activity; this time was also devoted to establish a glossary (triangle, side, isosceles triangle). We can place this activity at Van Hiele levels 1 and 2, cf. (Van Hiele, 1986). We chose to use a very poor geometrical language only containing the words "triangle", "side", and "isosceles", in order to avoid interference with possible physical models.

In the second step we submitted a training booklet containing 10 triangles in the "roof" position (III C), and in the "flag" position (III D); the training booklets presented the same triangles, in the same order: 5 of them were isosceles and the other 5 were not. The booklets used in each class differed only for the triangles orientation, consistent with the respective presentations given on the blackboard. We specified in each case if the triangles were isosceles or not. Because of the novelty and complexity of the word "isosceles", our verbal presentation tried to facilitate interiorization in the sense of (Sfard, 1991) of the geometrical term and its related concept by slowly telling: "isosceles", "isoscel...", "iso...", "i...", and waiting for the pupils to complete the word. We expected that a progressive devolution to pupils of the repetition of the word "isosceles" would have succeeded in creating their own correct mental images.

At the third step, we gave a test booklet, the same for the two classes, in which on each right page there was one of the 20 triangles of fig. 2 (no. 1, 2, 4, 5, 7, 10, 14, 15, 17, and 19 are isosceles), and on each left page a question with double-choice answers ("isosceles" or "not isosceles") to be checked. The test lasted less than 10 minutes and we allowed pupils to use only a pencil.

In the second session (November 21, 1997) we directly submitted the same test with the same methodology and made written interviews, asking pupils if the task had been easy or difficult.

Dimensions, orientations and triangles succession were established with M. Cooper, who presented the same booklets to Australian pupils. The lengths of triangles sides vary from 8 to 12 centimetres approximately and the differences between the lengths of two sides are small enough to not be easily perceived and therefore to require some measuring strategies from pupils.

### **Results and statistical analysis of the experiment**

The paper (Marchini *et al.* 2002) reports and comments the average results of the experiment. On the basis of results we classified questions in "very easy" (with success rate  $\geq 75\%$ ), "easy" (with success rate  $\geq 50\%$  and  $< 75\%$ ) and "difficult" (if the success rate is  $< 50\%$ ). According to this classification, questions 2, 5, 7, 8, 9, 11, 12, 13, 14, 16, 18 came out as "very easy"; questions 3, 6 as "easy", and the remaining questions 1, 4, 10, 15, 17, 19 and 20 came out as "difficult". The underlined numbers stand for isosceles triangles, so it is evident that isoscelity is difficult to recognize in spite of the embodied cognition of the balance metaphor (Núñez *et al.*, 1999).

The global highest rate of correct answers was obtained in Q9, the lowest in Q1. It is noteworthy that Q3 was "easy" for A pupils, and "difficult" for B pupils. The most difficult question was Q1 for both trainings and the easiest ones were Q9 for "roof pupils" and Q7 for "flag pupils". The fact that the results in Q7, presenting a typical isosceles triangle in "roof" position, was better for "flag pupils" in both sessions was unexpected.

Another interesting feature is that in all "difficult" questions, except for Q20, the "flag" training gave best results.

Amongst the first sixteen questions only four of them came out as "difficult", and in the remaining four questions, three of them were "difficult". We suspect that the test had made pupils tired. The isosceles triangles 2, 5, 7, 14 have horizontal or vertical axis of symmetry and this configuration appeared to be closer to pupils' mental images. Wrong answers relative to isosceles triangles 1, 4, 10, 15, 17, 19 seem to be due to their "strange" position, corroborating the hypothesis of the entire research, i.e. that the drawing orientation affects the perception.

In the second session the average achievements improved, particularly for the B training. The increments of average values required a deeper analysis: they should be grounded on the longer persistency of pupils' attention to the test in the second session than in the first one, as

researchers pointed out. A second hypothesis is that training was good enough to establish the learning. This hypothesis is supported by:

- the time between the two tests,
- the absence of comments on the first test and of teaching of geometrical contents in the meanwhile,
- (and mainly) the short time devoted to the test (15 seconds for turning the page, looking at the drawing and answering each question) prevented the possibility that, in the second test, pupils relied on their own recalling of the test.

Therefore, the existence of an improvement of results seems noteworthy even without a statistical relevance.

In order to investigate with statistical tools if the answers to the first test had affected the results of the second one, we used the Larher's *crédibilité* (believableness) of implication index, (Larher, 1991; Gras & Larher, 1993),

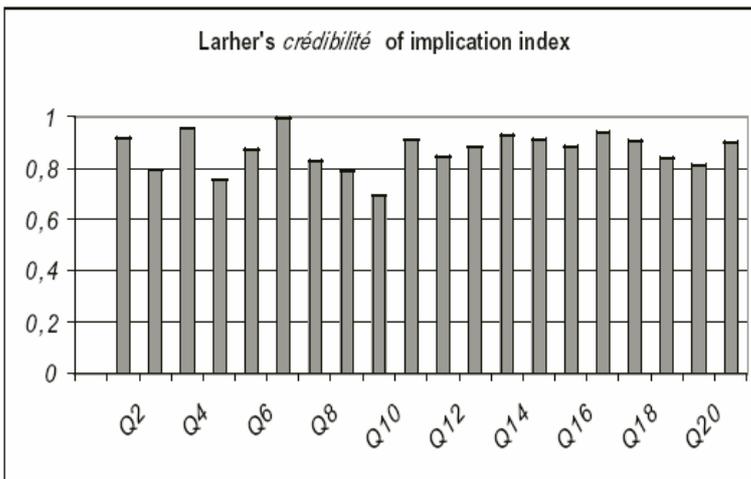


fig. 3

cf. fig. 3. This index is a probabilistic measure (ranging from 0 to 1) of the implication of two attributes. The use of this index is possible since the results of the two sessions were given as two sets of data, expressed with 0 or 1, and therefore they may be viewed as characteristic

functions of subsets or attributes. The values of the index varied between 69.35% (Q9) and 99.88% (Q6): we interpret these values as a corroboration of the fact that the first attempt influenced the second one and that the activity helped pupils to obtain a sufficient learning about isosceles triangles. We could not assess the persistence of this kind of learning with a third instance of the test due to the treatment of geometry topics in the school curricula.

## Results and comments on the Viadana experiment

Table 1 contains some statistical results relative to the Viadana experiment (in italics), compared with the achievements of the whole research in order to prove that Viadana sub-sample is coherent with the whole sample. The average values we observed are not far from averages of the whole experiment: the "flag" training got a lower success rate in each instance, while the "roof" training was better. Thanks to "roof" pupils the result of the second test was higher than the total average; for the same session pupils

	Global	A training	B training
Pupils' number	105	49	56
<i>Viadana experiment</i>	<i>35</i>	<i>15</i>	<i>20</i>
Answers	98.81%	98.72%	98.88%
<i>Viadana experiment</i>	<i>99.43%</i>	<i>98.83%</i>	<i>99.88%</i>
Correct answers	64.53%	64.39%	64.65%
<i>Viadana experiment</i>	<i>64.21%</i>	<i>65.67%</i>	<i>63.13%</i>
1st test correct answers	63.29%	63.66%	62.98%
<i>Viadana experiment</i>	<i>61.99%</i>	<i>63.82%</i>	<i>60.65%</i>
2nd test correct answers	65.82%	65.12%	66.31%
<i>Viadana experiment</i>	<i>66.57%</i>	<i>67.67%</i>	<i>65.75%</i>

improvements were comparable for both trainings, with a greater value for "flag" pupils.

Table 1

For Viadana pupils questions 5, 7, 8, 9, 11, 12, 13, 14, 16 and 18 were "very easy"; questions 2, 3, 6, and 20 were "easy" and questions 1, 4, 10, 15, 17, and 19 were "difficult". The easiest questions were Q18 globally, Q8 for "roof" training, with 100% of correct answers, and Q11 for "flag" pupils. The worst failures were Q1 globally and for "flag" pupils, and Q4 for "roof" pupils. Compared to the entire experiment, in the Viadana environment, Q2 was "easy" instead of "very easy", and Q20 was "easy" instead of "difficult". The presence of better results for "flag" training on "difficult" questions was confirmed, except for Q10, as well as the better results in Q7.

With reference to the three levels stated above, the test was "very easy" for 3 pupils (A: 1, B: 2), "easy" for 31 pupils (A:14, B: 17) and "difficult" for one flag pupil. The minimum frequency of correct answers was 42.5% and the maximum was 82.5%, both obtained with B-training.

