

WORKING GROUP 11

Different theoretical perspectives and approaches in research in mathematics education

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DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN RESEARCH IN MATHEMATICS EDUCATION

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The program committee assigned a very wide theme to this working group: different theoretical perspectives and approaches to research in mathematics education. In order to keep the work of the group focused and coherent, we published a somewhat narrower call for papers and, before the conference, decided, on the basis of the papers accepted, to concentrate discussion on research paradigms and/or theories within the context of their effect on empirical research. Specifically, we encouraged the working group participants to concentrate on one or more of the following:

1. The influence of different theories on data analysis by:
 - a) considering a given set of data or phenomena through different theoretical lenses and analyze the resulting differences;
 - b) analyzing the interactions of two or more theories as they are applied to the same empirical research study.
2. The relationship between theory and empirical research by:
 - a) analyzing how a specific research paradigm influences empirical research and,
 - b) exemplifying how empirical studies contribute to the development and evolution of theories;
3. The relationship between research and practice by analyzing how research influences practice and vice versa.

The over-riding theme during the group discussions turned out to be the need for a convergence in research, whether or not such convergence was desirable and possible, and, if so, how it may be achieved. In its research stance, mathematics education is multi-disciplinary, in the sense that researchers from different research communities - psychology, sociology, anthropology, mathematics, linguistics, and epistemology - contribute to it. It is also multi-disciplinary in the sense that though the theoretical frameworks built and used by the community of mathematics education researchers are strongly influenced by theoretical constructions and approaches initially developed outside the field, they progressively become genuine constructions of mathematics education.

As a consequence, it is not easy for researchers in mathematics education, even if they restrict themselves to the learning and teaching processes in mathematics, to delimit the pertinent objects for their research after taking into account the diversity of the determinants for these processes. Choices at this level result also from theoretical choices, and from the basic principles underlying the researcher's theoretical positions.

Beyond diversity emerging from multi-disciplinarity, there is also a more intrinsic diversity linked to the diversity of educational cultures, and to the diversity of the institutional characteristics of the development of the field of mathematics education in different countries or global areas. This diversity is both a source of richness for the field – it helps us to question what we often tend to consider as the normal or only way of thinking about or acting upon educational systems – and a source of fragility for research if we don't make specific efforts to counterbalance the difficulty that stem from communication. This is all the more so since the theoretical explosion we see today, the inflation of terms and notions, goes beyond what can be seen as a logical consequence of the sensitiveness of mathematics education to cultural differences.

Although there was a general, if cautious, agreement that convergence in research would be beneficial, the view that diversity implies richness, and should therefore be maintained, was also expressed. Indeed, Cestari, Daland, Eriksen & Jaworski¹ implicitly contributed to this view by presenting a developmental research paradigm. However, it was agreed that to be too general could run the risk of losing the specificity of mathematics education, including the requirement that research in mathematics education should deal in an essential way with mathematics.

The question thus arose whether or not there is a default research style or even a "mathematics education research paradigm" that can identify research in mathematics education. Kaldrimidou and Tzekaki gave hints of what we, as a community, may need to think about in generating such a paradigm and developing an all-embracing theory. It was a difficult conception to consider and no consensus was drawn, partly because of general problems of communication, linguistically, methodologically, and philosophically.

For example, research paradigms emphasized on one hand the social context and institutional practice (Bosch, Chevallard & Gascón) and on the other cognition (Poynter & Tall), but the two positions hardly converged. These two presentations soundly illustrated the degree within which the basic principles underlying a theoretical position shape, what we consider to be, deserving research agendas in mathematics education. From the perspective of Bosch et al., a basic assumption is that the key to understanding the teaching and learning processes in mathematics lies in institutional practices; the mathematical thinking of individuals is tightly shaped

¹ In this overview, we will frequently refer to the contributed papers that follow the overview. Such reference will be made simply by author names.

by these. What is known and how it is known is, in a sense, a by-product of these practices — a learner's knowledge reflects what it is the institutional practices allows them to know and learn. Thus, investigating and establishing theories in the cognitive development of individuals is of minor interest for research that wants to understand the life of mathematics in educational systems.

Poynter and Tall, on the other hand, placed an emphasis on the cognitive growth of individuals, in an attempt to develop theory derived from the way in which individuals engage in mathematical activity. Though Tall and Chevallard agree that things, which look complex, may have a pattern that suggests that theory may be developed, they do not look for explanations in the same way. However, they agree on the importance of the mathematical component within their analysis, but once more they diverge because they are not led by the same intention.

Most researchers don't adopt such radical positions. The most prevalent practice is that of cross-breeding theories in varying degrees. Such cross-breeding often involves theories that are not exactly of the same nature and do not possess the same detail. This makes it possible to see them as either closely related or simply complimenting each other. Within then the working group several examples illustrated this feature. Some of the contributions suggested how context — including social context — and cognition might be brought to interact more closely. Bloch's work, for example, introduces semiotics into the theory of didactic situations. The integrated use of theories associated with cognitive and social perspectives was demonstrated by Bikner-Ahsbabs, whose contribution suggested how this might be done in one area of study. Kidron's contribution carries the implication that the relationship between the strengths associated with theory derived from a social context and theory derived from a cognitive one may be mediated by a theory outlining cognitive construction for abstractions in context, whilst Arzarello and Olivero indicated how a combination of theories on a larger scale could possibly work. Particular frameworks are most clearly seen in approaches to data collection and analysis but a comparative analysis of data that emphasizes different disciplinary frameworks can be illuminating (Lenfant, Roditi & Artigue; Leron).