Before the experiment we had asked the maths teacher to give us her assessment for each pupil. These teacher's assessments were compared with the results of our test and showed a higher dispersion, contrasting with the general homogeneity observed in the results of our test. A possible interpretation of this dissimilarity is relative to the nature of the learning we

induced by the training. Teacher's assessment collects a lot of information on the pupils (linguistic competencies, mathematical skills, diligence, ...). On the other side we tested one aspect only, recalling visual abilities and totally avoiding linguistic competencies: this can justify the similarity of achievements among pupils and the dissimilarity between test result and teacher's assessment. For instance, a pupil may be disadvantaged in his/her teacher assessment by scarce competences in subjects, e.g. linguistic ones, that do not require visual competences.

The presence of the "roof" pre-conception in pupils clearly appeared during the test: we observed most of the children rotating the booklet or their head in order to place the triangle in the "roof" position comparing in this way the drawings with their archetype of triangle, (Medici *et al.*, 1986); this also happened in the "flag" class. This pre-conception should be originated by the experience: the true roof might be an object representing in a better way the concept of isosceles triangle, in the sense of (Collins & Loftus, 1975). The perceptive difference (Arnheim, 1974) with the "roof" pre-conception seems to justify the presence of a flexible and dynamic learning structure, in the sense of (Singer, 2001). The presence of the "roof" triangle pre-conception is confirmed by (Vighi, 2003a and 2003b), as a portion of a *concept image* (Tall & Vinner, 1981). In our opinion, the "roof" training supports an interiorization following rigid structures, as stated in (Singer, 2001), by re-enforcing a preconception, that is probably why "flag" pupils got better results in the most difficult questions.

The small difference in the length of the sides of triangles combined with the "strange" orientation of the proposed figures activated the pupils' pre-conceptions relative to measuring. We observed some pupils using the pencil and the fingers as compasses for comparing the lengths by "transportation" as painters do; in other cases, pupils preferred to "build" the compasses by using only the fingers even if their teacher had never used the "mechanical" compasses before in school. Another interesting pupils' strategy was to use fingers as a ruler for measuring: pupils "covered" the sides proceeding in jerks, in a sort of subdivision of the length, therefore in this case pupils used the "measure", and teachers had never introduced measure before. In some cases the same pupil used different strategies simultaneously, strengthening Vergnaud's statement that when a subject does not have the required competences, s/he uses different schemes at the same time (Vergnaud, 1990).

### **Conclusion**

The whole experiment and the Viadana sessions too revealed that pupils reached the learning with short activities. We ascribe these results to the presence of pre-conceptions that help pupils in the learning;

therefore the teacher must recognize the pupils' pre-conceptions. Moreover the experiment revealed the intuitive embodied measure "tools", showing us how metric geometry takes root in geometrical pre-conceptions or knowledge ripened out of school. In our opinion the experiment stressed important pre-conceptions that teacher cannot neglect; this would be a starting point for the teacher that must legitimise pre-existing knowledge, or better beliefs, cf. (Marchini, 1999).

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# USING DYNAMIC TRANSFORMATIONS OF SOLIDS TO PROMOTE CHILDREN'S GEOMETRICAL REASONING

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**Abstract:** *This paper is part of a research project on the study of children's thinking about geometrical solids in the context of dynamic transformations. The context of this study is defined by the mental transformations of an orthogonal parallelepiped. Twenty 6<sup>th</sup> grade children who had previously experienced dynamic transformations of physical models of geometrical solids in their classroom were clinically interviewed. The analysis of the data resulted in a modeling of children's thinking and indicated a development from holistic to a relational consideration of geometrical solids. Moreover, there is evidence of the significant role played by the dynamic transformations in this development.*

A number of research studies have focused on children's thinking on three dimensional solids. Most appear to be on the nets of solids (Mariotti, 1989; Potari & Spiliotopoulou, 1992), on their plane representations (Cooper & Sweller, 1989) and on constructions of the solids by unit cubes (Battista & Clements, 1996). This research studies children's thinking about geometrical solids and their properties. The research on this domain exploits van Hiele theory (1986) and uses it as a way of interpreting and classifying children's thinking using methods such as questionnaires, test or interviews (Lawrie, Pegg & Gutierrez, 2002; Pegg and Baker, 1999). For example, Pegg (1997) extended van Hiele's work by highlighting the differentiation between Levels 2A and 2B in the case of three dimensional solids.

## **The rationale of the study**

The studies mentioned above are mainly exploratory and do not indicate the conditions under which children's thinking on three dimensional solids develops. In a research project, part of which we present here, we attempt to investigate the development of children's thinking concerning geometrical solids in three different contexts. The first context involves children's manipulations of physical materials of three dimensional geometrical models. The second is defined through children's interactions in a computer-based environment and the last one is formed through children's involvement in imaginative situations concerning dynamic transformations of the solids. These contexts are characterised by the dynamic manipulation of geometrical solids, a process where "the solid changes its form through the variation of some of its elements and the conservation of others" (Markopoulos & Potari, 2000). This seems to be related to the concept of invariance which promotes intuitive reasoning (Otte, 1997). These contexts vary in the type of transformations which they

require through children's actions. Thus, through these contexts the children move from physical to visual and finally to mental actions.

We distinguish two main phases in the research process. The first concerns a classroom teaching experiment in three classrooms of the 4th grade and three of the 6th grade in three different schools in Patras, Greece, along the same lines described by Cobb, Yackel and Wood (1992). In this phase dynamic materials were developed by the researchers and tasks based on these materials designed by the teachers in cooperation with the researchers. The whole experiment took 4-5 teaching periods for each class. In the last phase the children of the 6th grade were interviewed one by one for an hour on tasks referring to an imaginary dynamic transformation of a cube and of an orthogonal parallelepiped. The data from the two phases consisted of video recordings which have been transcribed. In this paper we focus on the data coming from the last phase.

The whole philosophy of the project and some initial findings from interaction with a pair of children are presented in Markopoulos & Potari (1999), while the actual implementation of the dynamic environment in the mathematics classroom and the issues which emerged are discussed in Markopoulos & Potari (2000).

In this paper, we extend our work by studying more systematically how the children who participated in our previous study think about geometrical solids and their properties in a context defined by the mental transformations of geometrical solids without the use of physical or computer manipulatives. This context is related to the children's visualization process that involves the recall or the construction of the solid's mental image, its representation and its appropriate transformation if necessary (Weatley, 1990).

## **Methodology**

The research methodology is the *clinical interview*. The clinical research methods are based on the principles of constructivism and aim to investigate children's conceptions. The researcher acts as a teacher interacting with the children while aiming to investigate their thinking. By reflecting on these interactions, the researcher tries to interpret the children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised. (Bell, 1993; Hunting, 1997).

Participants: Twenty 6<sup>th</sup> grade children participated in this experiment. These children had already studied geometrical solids through the use of dynamic models in a classroom teaching experiment. These models were actual and computer-generated representations of geometrical solids and some of their properties could be varied dynamically (Markopoulos & Potari, 1999; 2000). In this environment the children faced tasks which involved the manipulation and the study of dynamic transformations of these models.

The process: The children were interviewed individually for about an hour on a number of similar tasks to those they had faced in the classroom but this time without

any visual reference. Nevertheless, the context of the activity differed, as neither the dynamic models nor the dynamic computer representations were used. The context was defined by the use of the mental images that the children themselves constructed and manipulated. Although the tasks were not predefined, they had a common structure in each interview. In particular, the children were asked specifically:

to imagine and describe an orthogonal parallelepiped

to propose a mental dynamic transformation of the solid

to focus on the mental dynamic transformation.

A dynamic transformation of a geometrical solid involves three dimensions: the initial solid, the process of the transformation and the produced solid. The study of the dynamic transformations depends on which of these three dimensions is the focus of attention. The dynamic transformations that took place during the interviews had an orthogonal parallelepiped as the initial solid. The children were asked to transform this solid, mentally defining either the process of the transformation or the produced solid.

*Analysis of the data:* The data consists of the twenty transcribed video recordings. Initial attempts to analyze the transcribed teaching experiments were made through the coding of the types of transformation performed by each child. Then, by scrutinizing the data line by line, we identified the children's conceptions and we formed categories that describe the children's thinking about geometrical solids. Finally, we re-examined the data for each child separately, looking for possible individual development in terms of the produced categories.

## **Results**

*The type of transformation:* From the initial analysis of the transcribed interviews, two main types of transformation emerged. The first type (A) involved the dynamic transformation of the orthogonal parallelepiped through the modification of its properties. For the children who performed this kind of transformation, the transformation process was characterised by their specification of the modified property or properties and the degree of modification. In this case, only the first two dimensions of the transformation were considered. The geometrical elements that the children modified through the transformations were the salient ones such as the edges, the faces, the dimensions and the angles of the orthogonal parallelepiped. In the second type of transformation (B), the transformation process was characterised by the children's reference to all three dimensions. The target solids were an orthogonal parallelepiped with two square faces and a cube. Although this type required the child to anticipate the specific modified properties, a process which is rather complex conceptually (Markopoulos & Potari, 1999; 2000), most of the children carried out this type of transformation (eighteen out of the twenty pupils). This tendency probably reflects the effect of the children's experiences in the classroom with the manipulation of the dynamic models. However, as will be

illustrated below, the degree of awareness shown by the children regarding the modification involved in the transformation was not the same for all of them.

*The categories of conceptions:* In relation to the children's thinking about geometrical solids, four categories emerge. In the first category (C1) the children conceive of a solid as a total entity and are unable to identify its properties. Although the dynamic transformation of a solid involves the modification of its properties, the children studying the transformation identify the two solids holistically and limit their description to the name of each solid.

The second category (C2) describes the children's geometrical thinking based on comparisons between the modified properties. The solid is no longer considered as a total entity but is related to the properties which have been modified directly by the children during the transformation. For example, when modifying the edges of an orthogonal parallelepiped to get a cube they compare the two solids in reference to this particular property.

In the third category (C3), the children correlate the properties of the solids. They also relate the properties that they have modified to the solid, though they do not seem to realize the role of the subsequent modification of the properties in the formation of the solid. For example, the modification of the length of the edges of an orthogonal parallelepiped results in changes to the area of its faces. Although the children identify the change in the faces, they do not relate these changes to the process of transforming the initial solid into a different one.

In the fourth category (C4), the children seem to realize the role of the properties in the solid's form and are led to make generalizations. They build relationships between the geometrical solids and their properties and between the solids themselves. For example, when transforming an orthogonal parallelepiped into a cube, the children relate the modifications made to the dimensions to the subsequent changes in the length of the edges, in the form and area of the faces and in the two solids.

These four categories of geometrical thinking describe the children's conceptions concerning the concept of geometrical solid through its dynamic transformation. They are presented in a developmental way and could be related to the Van Hiele level of geometrical thinking with regard to the discrimination between level 2A and 2B that was proposed by Pegg (1997) and Lawrie, Pegg and Guiterez (2000).

#### *The development of geometrical thinking*

In table 1 we demonstrate the children's conceptual development during the experiment with reference to the type(s) of transformation that each child performed.

Children	St1	St2	St3	St4	St5	St6	St7	St8	St9	St10
Types of transformation	A, B	B	B, B	B	B	B, B				
Category of thinking	C1 – C2, C2 – C3	C2 – C3, C3 – C4	C2 – C4, C2 – C4	C1 – C2, C2 – C4	C2 – C3, C2 – C4	C1 – C3	C1 – C2, C1 – C2	C1	C2 – C3	C1 – C2, C1
Children	St11	St12	St13	St14	St15	St16	St17	St18	St19	St20
Types of transformation	A	B, B	B, B	A, B	B	A, A	B	B	B	B
Category of thinking	C1 – C3	C1 – C2, C2	C1 – C2, C2 – C3	C2 – C4, C2 – C4	C2 – C4	C1, C1	C1 – C3	C2 – C4	C2 – C3	C2 – C4

Table 1: The children's conceptual development

The first row shows each individual child, while the second row details the types of transformation that each child experimented with. In the third row, the identified categories of conception are shown for each of the types of transformation. For example, St1's thinking during the study of a type A transformation developed from the first category to the second (C1-C2), while during the second transformation his thinking developed from the second category to the third (C2-C3). We discuss below some representative cases in order to draw attention to some issues that emerged concerning conceptual development and the role of the transformations.

Two children (St8 and St16) remained at a primitive level of geometrical thinking. For example, St8 could not evolve his reasoning beyond the first category (C1) since the form of the solid dominated in his conceptions. He also had difficulty in mentally transforming the initial solid and looked for a physical referent around him. Thus he constructed the mental representation of a physical model, a small box, and tried to transform it. In attempting to transform the initial solid into a cube, the child proposed a reduction in the length of the solid. When describing the transformation process, he focused on the modification of the external appearance of the solid: "to shrink it...", "to squeeze it...". When the teacher-researcher made identifying this modification the focus of the task, the child recognized the change of the one dimension at an intuitive level: "these two will go inside...". He also attempted, albeit unsuccessfully, to relate this change to the remaining faces. He seemed to recognise that the faces had to become squares but he could not justify his opinion. During the experiment the child identified the change in the form of the solid, but was at no point able to identify the faces or dimensions that varied. In conclusion, the student could not develop reasoning beyond a holistic consideration of the solid.

Three children (St7, St8 and St12) started by considering the solids in a holistic way and moved towards a recognition of the specific modified properties. For example, St7 used only one type of transformation (B) and reached the second category (C1-C2). He wanted to transform an orthogonal parallelepiped into another one with two square faces. Initially, he approached the solid in a rather holistic way and focused on the modification of its form without taking into account the modification of any of its properties. He described the modification of the solid's dimensions using expressions like: "I make it higher...", "I will make it smaller and higher...". Then, when he was asked to specify the changes that would be caused by this transformation, he developed his reasoning and referred to the alterations to the faces that would occur. He clarified his thinking by using a physical referent. The following dialogue shows the child's attempts to mentally construct a cube from an orthogonal parallelepiped.

St7: I will raise it, if it is possible, I will make it smaller otherwise it will look like an orthogonal parallelepiped.

Researcher: But you don't want it to look like an orthogonal parallelepiped, do you?

St7: I want it to be a cube.

R: And why do you have to make it smaller? How much smaller?

St7: A little bit. [showing with his hands the transformation of the parallel faces to squares]

R: Why is that so ?

St7: Because it will be higher as well, and if I make it small enough it will not be a square.

R: Why so much? Why does it look like a square?

St7: Yes, this way they will be two squares and then I will make it higher. [Showing the dimension of the height]

The student rather intuitively related the cube to its square faces but he could not focus on their properties. His conceptual development was restricted to the identification of the specific modified properties (the faces) and to the consideration of the form of these properties.

The conceptual development of a number of children reached the third category (C3). For example, during the experiment St13 proposed the transformation of the orthogonal parallelepiped into an orthogonal with two squared faces and a cube (B, B). The conceptual level demonstrated during the study of the first transformation developed from the first category to the second (C1-C2), whilst that demonstrated in relation to the second transformation developed from the second category to the third (C2-C3). The first mental transformation of the initial solid into an orthogonal parallelepiped was considered by the student in a rather intuitive way similar to St7. However, in his attempt to transform the solid into a cube, he started to consider the alterations resulting from the specific modified properties. One such alteration was

the change of the form of the faces to square. The child again used a physical referent, the dimensions of which were different, to justify his opinion regarding the transformation that caused its faces to become square. He proposed that the equivalent enlargement of the edges was a prerequisite for the change to the faces.

Finally, a number of children reached the most advanced level of thinking about a geometrical solid. St3 was the student whose thinking developed as far as the last category (C4) for both the transformations he performed (A, B). Initially, considering the transformation of the orthogonal parallelepiped through the modification of its angles, he focused on the specific modified angles, though he related these changes to the formation of the whole solid (category C2). At the next stage of the experiment he related the specific modified angles to the indirect change in the volume of the solid. The following dialogue demonstrates this development.

St3: Two of its faces and its angles will be modified.

R: Will anything else?

St3: It will simply become oblique.

R: What else will change?

St3: Aha! Its volume will become smaller?

R: Why do you believe that its volume becomes smaller? What do you change?

St3: Its angles.

R: How much do we have to change the angles?

St3: As much as we want. It could go there (Using his palms, he represents an extreme transformation where the solid becomes flat)

R: If it reaches there, will the solid exist?

St3: It becomes flat?

R: So, what will happen to the volume?

St3: It will have no volume. It becomes a plane (figure).

The justification for considering the reduction in volume as a result of the modification of the angles was based on the student's experience in the classroom teaching experiment with the dynamic model of orthogonal parallelepiped, which allowed a gradual transformation of the solid into a series of different solids. The initial orthogonal parallelepiped became oblique and finally flat. As was emphasized in our previous work Markopoulos & Potari (2000), the students developed an intuitive appreciation of volume through the comparison of this series of solids using a dynamic physical model of an orthogonal parallelepiped during the classroom teaching experiment. St3 seemed to transfer this kind of experience and adapt this approach to the case of mental transformations.