A further difficulty in comparing, connecting or even unifying theories is presented by the fact that there exist different levels of theories. Researchers use theoretical frameworks as paradigms, perspectives, background theories, foreground theories, empirically grounded theories or local theories, to mention a few. Often, the theoretical level on which a researcher operates is implicit rather than explicit. Nevertheless, in this overview, we refer to all of the above by the collective term "theories".

Finally, even researchers who are quite explicit about the theoretical frameworks they use, are usually not explicit about, and can even be unaware of the assumptions underlying their theoretical approach. One possible exception to this lack of explicitness is the contribution by Wilhelmi, Godino & Lacaste. The approaches

discussed above, namely whether knowledge is constructed individually or socially, is one important example for underlying and often unquestioned paradigms. Other underlying assumptions concern ontological or epistemological questions such as the nature of mathematical objects, or how we can perceive the world by means of empirical research. If underlying assumptions are unclear, or even contradictory, there can be little hope of comparing theories, and even less for integrating them.

On the unifying side, all working group participants appeared to aim to make a difference in the quality of learning as a result of their research. This difference was explicit within the several papers that constructed theory from practice (Cestari & al.; Ferrara, Robutti & Sabena; Assude, Paquelier & Sackur) and the way in which theory could be transformed into practical use (Ejersbo & Leron).

The group-work provided an opportunity to examine ways in which theories that were new to individuals interacted with those that were known. It was an opportunity to restructure personal opinion. The meeting thus provided opportunities to become aware of and compare theoretical standpoints.

In conclusion, the central term that emerged from the working group was *networking*. The overall conclusion was that because of the reasons cited above, there was no expectation that theories would be integrated into a “grand unified theory” in the near future. In fact, even in such a long established science as physics, the desire to integrate physical theories dealing with forces at different orders of magnitude have met, so far, with only partial success. Therefore, though we should maintain high hopes for future integration, we should also be realistic. If we can develop and maintain a certain degree of networking between some of the advocates of the different theoretical stances that are currently evident within mathematics education, this will constitute an important step on the path towards establishing mathematics education as a scientific discipline.

The idea of networking theories thus appears as more realistic than integration. On the other hand, as a research community, we need to be aware that discussion between researchers from different research communities is insufficient to achieve networking. Collaboration between teams using different theories with different underlying assumptions is called for in order to identify the issues and the questions. Such collaboration could take the form of separately analyzing the same data and then meeting to consider and reflect upon each other’s analysis. The project presented by Cerulli, Pedemonte and Robotti is a start in this direction. It aims at building an integrated frame for research and design on technology-enhanced learning. While this may be too ambitious a goal for the present, the project strategy is interesting. Beyond finding tentative integrative lines by reading and analysing research work, each of the six teams involved in the project will analyse a piece of software produced by another team and build an experiment around it, relying on the team’s own theoretical frames, thus allowing later comparison with the analysis and experiments built by the team having produced the software.

In order to promote the networking of theories, we suggest two foci of discussion for the theory working group at the next CERME conference. First, it would be useful to make explicit the level at which a theory operates. This might be helpful in assessing the possibility of comparing, networking or integrating theories. Second, in any attempt to network theories, it is crucial to have an awareness of the underlying assumptions of each theory. Only on the basis of such awareness, can a discussion on the possible coherence of underlying assumptions begin to take place so that a common language supporting such networking can be developed. We therefore recommend that a second aim of the theory working group at the next CERME conference would be to work in teams with the objective of identifying and making explicit the underlying assumptions of some current theories. Finally, and possibly more importantly, we reiterate that collaborative work between teams using different theories is necessary for substantial progress towards networking theories, not only during but also in between conferences.

THEORETICAL ISSUES IN RESEARCH OF MATHEMATICS EDUCATION: SOME CONSIDERATIONS

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Abstract: *In this paper, we use two key readings to demonstrate the importance of describing clearly the terms and the models presented in research of Mathematics Education: the term “conception” and the model “norms”. Both examples were chosen to reveal the specificity and the complexity of mathematics, mathematical knowledge and the mathematics classroom interplay. Based on the researchers’ explanations, we attempt to raise some questions that the presentation of these terms and models puts forward.*

Keywords: research terms, research models, conceptions, norms.

Introduction

Theory or theorizing is the essential product of research activity. The great development of this activity in the field of Mathematics Education has led to an important production of terms, theoretical frameworks, models and methodological tools. Thus, the need of convergence of the different theoretical perspectives /approaches in the research is recently raised in the community of M.E.

In 1996, Serpinska and Lerman in their article “*Epistemologies of Mathematics and of Mathematics Education*” attempted to present the various theories that exist or are under development in the scientific field of Mathematics Education (Sierpinska & Lerman, 1996). Moreover, in 1998, an ICMI Study pinpointed a number of important theoretical questions concerning the aims, the objects, the specific theoretical questions and the research results in Mathematics Education (Sierpinska & Kilpatrick, 1998). A similar attempt was made in the Research Forum of PME26 “*Abstraction: Theories about the emergence of knowledge structures*”, although it was more focused on the “*description of processes during which new mathematical knowledge structures emerge*” (Dreyfus & Gray, 2002).