The role of the dynamic transformation of solids is crucial in the development of children's thinking concerning the geometrical solid. As the children focused on the process of the transformation, they seemed to become aware of the role of the specific and subsequently modified properties on the formation of the solid. The development of their thinking from a holistic consideration to an understanding of the abstract relationship between the properties and the solid involves two intermediate levels of thinking: initially, an intuitive understanding of the relationship between the specific modified properties and the solid and, subsequently, an awareness of the changes that the former cause to the solid during its transformation. The role of the dynamic transformation in the development of these two intermediate levels was significant. When focusing on the transformation process, the children started to become conscious of the relationship between the properties of the solid and the effect of their modifications.

What is more, in the complex context of the mental manipulation of the solid, most of the children resorted to using a physical referent. The way it was used differed from child to child depending on their level of thinking. For example, the children who conceived of the solid in a rather holistic way used the physical referent as a prototypical resource in order to construct a mental image of the solid. The dynamic manipulation of this mental image was impossible as they could not focus on its properties. The children who could only relate the solid to its specific modified properties used this physical model in the creation of the mental image as well as in the process of transformation. The children who reached the third level of conceptual ability used the physical referent as an instrument in their attempt to communicate their thinking. Finally, the children whose thinking reached the most advanced level utilized the physical model to exemplify the transformation process and the relationship between the subsequently modified properties and the solid. It would appear that, the use of the concrete model does not necessarily imply primitive thinking but can act as an intuitive tool that in the process of transformation can possibly support abstractions and formalizations to geometry education (Meira, 1998).

### **Concluding remarks**

The study indicated that in the context of mental transformations children's thinking concerning the three dimensional geometrical concept can develop from a holistic to a relational consideration but not necessarily in a linear order. Although not all the children reached an advanced level of thinking, the context of dynamic transformations promoted the development of most children's geometrical thinking. There is also an indication that the children's experience with dynamic transformations of physical models in a mathematics classroom environment can probably act as a means that can allow children to transfer experience from one context to the other (Evans, 1999). Moreover, the whole process of transformation can be considered as a metacognitive activity (Pandiscio & Orton, 1998) that can help

children to become aware of their actions, either physical or mental, and lead to a deeper understanding of the concept of geometrical solid.

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# COMPARING PERIMETERS AND AREAS CHILDRENS' PRE-CONCEPTIONS AND SPONTANEOUS PROCEDURES<sup>1</sup>

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**Abstract:** *In this paper we present research carried out into the concepts of perimeter and area among fourth and fifth year Primary school pupils. The paper focuses on children's pre-conceptions and spontaneous procedures. Researches show that the conflict perimeter-area is an epistemological obstacle causing difficulties in the comprehension of these important geometrical concepts. In fact it is only by recognising and understanding these that we can identify teaching strategies for overcoming the 'perimeter-area conflict'. We present and discuss two worksheets aimed at revealing pupils' reasoning and behaviour, especially unpredicted. We make a detailed analysis of how children completed the worksheets, and of subsequent individual interviews with them.*

## Introduction

In this work we discuss two worksheets concerning an experimental activity which is still currently in progress, inserted at the start of a Mathematics Laboratory Project (MLP). The activities supplemented traditional curricular teaching with different new activities, in terms of both methodology and contents. It is fundamental from the teaching point of view to observe strategies children use. This allows teachers and researchers to identify activities, which can help pupils to distinguish between the two concepts of perimeter and area.

For our discussion it is necessary to solve a translation's problem. In Italian two different words are used to translate the English word *quantity*. The first, *quantità*, is used as a sort of measure of something; it denotes numerosness in cases where natural numbers are used. The second Italian term *grandezza* emphasises qualitative aspects of the same mathematical entities. This paper focuses on the comparison between quantitative and qualitative aspects of lengths and surfaces, so in order to distinguish the two concepts in the English translation we use *q-quantity* for *quantità* and *g-quantity* for *grandezza*.

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<sup>1</sup> This work was carried out in the Local Research Unity in Mathematics Education, University of Parma, Italy.

## Theoretical framework

Great importance is given to the concepts of perimeter and area in geometry teaching in Italian primary schools. The concepts are introduced with pupils 9-11 years old, but usually only measuring work is done. Measuring length would be simplified by using a ruler, but no artefact (Bonotto, 1999) as simple as the ruler exists for the measurement of area, even for simple polygons. In our opinion the early introduction of measurement in geometry presents the risk of confusing a  $g$ -quantity with its measure ( $q$ -quantity), whereas it is in fact necessary to emphasise that an object has a length or a surface, even if these are not measured. Teaching research shows that it is essential to work on this aspect (Moreira Baltar, 1996-97): “Allowing pupils to discover that length is a property of objects, and can be considered apart from numerical considerations, requires specific preparation.” (Chamorro, 2002). The same observation can, of course, be made about surfaces. Work on  $g$ -quantity should precede work on  $q$ -quantity: “The two stages need to be distinguished, as two separate skills, each using a distinct level of abstraction, are needed for the two aims.” (Marchini, 1999).

When a pupil “... has to differentiate on a physical object itself or on a geometrical representation, between the  $g$ -quantities in either one or two dimensions” (Jaquet, 2000), an obstacle inevitably arises. This is the perimeter-area conflict. The research shows that younger children (6–8 years old) tend to identify the **largest** shape with the widest or highest (Montis et al., 2003) and older children perhaps add the measurements of width and height rather than multiplying them. This attitude can be ascribed to the pre-eminence of the additive conceptual camp (Vergnaud, 1990). Moreover “For a child it is an entirely new operation, perhaps even surprising; he knows how to add measurements of length, the sum of which is still a measurement of length, he now has to multiply two measurements of the same type to obtain another measurement of a completely different type.” (Jaquet, 2000).

The contemporaneous presence of many shapes could be used to compare  $g$ -quantities related to these shapes. We suggest that the learning of geometry is helped by working on comparison between two or perhaps more carefully chosen shapes. Moreover it may be that, before geometry had become institutionalised as a school topic, the pupil is able to compare the areas of two surfaces, for example by superimposing the sheets of paper on which they are drawn, but not be able to calculate the area. Similarly, he may be able to compare the lengths of two curved lines, but not be able to calculate the measures. Pupils’ normal school work on only one shape, instead, is a symptom that there is only the perimeter or area of this one shape to be measured, placing  $q$ -quantities before  $g$ -quantities. The consequence is that even if the question is about congruence of shapes, “Ascertaining equivalence ... is carried out ... translating the comparison into the field of numbers ...” (Chamorro, 2001). At school, more attention is usually paid to  $q$ -quantities, mainly for reasons of time and availability of instruments. But in order to facilitate pupils’ understanding and not present them with unnecessary obstacles, it is important for the step to

measurement to be taken only in the next stage. Early introduction of formulae for measuring contributes to the formation of an obstacle<sup>2</sup> famous as “perimeter-area conflict”.

## Methodology

To make a preliminary investigation of pupil’s spontaneous procedures, we gave them two individual worksheets. Worksheet A concerned comparison of perimeters, and worksheet B concerned comparison of areas, without referring to their measurements. These worksheets were distributed to pupils at the start of the activity of MLP, to four classes (9-10 years old) and two (10-11 years old), for a total of 130 pupils<sup>3</sup>. The pupils in the fourth year had not yet been introduced to the concepts and were given first worksheet A and then worksheet B. The fifth year students were given first worksheet B, and then worksheet A, with necessary modifications. The second worksheet was given several days after the first. As shown in the next section, each worksheet contained pictures of 11 shapes. For worksheet B it was presumed that, as it usually happens, the children would understand the shapes as two dimensional, even though only the boundary was drawn. As the drawing of the boundaries of a two-dimensional shape can constitute an obstacle, we chose a context, which allowed differentiation between the one- dimensional and the two-dimensional aspects in the same shape. The context is described below. We got pupils to work with shapes drawn on paper rather than on concrete objects for the following reasons: the high number of children involved, the need to find out each individual child’s own ideas without external influences and the need to have a written record of these ideas. Each pupil worked individually on an enlarged copy of each shape. All shapes were drawn on an A4 sheet of paper. The teacher gave no instructions apart from telling the pupils to do what was on the worksheet, so they were free to use any instrument they wanted. After the tasks, individual oral interviews were carried out and pupils asked to explain their choices, strategies and reasoning. The worksheets will be distributed again in another year’s time, as a long-term check, at the end of the project.

## Worksheets presentation

The worksheets (in appendices) describe the problem of two shepherds, Mario and Pino, who have to build a fence to enclose their sheep. This context seemed suitable for an examination of ‘perimeter’ and ‘area’<sup>4</sup>. Worksheet A deals with determining a fence: the smallest possible quantity of barbed wire is to be used. Worksheet B deals with the greatest possible quantity of grass for the sheep to graze on.

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<sup>2</sup> Research reveals that the damages of this sort of teaching persist in secondary school students: if the shape is out of canonical geometric shapes, many students answer negatively to the question whether there is a measure of the length or of the area. (Grugnetti, Rizza, 2003).

<sup>3</sup> Istituto Comprensivo (Primary and Middle School) in Collecchio, Parma, Italy. We wish to thank the teachers in whose classes the pupils took part: A. Balestrieri, D. Bazzarini, E. Nocera, and M. Zanetti.

<sup>4</sup> Modern pupils do not usually have direct experience of sheep and fences, but they are usually familiar with them from stories.

In order to focus pupils' attention on the drawings, the first question is about the fence itself: it is based on the concepts of closed or open lines.<sup>5</sup> The question about safety is intentionally vague with the aim of investigating pre-conceptions. Pupils were asked to explain their choices. It was presumed that they would select as safest those fences, which prevent the sheep from escaping, and would thus exclude lines D and M, which are open. We used the representation normally used to show open lines; a possible gate is not drawn. When the unsafe fences have been excluded, nine shapes remain for comparison of perimeter and area. Most of the shapes are not easy to compare. The aim was not for the pupils to use intuition, which is often the case in introductions to the subject. We worked pupils to compare shapes precisely where comparison is not immediate. To simplify the task, the worksheets ask pupils to look at terns groups of shapes, which are intended to help them make meaningful comparison, and graded in increasing order of difficulty. Within each group of three, two of the shapes are easy to compare, while the comparison with the third requires pupils to identify suitable strategies. Overall, the pupils have to find suitable ways of comparing and possibly find measurement tools, and the activities are fairly complex. We predicted that pupils would use various methods: *perceptive comparing* (e.g. 'you can see' that B is smaller than A), *superimposing shapes* (e.g., if you put H over I it is clear that H has a larger area), *dividing into equal pieces* (equi-decomposability) (e.g. G can be transformed into A by cutting off and moving a piece) and *measuring* (e.g. the perimeter of F or H can be evaluated with a piece of string). It would be difficult for pupils to measure the perimeter of the shapes with curved boundary, even for older pupils who know how to approach the exercise with polygons. Shape H in particular is irregular and allows only estimates of measures. It is important to let primary school pupils realise this so that they do not grow up thinking that perimeter and area can only be measured when handy formulas are available. Another reason for using H in the exercise was because it is similar to a fence in reality, it also enabled pupils to think about perimeter and area without using calculations and formulae. Overall, various different comparison strategies are necessary; this is another reason for the complexity of the exercise.

The next questions on the worksheet concern the possible existence of isoperimetric or equi-extended shapes. Lastly, pupils are asked to consider all the shapes and rank them in increasing order of perimeter or area. This last task is particularly complex, as it requires numerous comparisons and ordering of a set of nine members. Pupils are not used to dealing with such numerous sets.

### Results and analysis

The experiment yielded both quantitative data (the percentage of correct replies) and qualitative data (reasoning, pre-conceptions etc.). We follow the numbering scheme of the worksheets.

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<sup>5</sup> This question appears on the first worksheet pupil's meet, either in worksheet A as illustrated or on worksheet B.

## Worksheet A

### 1) Concept of open vs. closed

We predicted that the first question would not be problematic as it concerned activities the pupils had carried out previously. But it yielded significant results. The table shows the percentages relating to shapes chosen, respectively, least or most frequently.

	M	D	A	E	I
9 -10 years old	63%	62%	7%	7%	9%
10 -11 years old	91%	89%	6%	11%	17%

A few 9-10 years old saw D and M as unsafe fences because they mistook the ‘gap’ in the line as a gate, necessary in their view to let the sheep in and out. The 10-11 years old excluded D and M in a higher percentage,

perhaps because they are more used to working with traditional geometrical shapes given by closed lines. It is interesting to note that the least frequently excluded shapes are A, E and I, in other words, those with rectangular or almost rectangular shape. This was not the only criterion used. The context of our questions influenced the pupils a lot. By the word ‘safe’ they understood not only that the sheep should not escape, but also that they should not hurt themselves. We researchers used the word ‘safe’ in order not to explicitly name closed lines, but the pupils’ interpretations were different based on the distinction between closed and open lines. The choice of an ambiguous word allows various different interpretations. If the pupil “sees” the space inside and outside a line, s/he has the concept of “independent space”, in which the line is placed (Speranza, 1997). The pupils who “see” closure in the open fences perceive as space only that internal (intra-figural non-independent space). Another important aspect is the presence of sharp corners: children distinguish between internal and external corners (Vighi, 2003). The question ‘Which (fence) do you think they should choose?’ does not necessarily force pupils to choose only a closed line. For example, B and F were often excluded because they were too thought to be too small. *“F and B are not wide enough for the sheep, and if they are all squeezed up together they might climb over the fence by climbing on each others’ backs”*. Some pupils excluded C, F, H, G and L *“because if they run the sheep might hurt themselves”*. In the oral interview they explained that they meant the *“parts with corners”* which might *“hurt them”*. The drawings do not give the idea of a fence for some of the pupils. *“C, L, G and H are a funny shape.”* *“Sheep usually have a fence like I”*. *“I have never seen a fence like F”*. Pupils with greater experience of geometry wrote things like, *“I excluded C, F, G and L because they are not geometrical shapes and so they are more difficult to control”*. So pupils opted for rectangular shapes for reasons of regularity, symmetry, absence of ‘inside corners’, spaciousness as well as aesthetics.

2) Comparing perimeters*Finding the shape with the smallest perimeter*

	<i>B</i>	<i>F</i>	<i>C</i>
<i>9 -10 years old</i>	95%	87%	79%
<i>10 -11 years old</i>	85%	94%	88%

*Comparing perimeters*

	<i>A-B-I</i>	<i>E-F-L</i>	<i>C-G-H</i>
<i>9 -10 years old</i>	80%	26%	16%

The percentages in the first table show that the fence with the smallest perimeter was identified fairly easily. As the second table shows, only the 9-10 years old carried out the comparison of terns activity, as the older ones had already worked on the concept of perimeter. In the first group of three, the order of the perimeters is the same as the order of the areas, and this meant that most of the replies were correct even if the reasoning was faulty. Many pupils in fact put one shape on top of another and seeing that shape I ‘fits inside’ A, deduced that I has a smaller perimeter than A. This is a clear instance of the area-perimeter conflict, as is the following remark: *‘B is smaller, so it covers less area so less barbed wire is needed’*. Some pupils used the concept of perimeter correctly: *“I went round the shape with string ...”* and *“Shape I has rounded corners so it needs less barbed wire”*. In the second group of three, the shape with lowest perimeter and area is F, and reasoning correctly, in terms of perimeter, or incorrectly, in terms of area, it is in either case identified as the ‘smallest’. The other two, L and E brought to light the confusion between the two concepts. Shape E in fact has the biggest area, but the smallest perimeter. The following are some of the interesting comments from pupils: *“I traced the perimeter and counted every second, and E is the biggest”*. There is the idea of movement and the dynamic geometry. *“I measured the length and the height with a ruler, and L is the biggest”*, *“F has sides that come in so it needs less barbed wire”*. In the third group of three, the most common mistakes were to break up G and transform it into a rectangle equivalent to A, and say that C has one ‘piece’ less than A. This method is valid for measuring the area, but misleading for measuring the perimeter. In this case too, the shape with smallest perimeter, C, coincides with the smallest area. It is thus important to compare G and H. Not many pupils used string to measure the perimeter of H. A few used a ruler to measure the longest dimensions, many made a rough assessment just by eye, seeing which shape was the biggest.

3) Isoperimetric

The results show that pupils do not often recognise that different shape can have the same perimeter. Only a few identified the group A-E-C or two of its members. The percentages regarding the two shapes A-E are surprising: 19% (9-10 y. o.) and 12 % (10-11 y. o.). This was because they failed to recognise or use the congruence of the two shapes. A-C were recognised as having the same perimeter by only 9% of the

older pupils. Shape C was matched with L by 9% as having the same area (Tirosch, Stavy, 1999). The higher percentage concerns shapes A-G: the equi-decomposability prevails, as for B-F (18%).

#### 4) Comparing the perimeter of all the shapes

This question provided the opportunity to look over replies to previous questions and find possible mistakes. But in fact pupils found it hard, as they are not used to comparing so many elements. There were no correct replies, and only 28% of the 9-10 years old gave a reply coherent with the comparisons they had made in previous activities.