The issue of developing theoretical frameworks is proving exigent and difficult in the field of Mathematics Education, because the phenomena under study can be approached at different levels and from different perspectives. Even limiting the focus of our interest on the teaching and learning of Mathematics, inside and outside the school system, at different cognitive levels, it can be seen that:

1. Research questions can be categorized in many different ways: according to the mathematical content, the cognitive level and the object of study or the aim of the research (theoretical or practical).
2. The existing research or theoretical knowledge comes from inside or outside the Mathematics Education (mathematics, history, epistemology, psychology, sociology, pedagogy, etc.).
3. Research in Mathematics Education uses theoretical terms, frameworks, models of analysis and methodologies borrowed from other scientific fields (i.e. the psychology of Mathematics uses the tools of psychological research, the social interactionism the tools of sociological research, etc).

Studying research outcomes, as far as the theoretical terms and the theoretical tools are concerned, we detect at least two important phenomena: the use of a single term with different meanings and the construction of similar models that researchers utilize in parallel. Although it could be argued that these phenomena are expected, because of the complexity of the questions about the teaching/learning of Mathematics, it is apparent that the use of ill-defined or polysemic terms and models is problematic in the research in Mathematics Education. In this paper, we present two examples to demonstrate the importance of describing clearly the terms and the models used in a piece of research for the validity of its results. These examples are as follows:

(I) With regard to the existence of terms with different/multiple meanings, we examine the term “conceptions” as found in the literature. This term can be found in a large number of studies about “conceptualization” and the learning theories of Mathematics.

(II) With regard to the construction of similar models in attempting to analyze classroom phenomena, we focus on the “socio /mathematics norms”, a notion found at the heart of the research concerning the study of the mathematics classroom.

I. The term “Conceptions”

The term “conceptions” is used in the relevant literature (Thompson, 1984) with various/multiple meanings, at least since 1984. In the following, an attempt is made to organize the meaning attached to the term by various researchers.

- A) In a first use, the term “conception” is used to refer to the different/multiple approaches (expressions and meanings) of a mathematical concept. Thus, in a number of research papers, the term “conceptions” is employed to discriminate between different aspects of a mathematical concept, according to its definition and the context in which it appears.

The following citation gives an example of this use that can be traced in Selden & Selden’s article (1992) “*Research Perspectives on Conceptions of Function*”. Analyzing the concept of “function” according to the domain in which it occurs (set theory, calculus, mathematical structures, vector spaces) and its role in a context (description of relationships, operation on a structure, transformation, object of a set),

the authors claim that there are several conceptions about “function” since “*a function can be regarded as a set of ordered pairs, a correspondence, a graph, a dependent variable, an action, a process or an object (entity)*” (p.4).

Since the word “conception” is related to the definition of the function and the context in which it appears, it could be argued that this use of the term expresses the differences in the mathematical nature of the concept and/or the concept category in which it belongs (set, correspondence, relation, etc).

- B) In a different approach, the term is employed to identify the difference between the meaning that students construct about a mathematical concept and the concept itself. It is related to the individual’s knowledge, usually erroneous or limited, about the concept. In this case, derivative terms like “misconceptions” are also used.

The work by Breidenbach et al (1992) can be cited as an example of this kind of use of the term “conceptions”. In their article, the authors described as “object conceptions” (pp. 253-254) the students’ examples of the functions such as “ $F(x)$ =some algebraic or trigonometric expression”. They explain that these conceptions do not “*represent ‘stages in development’ of the function concept, but rather, different ways of thinking about functions*” (p. 253), thus attaching to the term the meaning that the students assign to the concept of “function”. Breidenbach’s research is related to Dubinky & Harel’s (1992) approach for the function concept, which “*adopts, for describing a function conception, the terms pre-function, action, process and object conceptions*” (p. 85). This approach attaches the same meaning to the term.

A similar use of the term “conception” can be traced in the process-object theories, in which Sfard (1992), considering “*the ontological duality of mathematical conceptions ...regarding the formation of such [mathematical] notions as number, set or function*” (p. 59), identifies, in students’ answers, three categories of conceptions for the notion of function: the operational conceptions, the structural conceptions and the pseudostructural conceptions.

In the same context, a different meaning of the term “conception” can be recognized. In the theory of conceptual fields, Vergnaud (1991, 1994) considers a “conception” as the equivalent to the individual’s mental construction of a concept.

Balacheff & Gaudin (2002) give a formal definition of the “conception” as the quadruplet (P, R, L, Σ) , in which P is a set of problems, R a set of operators, L a representation system and Σ a control system. In this approach, they consider knowing “*as a set of conceptions, which refer to the same content of reference and a concept as the set of all knowing sharing the same content*” (p. 18). Following their analysis, knowing can be considered as the projection of a concept in the individual’s mind, the equivalent to the individual’s mental construction of a concept; thus, “conception” that is “*the instantiation of the knowing of a subject by a situation*” (p.

19), does not characterize only the subject's knowledge but also the subject/milieu system in a situation.

In the aforementioned citations, the researchers use the term "conception" in their approaches, referring to the individual's knowledge. Dubinsky and Sfard employ it to identify differences in the mathematical nature and the role of the corresponding intellectual development, related either to a specific mathematical concept or to all the mathematical notions (conceptualization). Vergnaud, on the other hand, employs the term in order to pinpoint the difference and the partiality of the individual knowledge concerning a scientific notion versus the scientific concept itself, while Balacheff and Gaudin employ it to identify the individual knowledge in a specific situation.