### Worksheet B

#### 5) Comparing area

*Finding the shape with the largest area*

	H	G	A	E
9 -10 years old	55%	17%	13%	11%
10 -11 years old	60%	20%	46%	57%

*Comparing the areas*

	A-B-I	E-C-F	G-L-H
9 -10 years old	72%	64%	56%
10 -11 years old	66%	83%	69%

The children generally found it easier to compare areas than perimeters, or at least tackled the problem with more appropriate strategies. Most pupils recognised shape H as the largest, but although the question 5) suggests there is only one, they indicated two or more fences although they are not in fact equivalent. The following methods were used by pupils to compare the area of shapes, as recorded in the interview:

- superimposing shapes: “I placed H on top of E, H sticks out more, so it is bigger”
- decomposing: “In G we moved a piece and it became the same as A”
- measuring and comparing the ‘main dimensions’ (the main length or height): “H is longer than A, so it is bigger”, “I only measured the height of E and G, and they are the same”, “A, E and L have the same height and the same width so they are as big as each other”. The pupils used the traditional shape the rectangle and measuring methods.
- measuring and comparing the perimeter: “in the straight parts I measured the sides with a ruler, and the curved sides in my head, then I added up the sides”
- perceptive comparing: “I measured by eye”, “L is the biggest by eye”

#### 6) Equi-extension

We did not predict that determining shapes of the same area should be difficult, at least as far as the simplest pairs such as A-E and C-L are concerned. But in fact there

were cases where pupils failed to recognise that two congruent rectangles, set in a different way on the sheet of paper, have the same extension. Only 18% of 9-10 years old recognised rectangles A-E and only 6% of the 10-11 years old identified pair A-E. So isometric invariance, important property of equi-extension was not recognised.

Visual perception tricked some pupils in the pair C-L; the missing rectangle in L appears to be larger than C. Cutting up pieces allowed pupils to see the equi-extension of pairs B-F and G-A, and also E by transitivity. But none of the 9-10 years old used this method and the 10-11 years old used it only for G. So 20% recognised A and G as equivalent in area, but only 6% A, E and G.

#### 7) Comparing the area of all shapes

There was a higher percentage of correct replies than for the question comparing perimeters, although the percentages were still low (14% and 16%). Older pupils however showed a higher percentage of replies coherent with comparisons given to previous questions: 46% for the 10-11 years old and 30% for the 9-10 years old.

### **Conclusions**

The experimentation clearly confirms the existence of difficulties with the concepts of perimeter and area, widely discussed in teaching research. It yielded useful findings on children's approach and procedures, discussed in our analysis of the results. Rather than definitive conclusions, it furnished us with ideas for further research. The pupils worked better on the area of a shape than the length of its boundary. In general, the bidimensional aspect predominated over the unidimensional: the fence with the largest area is usually thought to be that with the longest perimeter. But as we remarked above, it could be that the context we chose affected their judgement. In fact the adjectives "bigger" and "smaller" are more familiar to children in the context of sheep and fields than "shorter" and "longer". Sometimes the unidimensional aspect prevails: to compare some areas, some children compare heights, (Montis et al., 2003). The context we selected involves perhaps too many shapes and comparisons, but precisely for this reason it is more meaningful for the children as well as the researcher.

An analysis of the children's work confirmed the importance of working first on the concept of perimeter and area and only subsequently on measuring them. In fact measuring can prevail over the reasoning: for example, some children patiently measured every section of shape H, the most unusual, and added all the numbers together. The habit of using a ruler led them to identify the measurements they took with the lengths. They also used the verb "measure" inappropriately in cases of simple comparison. We also noted that pupils are accustomed to working on single shapes and not on the comparison of geometrical shapes. It would therefore be opportune to carry out work on geometrical transformations, particularly congruence, earlier.

The activities caused difficulties to the pupils in that there was a conflict between their already acquired knowledge and skills in geometry. They did not think of switching register, so there were few 'common sense' observations. But when they looked at area, which they had not previously studied at school, the children approached the problem in a more relevant and adequate manner.

The procedures they used yield important information for designing activities to help them distinguish the two concepts of perimeter and area.

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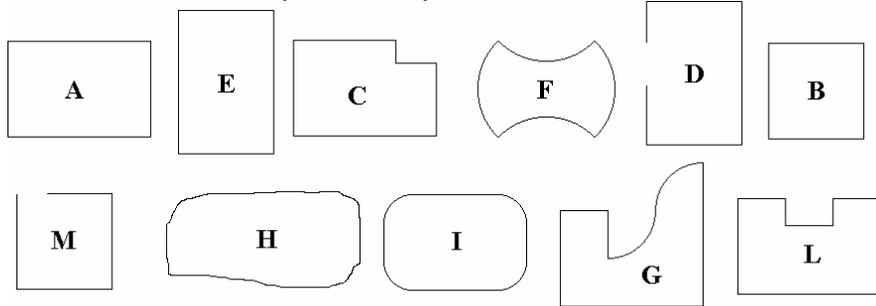
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**WORKSHEET A**

- 1) Two shepherds, Pino and Mario, want to find a safe fence to keep their sheep enclosed.

Which of these fences do you think they should choose?



The best fences are: .....

If you excluded any of the fences, say which ones and write down why you excluded them.....

- 2) Mario has a problem: he needs to use barbed wire for the fence and he wants to buy the smallest possible quantity.

Which fence uses the least barbed wire, A, B, or I? .....

Write them in order here, starting with the smallest:

Which fence uses the least barbed wire E, F or L? .....

Write them in order here, starting with the smallest:

Which fence needs the least barbed wire C, G or H?.....

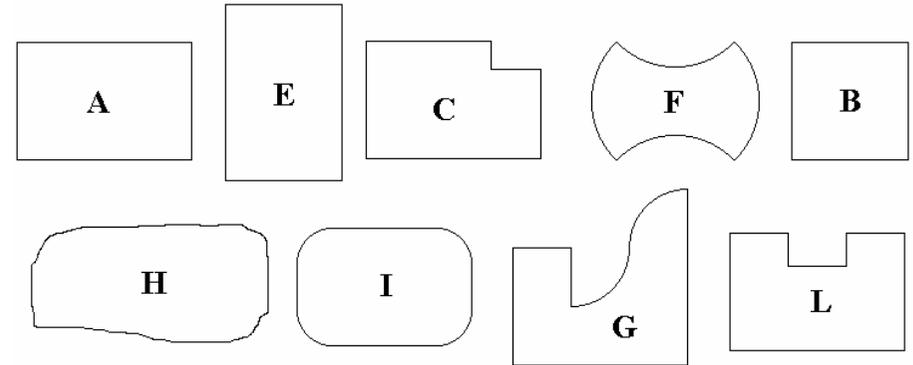
Write them in order here, starting with the smallest:

- 3) Are there any fences, which need the same length of barbed wire? If so, write them here and explain why.....

- 4) Now put all the fences in order, from shortest to longest.....

**WORKSHEET B**

- 5) Pino wants the fence to be as safe as possible, but he wants the sheep to have as much grass to graze as possible. Help him to find a safe fence, which encloses as much grass as possible.



Which fence do you think he should choose? Why? .....

Now look at fences A, B and I and compare them according to how much ground they enclose.

Write them in decreasing order. ....

Do the same for fences E, C and F. ....

Do the same for fences G, L and H. ....

- 6) Are there any fences, which enclose the same area of grass? If so, write them here and explain why. ....

- 7) Now put all the fences in order, from the biggest to the smallest, according to how much ground they enclose.....

# “MEASUREMENT” ON THE SQUARED PAPER<sup>1</sup>

Paola Vighi, *University of Parma, Italy*

**Abstract:** *In this paper we present an activity, experimented in an Italian primary school, focused on drawing ‘diagonal’ and congruent line segments on squared paper. Squared paper is a material tool used to mediate geometrical concepts, among other things. It embodies rich mathematical knowledge and in some cases facilitates learning, while in other cases can be an obstacle. After an analysis of the problem on which the activity is based, we discuss results of an experimentation that have implications for teaching. The long-term risks relating to the lack of understanding and conceptualisation of some aspects embodied in squared paper are discussed.*

## Introduction

Squared paper is nowadays an ordinary tool in schools. Indeed it is somewhat surprising to find that it came into use only at the beginning of the Twentieth Century, when manufacturers started to sell it at affordable prices. Brock and Price (1980, p. 366) give a full description of the causes and stages of the gradual adoption of squared paper: “It is clear that until the 1820s and 1830s the graphical method and, in particular, squared paper were *rarely* used. Moreover, until the 1870s ... such methods remained in relatively uncommon use in Great Britain and were not employed by students as part of their elementary scientific and mathematical education”. Only by the end of the first decade of the Twentieth Century, squared paper came into use in mathematical activities. Squared paper is now used in primary schools in Italy for different activities and purposes. In the early years it is used to support drawing activities. The points where the lines cross are used to pick out “stylised” drawings, with consecutive line segments as boundary. Exercise books of squared paper are commonly used in mathematics; the grid help, for example, to keep numbers in columns for additions and subtractions. The squares are useful for drawing geometric shapes, for the perimeters of rectangles, squares and shapes with straight sides, and they are particularly used for area. In the third year of primary school, a ruler replaces the “square” as a unit of linear measure, but the “square unit” is used for measurement of areas. Squared paper is also used for drawing symmetrical shapes. Though the syllabus may not be followed in practice, the maths programme in Italian primary schools explicitly refers to “the use of squared paper”, since, as Speranza writes (1989 a, p. 20), “the idea is extremely fertile, it gives us the basic idea of coordinates”. Squared paper embodies rich mathematical knowledge and can be useful to facilitate learning. But I believe that its full potential as a useful and important instrument in mathematics has

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<sup>1</sup> This work was carried out in the Local Research Unit in Mathematics Education, University of Parma, Italy.

yet to be explored. Nowadays a lot of attention is given to technological tools, but it would be interesting and constructive to investigate more thoroughly how to exploit the humble squared paper.

This paper presents an activity based on the concept of length of a diagonal line segment drawn on squared paper. Line segments of equal lengths will be compared without superimposition or measuring. The research is a preliminary enquiry.

### Aims

The main aim is to find out whether children understand how to draw “diagonal” line segments on squared paper without measuring the length directly, but using the squared paper to measure “indirectly”. We also aim to identify possible difficulties and preconceptions about this.

A more general aim is to study how to pave the way for the concept of gradient of a straight line.

### Theoretical framework

We can consider squared paper as a *cultural artifact*. By *cultural artifacts* we mean “historical products that can be conceptual (e.g. scientific concepts), symbolic forms (for ex. numerical systems) or material (for ex. tools)” (Saxe, 1991). In other words, an artifact is a man-made instrument present in our culture; it is a means of communication. As research shows, “Pointing out and working on some «mathematical facts» that are present and encoded in opportune cultural artifacts can prove to be the keystone to create learning situations having a strong educational impact in school practice. In this way mathematical knowledge can be inserted into a common knowledge, pupils can be stimulated and motivated and led toward a more conscious learning” (Bonotto, 1999).

Traditionally on squared paper the pupils measure only horizontal or vertical line segments. But clearly squared paper can be used for other purposes, apart from counting horizontally or vertically. It can for example be used to work with “diagonal line segments”. By “diagonal line segment” we mean any line segment which is neither horizontal nor vertical compared to the lines of the grid. Of course, for measuring diagonally, Pythagoras’ theorem is necessary, and obviously this is not taught at primary school. Let us, however, investigate how it can be used implicitly as an in-act theorem (Vergnaud, 1990).

See the following line segment:

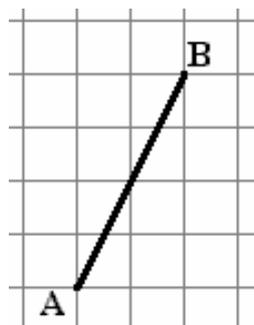


Fig. 1

Even without measuring the length of a segment by a ruler, it can be “identified” as the diagonal of a rectangle. The line segment AB, for example, can be seen as a diagonal line of a rectangle with the horizontal sides measuring “two” and the vertical sides measuring “four”. Consequently any other line segment identified as “movement by two and four sides” will be congruent to AB. We call this strategy “identifying a diagonal line segment”. As research shows, “the idea of using two ‘components’ (horizontal and vertical) to identify a shift is not immediately obvious. Children tend to speak in imprecise terms and may confuse the length of a line with its projection.” (Speranza, 1989 b). In this case the custom of using the squared paper for horizontal or vertical measuring constitutes an obstacle (Brousseau, 1983) and leads pupils to make mistakes.

It is also important to communicate that it is possible to work on length without measuring, that is to present the concept of length as g-quantity (Marchetti, 2005), as intrinsic property of an object, in order to avoid the “reductive effects of a didactic transposition which converts important mathematical concepts into useless concepts, through algorithms.” (Chamorro, 2001).

We opted to focus on this subject in class through an activity based on the drawing of Fig.1. This is not a simple activity, in fact it is an *a-didactic situation*, in Brousseau’s sense (*Theory of Didactic Situations*). It is a situation where the pupil interacts in an environment (milieu) set up by the teacher for the learning of a particular knowledge: “The pupil knows very well that the problem has been selected to allow him to acquire a new knowledge, but he must also know that this knowledge is wholly justified by the internal logic of the situation and that he can construct it without making recourse to didactics.” (Brousseau, 1986). He acts as researcher and the teacher leaves him to work on his own initiative. An a-didactic situation is one where the pupil tries to find the answer using his own knowledge, which is however inadequate. This means that he has to take decisions, adjust his scheme of knowledge and sometimes retrace his steps to correct and modify his actions. This paper is based essentially on the pupils response to an a-didactic situation of *action* and on the analysis of their behaviour.

### Methodology

In order to investigate, we designed an experimental activity based on drawing congruent diagonal line segments by asking pupils to draw isosceles triangles. We did not allow the pupils to use measurements in the task. The complete experiment was carried out as follows:

Planning and identification of a task which would lead pupils to think about the length of diagonal line segments: we used the concept of isosceles triangle, which has two sides of equal length, in order to work on equal lengths without explicitly naming them.

Experimentation in the classroom: the pupils were asked to draw individually and freehand onto squared paper (squares 0.5 cm) following the teacher’s instructions.

Analysis of results: The pupils’ drawings were studied and classified.

### **The classroom activity**

Sixty pupils 9 – 10 and 10 – 11 took part in the experiment<sup>2</sup>. Both classes had already worked with triangles, including isosceles triangles.

The teacher drew the shape in Fig. 1 on a squared blackboard. The pupils were asked to look carefully and copy the shape onto squared paper.

The next task was: “AB is one of the equal sides of an isosceles triangle which has all its vertices at the intersections of the lines on the paper. Complete the triangle”<sup>3</sup>.

The next task was: “Could you have completed the triangle in a different way? Copy line AB again and try again”.

So the teacher merely asked the question. Each pupil took his own decisions and tried out drawings. At the end the teacher led class discussion.

### **Analysis a priori**

“The a priori analysis of a situation attempts to determine if the situation can be lived as a didactic by the pupil. It is a search for necessary conditions” (Margolinas, 1990).

The first task is for the pupil to copy a line segment drawn on the board by the teacher. The second task is to draw an isosceles triangle, following a precise task: one side is given and the triangle vertices can only be at certain fixed points. Pupils are not told if the triangle has to be isosceles in A or in B. The pupil therefore needs to know what an isosceles triangle is, and use this knowledge opportunely. As mentioned above, we adopted this approach of drawing congruent shapes in order not to mention the lengths of line segments. It is likely that the triangle drawn will have a horizontal or vertical side, and that the pupil will make use of the figural concept (Fischbein, 1993) of an isosceles triangle (Vighi, 2003). In other words, the pupil will probably be influenced by the stereotype of the “roof” or “flag” triangle (Marchini, 2005). But this knowledge is not sufficient to carry out the task, it may even be an obstacle: squared paper can interfere with the mental image and the pupil may have difficulty in reconciling the two aspects. This is precisely the aim of the third task; leading pupils to acquire the skill to draw five different types of triangle: two acute-angled, two obtuse-angled and one right-angled, as shown in Fig. 2 (if we distinguish between inversely and directly congruent triangles there are altogether six possibilities). The pupil may attempt the task by trying out to draw different triangles, and then deciding if and how they are acceptable for the task.

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<sup>2</sup> I wish to thank teachers Tiziana Colla and Sara Ziveri warmly for their co-operation.

<sup>3</sup> We stipulated that the triangle vertices were to be at these intersections after a previous experiment where pupils had used the squares sporadically, drawing their shapes as though the paper had been a plain white sheet.

Recourse to the figural concept of the triangle allows drawing the first two triangles in Fig. 2.