- C) A different way of using the term "conception" is also related to the individual's knowledge, but expresses the differences in the ways a person conceives the epistemological and structural elements of Mathematics.

An example of this use can be traced in Sierpinska (1992), where the author calls "conception" the individual's partial or erroneous knowledge about a concept, such as "*conceptions of function*" (p. 46, 49), "*conception of coordinates*" (p. 51), "*conception of a graph of function*" (p. 52), "*conception of variable*" (p. 55). But, she also identifies, in the students' ideas, the "*conception of a definition*". As she explains in the same article, for the students, a "*definition is a description of an object otherwise known by senses and insight. The definition does not determine the object; rather the object determines the definition...*" (p. 47). Thus, she links the term "*conception of a definition*" to an erroneous or limited way of understanding not of a mathematical concept but of the epistemological and structural elements of Mathematics.

- D) Finally, in older papers, the term "conceptions" was employed as a synonym of the word "ideas" or "beliefs", describing general convictions of students and teachers about Mathematics and its learning.

For example, in 1992, Thompson presented the claim that "*students learn better listening to the teacher's explanation and answering to their questions*" (p. 111) as a "conception" about the learning of Mathematics, while Borasi (1990) wrote about "*the students' conceptions for the nature of Mathematics and their expectations,...since their beliefs are deeply rooted,...to change conceptions*" (pp. 175-176). In this use, a "conception" is a way of understanding or learning Mathematics.

Summarizing, it can be argued that, a systematic study of the literature reveals that researchers use the term "conceptions" referring to different and sometimes opposite elements:

- the mathematical concepts, but also the epistemological elements or more general ideas about the nature of Mathematics;

- specific concepts but also all mathematical concepts;
- the content of Mathematics, but also the mathematical knowledge;
- the individual knowledge, but also the knowledge shared between groups or individuals.

These multiple uses and meanings of the term raise several questions about the nature of “conceptions”:

- Are they elements of the conceptual knowledge and/or of the process of conceptualization (individuals’ mental constructs) or tools in the analysis of learning (researchers’ constructs)?
- Are they connected to specific mathematical concepts (like function, number etc) or can describe other elements of Mathematics (definitions, fields, roles)?
- Which of the above mentioned meanings is ascribed to the development of other terms expressing an individual’s inadequate or restricted or partial knowledge, like “concept-image/ concept-definition” (Vinner 1992), “embodied world/ proceptual world/ formal world” (Tall, 2004)?

It has already been argued that this polysemy of the term “conceptions” reflects the complexity of Mathematics and of the mathematical knowledge. It expresses and is related to the multiple approaches and aspects which a mathematical concept can have, depending on the aim of its use, the context in which it’s applied and the ways of its construction and evolution. The teaching and learning of Mathematics carries the same complexity (multiple meanings, aspects and approaches). Thus, this polysemy could be possibly explained by the existence of multiple underlying theories about mathematical learning and diverging epistemological perspectives about what constitutes a mathematical knowledge. But the question still remains: why does the same term have to be used? It could be supported that it would be enough for a researcher to clarify the meaning of the term. However, we think that simply clarifying the use of the term each time is not profitable in the course of the development of theoretical tools urgently needed in the field of Mathematics Education nowadays.

II. A model of analysis: “Norms”

The development of a model of analysis of didactical phenomena in the mathematics classroom is shown to be a very demanding work. In this section, we try to examine different models, as they are presented in relevant readings. Based on the researchers’ explanations, we attempt to raise some questions that the presentation of these models puts forward.

The models of analysis of the classroom activity attempt to explain the nature of the teaching and learning that takes place in the classroom and to explicate significant aspects of the teaching and learning situation. The assumption that something different, from a didactical point of view, happens in the mathematics classroom led

many researchers to bring to the foreground the specific practices and phenomena that are connected to Mathematics. An important model of such an analysis is based on what it is called “norms”, the “classroom norms”, the “mathematical norms”, the “social norms”, the “sociomathematical norms”, etc.

The following citations present how the authors understand and define the term “norms”. Yackel (2001) explains that “*norm is not an individual but a collective notion. One way to describe norms, in our case, classroom norms, is to describe the expectations and obligations that are constituted in the classroom. ...The understanding that students are expected to explain their solutions is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm*”(p.6).

Analyzing the *social norm*, Cobb (1998) clarifies that “*...(they) include explaining solutions, attempting to make sense of explanations given by the others, indicating understanding or non - understanding, asking clarifying questioning and articulating alternatives when differences in interpretations have become apparent*”. Still, “*These norms, it should be noted, are not specific to Mathematics but apply to any subject matter area*” (p.34). For this reason, the study of the mathematics classroom brought into light the necessity of introducing in social norms the characteristics that are specific to Mathematics. The socio-mathematical norms include “*... what counts as different mathematical solution, as sophisticated mathematical solution, an efficient mathematical solution and an acceptable mathematical solution...The analysis of sociomathematical norms has helped to understand the process by which the teachers...fostered their students' development of what might be called a mathematical disposition*” (p.34).

Sullivan & Mousley (2001), adapting this framework to the specificity of the mathematics classroom, identified two complementary norms of activity. They called the first “mathematical norms”, which refers to “*the principles, generalizations, processes and products which form the basis of the mathematics curriculum*”. The second, named “socio-cultural norms”, is related to the “*usual practices, organizational routines and modes of communication that impact on the approaches to learning teachers choose, the types of responses they value, their views about legitimacy of knowledge produced, the responsibility of individual learners and their acceptance of risk-taking and errors*”.