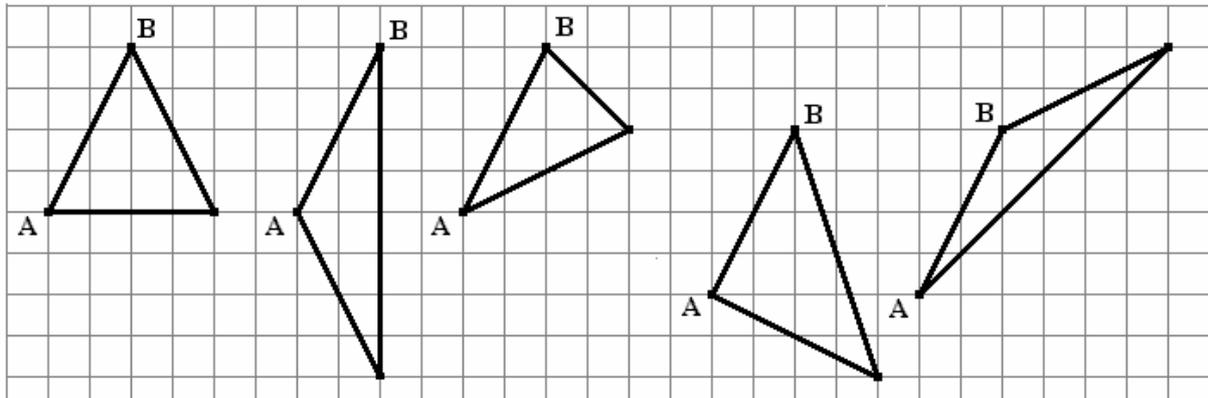


Fig. 2

We predicted that some pupils would stop at this, but that others would look for new strategies to complete the task. It is necessary to take into account the grid of the squared paper as an element that can help carry out the task. In other words, the pupil needs to realise that the strategy of “identifying a diagonal line segment” is the best way of solving the problem. In fact, in order to complete the task, the following observation is fundamental: to move from point A to point B there is a shift of two horizontal and four vertical squares. Moreover, starting from a point P each shift of two squares in one direction and four in the other gives point Q such that PQ is congruent to AB. This is the main learning point of the activity. We predicted two types of behaviour: either pupils ignoring the grid or otherwise finding that it interferes with what they have in mind, making their task more difficult rather than easier. Another strategy could be to place a pencil over AB so that one of the points coincides with A or B and find a second point on the pencil and rotate it so that it meets one of the intersections. If the pupils do not do this spontaneously, in the phase of validation, we present a dynamic model consisting of a short stick which can rotate round one of its ends fixed to a grid.

### Analysis of the results

A preliminary observation on the way the pupils drew the line segments is that when asked to copy a line segment like that in Figure 1 from the blackboard onto squared paper, many pupils mistakenly drew a simple diagonal line segment. Some pupils did not mark the end points of the line segment before they started to draw it.

Observations on the first task:

- Pupils found the task difficult as far as using the intersections of the squares was concerned, even though the points A and B they were given also coincided with the intersections.
- A high percentage (82%) of pupils first drew a triangle with a horizontal base. The figural concept of triangle as “roof” predominates (Marchini, 2005).

### Working Group 7

- Some pupils (15%) drew a right-angle triangle with AB as hypotenuse, thus not carrying out the task correctly and giving emphasis to horizontal and vertical directions (5 pupils in the lower class and 4 in the upper).
- Many pupils (39%) copied the line segment AB with a slope of  $45^\circ$  to horizontal. The pupils called this type of line segment a “perfect diagonal”.

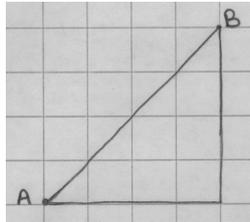


Fig. 3

#### Observations on the second task:

- Some pupils at first said they were sure there was no other possibility, but they thought again when they saw their peers drawing.
- Some did not succeed in overcoming the usual stereotype of the “roof” and drew other triangles congruent to the previous ones or only isosceles ones, with AB a different length compared to the length stipulated (43%).
- Other pupils (2 in the lower class and 9 in the upper one) opted for a “flag-style” triangle.
- Others gave up the idea of the squared paper and emphasised the quality of “isosceles”.
- Of the triangles shown in fig 2, the first was the most frequently drawn. The stereotype of the acute angle triangle prevented them from seeing the obtuse angle triangle, which moreover is “turned round”.
- Some pupils made a typical mistake mentioned above caused by their way of ‘measuring’ line AB. They counted the squares crossed by the diagonal line from top to bottom up to four and applied the same number to the horizontal. Sometimes they actually remarked ‘Oh, it’s an equilateral triangle!’ even though the shape was clearly not equilateral. Checking with a ruler or the teacher question, (“Are you sure? It is an isosceles triangle?”) showed them their mistake.
- Only two pupils in each class, after much thought and many attempts to carry out the task by exploiting the squared paper, found a way to “identify a diagonal line segment”. When they did, their faces lit up and they produced their drawings. The pupils have to anticipate the idea of a diagonal line as the hypotenuse of a right-angled triangle, which is not yet drawn. This activity is at the third, or representative level, according to Van Hiele (1986).

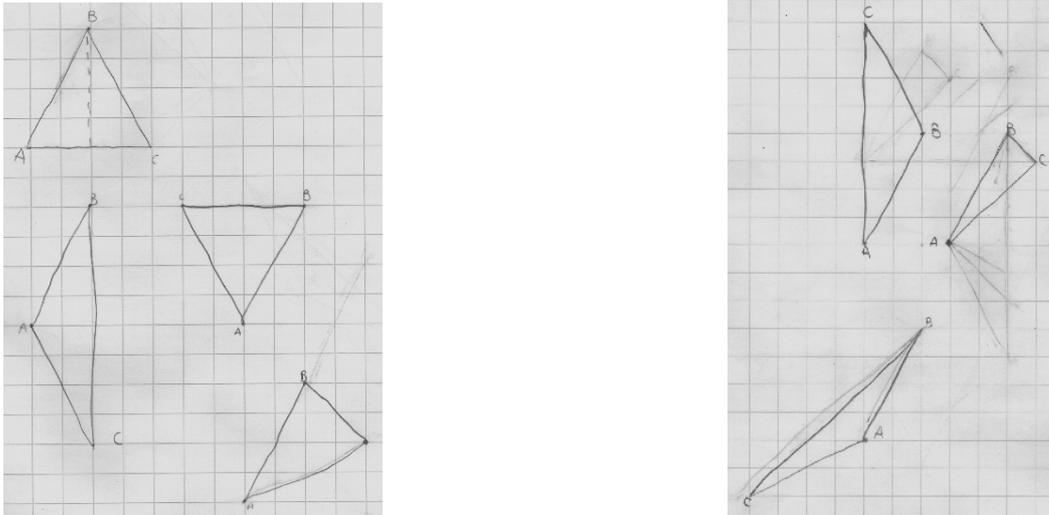


Fig. 4

The class activity was followed by a class discussion, led by the teacher. The pupils who had made the “discovery” drew it on the board and explained it to the others. Pupils debated and made argumentations about the solutions and tested solutions to find the ‘correct’ drawings. Naturally the discussion was concluded and institutionalised by the teacher. Exercises to test learning of the new concept were also used in subsequent lessons.

### Conclusions

The problem and the way the pupils approached it provide rich food for thought. The situation is a good *action's situation* in that the pupils assessed their own work as well as having the opportunity to improve it. But only about half of the pupils did this. The others were unable to overcome the stereotype of the “roof” or “flag” triangle and look for a new way of solving the problem. Learning took place in some cases. So the main aim was achieved: several pupils discovered independently the strategy of “identifying a diagonal line segment”. The others, although they appeared to understand it were often not able to apply it immediately. Further work is clearly necessary.

The artifact “squared paper” can lead the pupil to modify his actions and knowledge. But it can also constitute an obstacle. The requirement to draw a line segment with the end points at the grid intersections proved to be a difficult constraint. The traditional uses of squared paper typically gives rise to “taxi-geometry”, with horizontal or vertical movements. I feel however it would also be useful to work on diagonal lines and their lengths. It is a way of realising that in a right-triangle the hypotenuse is longer than each cathetus. Pythagoras’ theorem supplies a numerical explanation for this, while our activity involves the geometric aspect.

The idea, discussed by Speranza, that the length of a diagonal line is the same as its orthogonal projection is often mentioned in the literature. Our experiments showed its prevalence. The mistake can last beyond school years if it is not corrected. Even uni-

versity students are not immune: as can be seen in Fig. 5 showing two drawings by first year university science students.



Fig. 5

The triangle drawing activity allows teachers to point out this preconception and eradicate it before it becomes rooted in pupil's mind.

In conclusion, the low awareness among teachers of the mathematical facts embodied in squared paper means that the mathematical potential of the artifact may not be fully exploited. But I completely agree with Laisant who calls "squared paper, a marvellous instruction which ought to be in the hands of everyone who works in mathematics, from the kindergarten to the university" (Laisant, 1904, p. 23, quoted in Brock and Price, 1980).

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