These approaches of classroom norms and their specialisation to Mathematics show the increasing need to find the relationship between general models and Mathematics (this need turned the social norms to sociomathematical norms). Still, this adaptation gives rise to important questions:

- Who decides for the *legitimacy of knowledge produced, what counts as an acceptable mathematical explanation* etc. in the mathematics classroom? The content (that is, Mathematics), the schoolteacher, the classroom?

- How mathematics norms are shaped? By the school programs, their implementation in the classroom, the way the schoolteacher handles them? And, speaking of curricula, isn't it necessary to examine the *didactical transformation* of Mathematics (content, nature, epistemological characteristics of school mathematics, etc), as it is revealed by Chevallard (1985)?
- If social norms create the essential connection between individuals or groups (reciprocal expectations and obligations, what is expected from the schoolteacher, the students, how they interact), isn't it also indispensable to study the *knowledge produced* from this interaction, which, again, takes us back to the mathematical meanings developed from it?
- Finally, does the effort to bridge these elements using the model of the sociomathematical norms again put the social aspect in the foreground? In other words, isn't *the legitimacy of produced knowledge and what counts as Mathematics*, the result of the interaction in the classroom that gives a significant role to how the teacher handles this knowledge?

Trying to justify this last question we present an example used by Yackel (2001). In this episode, a teacher intervened in a student's solution because s/he decided that the other students would not understand it and s/he also wanted to cover future instructive needs. The whole course was oriented to the significance of the (mathematical) "*explanation*" that includes- according to the author - "*explicit and implicit negotiations*" in the classroom, as "*the meaning of acceptable mathematical explanation is not something that can be outlined in advance for students to 'apply'. Instead, it is formed in and through the interactions of the participants in the classroom*" (p. 6). But didn't this teacher's intervention destroy the mathematical characteristics of the explanation itself?

Examining the above elements (at least on the basis of the available examples and explanations), it can be seen that all these questions concern the mathematical aspect of the studying phenomena. If this is the case, more questions arise:

1. The "sociomathematical norms" concern all the elements of mathematical activity or only some specific procedures (justification, validation, problem solving etc.)?
2. What indications do we have that the organisation of these norms is a regular element of mathematical activity in the classroom? Does the term "norms" have different meaning from practices, habits etc, and if so, what is the difference?
3. Finally, what is the impact of these norms on the mathematical knowledge developed in this way?"

In fact, the main question is whether these approaches can support the identification, description and analysis of didactical phenomena in the mathematics classroom, as new or well adapted models coming from neighbouring sciences.

Trying to counter this requirement, Brousseau (1997) presented a model that is argued to cover the classroom interactions, but is specific to the mathematical knowledge: *“Then a relationship is formed which determinates –explicitly to some extent, but mostly implicitly- what each partner, the teacher and the student will have the responsibilities for managing and in some way or other, be responsible to the other person for. This system of reciprocal obligations resembles a contract. What interests here is the didactical contract”*, Brousseau continues, *“that is the part of this contract which is specific to the ‘content’, the target mathematical knowledge”* (p. 31).

This definition could be seen as very close to that of socio-mathematical norm. However, despite the closeness of the two models, no attempt detecting similarities and differences between them could be traced in the literature. Shouldn't this be necessary for two models concerning the same phenomenon? This would detect their limits and would therefore make possible their productive exploitation for further research.

In a series of studies (Kaldrimidou et al., 2000, Tzekaki et al., 2002), we attempted to analyze teaching and learning phenomena using the model of mathematical and social (even socio-mathematical) norms. Our findings revealed an important interplay between the epistemological organization of the mathematical content and the organization of the mathematics classroom. More specifically, in these studies, which particularly focused on the ways teachers manage the construction of meaning in the mathematics classroom (that is, on the ways they handle the epistemological features of Mathematics and deal with pupils' work and errors) and on the communicative patterns they adopt, we finally detected that the management of the mathematical content often distorts the mathematical meanings and it is dialectically related to the communicative practices employed.

Discussion

Summarizing, our analysis (an analysis that could be applied to other terms or models as well) denotes that all terms or models identified in the literature intend to be related to the mathematics knowledge or the mathematics classroom, but this relationship needs further elaboration. Sometimes, the interplay between individual and social, as well as between interaction and management of meanings is missing. Moreover, the question about their local or global character requires more clarification.

In particular, using the term “conception” as a key reading, we presented the polysemy of this word in the literature, arguing that this is connected to the different epistemological perspectives about what is Mathematics and also about what constitutes a mathematical knowledge. Similarly, analyzing the models of “norms”, we argued that all the presented approaches attempt to develop a framework specific to the mathematics classroom. However, the “models” are not clearly defined but are simply described and neither delimit common or varying aspects nor clarify which

part of what is happening in the mathematics classroom they refer to. Both examples were chosen to reveal the specificity and the complexity of mathematics, mathematical knowledge and the mathematics classroom interplay.

Research in Mathematics Education is aware of this complexity and that is why it develops multiple and different tools to deal with it. There is no reason to support the convergence of the different theoretical approaches, because the phenomena under study are exceptionally compound and admit different opinions and different analysis. However, it seems that the time has come for a systematic debate on some presuppositions. The range of the unanswered questions shows that the attempt to analyze, interpret and theorize the learning and teaching of Mathematics requires at least systems of knowledge, which are:

- clearly adapted to the specificity of mathematical knowledge, thus putting the limits between Mathematics Education and other sciences;
- more systematically organized in bodies with well defined terms and relevant models;
- carefully tested and evaluated with respect to their implications for the classroom reality.

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SCIENCE OR MAGIC? THE USE OF MODELS AND THEORIES IN DIDACTICS OF MATHEMATICS

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Abstract: *The struggle to eliminate “magic mentality” has affected the development of all scientific disciplines. This process of “de-magification” has been sustained in the use of models created by every discipline. In this sense, any scientific approach in didactics of mathematics uses –more or less implicitly– a general model of mathematical activity and specific models of the different mathematical contents that are taught and learnt at school. Here we summarise the models proposed by the Anthropological Theory of Didactics and the minimal empirical unity of analysis required to use them. The scope of this approach is illustrated through a single example about limits and continuity of functions at secondary school in contrast with the analysis proposed about continuity in terms of “embodiment cognition”.*

Keywords: Praxeologies, didactic transposition, didactics of mathematics, epistemological models, limits of functions, continuity, Anthropological Theory of Didactics, Theory of Didactic Situations, Embodied Cognition.

1. The Magician and the Scientist

In his presentation to the International Scientific Conference in Rome in 2002, Umberto Eco talked about “The Perception of Science by Public Opinion and the Media”. The Italian semiologist stated that, even if we believe ourselves to be living in the Age of Reason mastered by science, we are in fact submitted to the *magic mentality* that always re-emerges from its ashes and that is supported by the need of the immediate satisfaction of our wishes.

“What was magic, what has it been for centuries and what is it still today, even if under a false appearance? The presumption that we can go directly from a cause to an effect by means of a short-circuit, without completing the intermediate steps. For example, you stick a pin in the doll of an enemy and get his/her death; you pronounce a formula and are all of a sudden able to convert iron into gold; you call the angels and send a message through them. Magic ignores the long chain of causes and effects and, especially, does not bother to find out, trial after trial, if there is any relation between cause and effect.” (Eco, 2002, our translation).

An essential difference between the magician and the scientist is that, while the magician dares to give definite answers, the scientist tries hard and humbly to raise

questions that will only accept provisional answers. Whereas scientific theories are tentative models of some aspects of reality, magic expects to catch the whole reality to master and submit it. Scientific models are only tools (machines) that mediate between scientists –who cannot act directly– and reality. Magic, on the contrary, claims to act directly on reality through images or representations of it (for instance the doll that represents the enemy).

According to the German sociologist Max Weber, scientific progress can be described as a process of *de-magification* that has been going on for millennia in Western culture (Weber, 1959). This struggle to eliminate the *magic mentality* in the explanation of facts has been present throughout history and has become visible in the periods of emergence and consolidation of all sciences. It is easy to follow the tracks of this struggle at the origins of most of the disciplines: physics, chemistry, biology, medicine, psychology, anthropology, sociology, political science. In all these cases, “de-magification” has been accompanied by the *modelling of ‘a piece of reality’* by means of models that, far from being exact representations, turned out to be “machines” good at producing knowledge about the reality in question.

With regard to *didactics of mathematics*, and given that we are part of the founding generation of this discipline, we are still immersed in the “de-magification” process. It is still usual to find some “magicians” who offer “magic” solutions to the problems of mathematical education. Their proposals come up in terms of *general slogans* that obviously promise immediate, direct and complete solutions. These are always based on common-sense notions that, being easily accepted and shared by teachers, provide the illusion of a photographic representation of the educational system. On the opposite side, any scientific approach to problems related to the teaching and learning of mathematics needs to elaborate (or to adapt) its own specific models, based on its own primitive terms and basic assumptions, about the domain of reality concerned.

2. The Brousseauian revolution in the didactics of mathematics

At its initiation in the seventies, the Theory of Didactic Situations TDS (Brousseau, 1997) was one of the first, it seems, to state the necessity of a specific scientific approach to the problems of teaching and learning mathematics. In this sense, we can say that it performed a Copernican revolution in the field of mathematics education. It proposes a methodology that starts questioning mathematical knowledge as it is implicitly assumed in educational institutions: what is geometry, what is statistics, what are decimal numbers, what is counting, what is algebra, etc. It then proposes *specific epistemological models* of mathematical knowledge –the *situations*– that are to be experimentally tested: a mathematical notion can only be analysed as far as it appears as the solution to a situation. This is the fundamental methodological principle of the TDS: a piece of mathematical knowledge is represented by a “situation” that involves problems that can be solved in an optimal manner using this knowledge.

Thus appears a new general model of mathematics as an alternative to the *conceptualist* ones most commonly used –implicitly or explicitly– in mathematics education. Following the TDS, mathematics is described in terms of *situations* and consists mainly in “dealing with problems” in a wide sense. Teaching and learning mathematics is not considered as teaching and learning mathematical ideas, notions or concepts, but as teaching and learning a *situated human activity* performed in concrete institutions. Moreover, a situation includes the “raison d’être” or *rationale* that gives sense to the performed mathematical activity. And it also contains institutional restrictions that provide and limit the application of the corresponding mathematical knowledge. Therefore the TDS changes the old central questions in mathematics education: “How do students learn mathematics?” and “What can we do to improve their learning?” by more comprehensive ones: “What are the necessary conditions for a situation to implement the specific mathematical knowledge it defines?” and “How can situations be designed and their development managed in a given educational institution?”

Thus the TDS has led to a change in the notions used to study learning and teaching processes, and, what is more, in the particular way of questioning educational reality. It has changed the *problems*, the *models used* and the *system to study*, stating that the study of any didactic phenomena needs to question common epistemological models of mathematics. We have called *Epistemological Programme* the new research paradigm in mathematics education originated by these assumptions of the TDS that situates the modelling of mathematical activity in the core of the study of any didactic phenomena (see Gascón, 1998 and 2003).

3. The Anthropological Theory of Didactics

Within the Epistemological Programme, it was soon made clear that mathematical activities performed at school could not be adequately interpreted without taking into account phenomena related to the *reconstruction* of mathematics in educational institutions. We thus need to go to the place where these phenomena start, that is, the institutions of production of mathematical knowledge. This is the first contribution of the *theory of didactic transposition* (Chevallard, 1985). If we want to understand (and thus to model in an appropriate manner) what kind of mathematical activity is done at school, we need to know the other kinds of mathematical activities that motivate and justify the teaching and learning of the former. And we also need to know the way these other activities are interpreted in the different institutions. Thus didactic phenomena cannot be separated from phenomena related to the *production* and the *use* of mathematics. Mathematical activities done at school are then integrated to the broader domain of the study of *institutionalised mathematical practices*. The domain of didactics goes beyond educational institutions to all those that embrace any kind of handling of mathematical knowledge. It can be said that the didactics of mathematics –as it is now considered by the Epistemological Programme– studies *mathematical cognition* in the sense of the *conditions that make the production and development of mathematical knowledge possible in social institutions* (Chevallard, 1992).

3.1. The minimal unity of analysis of didactic phenomena

The Epistemological Approach considers that any didactic problem contains some mathematical activities that are being produced, taught, learnt, and practised. Even if these mathematical activities take place in a concrete institution (generally an educational one), their form of existence and their evolution depend mainly on educational constraints related to the process of *didactic transposition*. This process, first pointed out by Chevallard (1985), acts on the necessary changes a body of knowledge and its uses have to receive in order to be able to be learnt at school. It introduces a distinction between: (1) “original” or “*scholarly*” (in an ironic sense) mathematical knowledge as it is produced by mathematicians or other producers; (2) knowledge “*to be taught*” officially prescribed by the curriculum; (3) knowledge as it is *actually taught* by teachers in their classrooms and (4) knowledge as it is *actually learnt* by students. Figure 1 illustrates the various steps that compose the didactic transposition. It also includes the “*reference*” mathematical knowledge that constitutes the basic theoretical model for the researcher (Bosch and Gascón, in press) and is elaborated from the empirical data of the three corresponding institutions: the *mathematical community*, the *educational system* and the *classroom*.

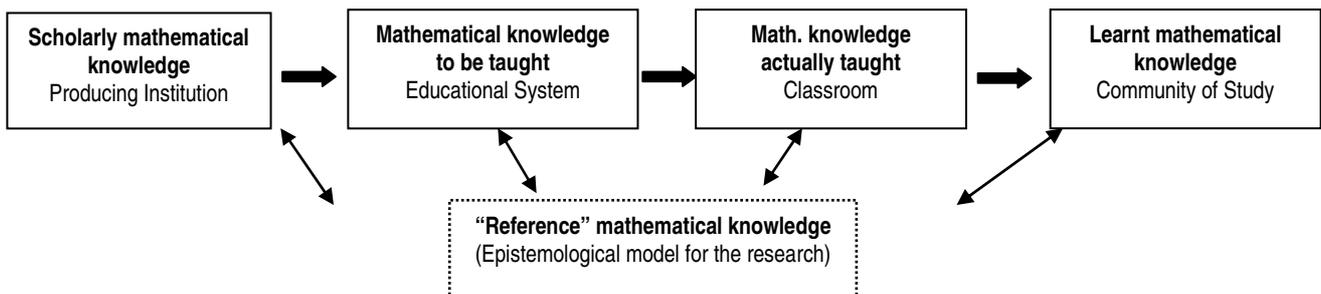


Figure 1. The process of didactic transposition

To take into account the process of didactic transposition means that the study of any didactic problem needs to adopt a particular standpoint (model) on the involved mathematical practices. For instance, what is this “content” we are considering? What is it for? Is it something existing in “scholarly mathematics”? In what way? In what practices? How has it become a piece of knowledge to be taught? In what school mathematical practices is it (or could it be) included? What kind of restrictions does it impose on the development of students’ and teachers’ practices? Etc.

The process of didactic transposition highlights the *institutional relativity of knowledge* and situates didactic problems at an institutional level, beyond individual characteristics of the institutions’ subjects. Its main consequence is that the *minimal unity of analysis* of any didactic problem cannot be limited to the consideration of how students learn (and teachers teach) mathematics. It must include all the steps of the process of didactic transposition, including data coming from each and every one of the involved institutions as an empirical basis. In this sense we can say that phenomena of didactic transposition are at the very core of any didactic problem.

3.2. Modelling mathematics in terms of praxeologies

The Anthropological Theory of Didactics, ATD, (Chevallard, 1997, 1999; Chevallard *et al.*, 1997) emerged as a natural consequence of the development of the theory of didactic transposition (Chevallard, 1985 and 1992). It states that mathematical activity must be interpreted (that is, modelled) as a human activity among others, instead of regarding it only as the construction of a system of concepts, the use of a language and/or a cognitive process (in the sense of cognitive science). The ATD takes mathematical activity institutionally conceived as its primary object of research. It thus must explicitly specify what kind of general model is being used to describe mathematical knowledge and mathematical activities, including the production and diffusion of mathematical knowledge.

The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical *praxeologies* whose main components are *types of tasks (or problems)*, *techniques*, *technologies*¹ and *theories*. The most elementary mathematical praxeologies consist of a *practical block* or “know-how” (the *praxis*) integrating types of problems and techniques used to solve them, along with a *theoretical block* or “knowledge” (the *logos*) integrating both the technological and the theoretical discourse used to describe, explain and justify the practical block. Thus any “piece of mathematical knowledge” should be described through the statement of what kind of mathematical problems and techniques are involved and what kind of description and justification is given to this “way of doing”. For instance, in the case of limits and continuity of functions we are considering later (as a knowledge to be taught), and regarding a concrete institution as Spanish secondary school, the practical block includes problems such as the calculation of the limit of elementary functions at a given point and at infinity through different techniques based on algebraic transformations of the functions expression. The theoretical block accompanying this practice contains some definitions, properties and general statements about limits, continuity and algebraic transformations on functional expressions.

Problems constitute the origin, the motor, of the process of producing mathematical praxeologies. However, doing mathematics does not only consist in solving problems. The resolution of a problematic question always produces much more than a single solution. It produces new knowledge (new problems, new techniques, new technologies and theories) and new arrangements of previous knowledge. Praxeologies can thus be used to describe mathematical knowledge as well as mathematical practice. Doing mathematics consists in trying to solve a problematic question using previously available techniques and theoretical elements in order to elaborate new ways of doing, new explanations and new justifications of these ways of doing.

¹ The term “technology” is here used in the sense of discourse (*logos*) about a technique (*technè*).

The use of praxeologies to model mathematical practices can be extended to any kind of human activity, in particular to the process of studying a problem and building up new mathematical praxeologies (for the subjects of the activity) or helping others to do so. These are called *didactic praxeologies* and cover the whole process of study, from the first formulations of a problematic question to the validation and institutionalisation (making public) of the knowledge produced. We are not developing this kind of models here (see Chevallard, 1999; Bosch and Gascón, 2002).

3.3. An example of a specific model of the taught mathematical knowledge

In previous works (Bosch *et al.*, 2004; Barbé *et al.*, in press) we have analysed the taught mathematical knowledge about limits and continuity of functions at Spanish secondary schools. We showed that mathematical praxeologies prescribed by syllabi and made explicit by official textbooks offer two *completely disconnected* mathematical praxeologies: an “algebra of limits” reduced to the calculations of limits of functions at a given point, and a “topology of limits” centred on the problem of the existence of this limit. The absence of a link between both praxeologies hinders the teacher’s interpretation of syllabi about what is the mathematical knowledge *to be taught* concerning the limits and continuity of functions. On the one hand, the “algebra of limits” becomes the practical block of the mathematical praxeology to be taught because it is closer to the set of tasks and techniques that appear in syllabi and textbooks. On the other hand, and due certainly to the “imposition” of a “scholarly” technology (ϵ - δ definition of limit, etc.), the theoretical block remains close to the “topology of limits” praxeology. The result is a hybrid praxeology with a theoretical block that does not really fit with the practical block.

This situation causes two kinds of difficulties, and even contradictions, in the teacher’s practice. The taught mathematical praxeology does not contain the technological elements needed to explain and justify the calculations of limits. Neither does it include a practice that would show the benefits of the theoretical definitions of limits and continuity. Therefore, it is rather impossible for the teacher to “give meaning” to the mathematical praxeologies to be taught, because the rationale of limits of functions (why we need to consider and calculate them) cannot be integrated in the mathematical practice that is actually developed at this level.

Another particular consequence is the difficulty for the teacher to avoid a circular argument about the notion of “function continuous at a point”. In effect, the main technique to determine if a function $f(x)$ is continuous at a given point $x = a$ consists of comparing the limit of $f(x)$ at $x = a$ with the value $f(a)$. However, in the “algebra of limits”, most of the techniques used to calculate the limit of a function $f(x)$ at a point $x = a$ are based on some algebraic transformations on $f(x)$ that lead to an expression of the function where x can be replaced by a (that is, the expression of a function continuous in a neighbourhood of a). So the implicit use of the continuity of some kinds of “elementary functions” appears as an essential tool for the determination of the continuity of a function at a given point.

The model of the “two-sided” or “hybrid” mathematical praxeology about limits and continuity of functions can thus explain some important “distortions” on the teacher’s and students’ practice that are entailed by constraints coming from the first steps of the process of didactic transposition.

4. Embodied concepts VS praxeologies: the case of “continuity of functions”

We will now contrast our results with the detailed analysis presented by Núñez *et al* (1999) in terms of embodiment cognition.

4.1. The “pedagogical” problem according to the Cognitive Science of Mathematics

Núñez *et al.* (1999, pp. 53–60) present a case study about continuity of functions to illustrate the bodily-grounded nature of cognition. The didactic problem that is taken as a starting point can be formulated in the following terms: Why is the teaching and learning of the concept of ‘continuity’ of a function such a difficult task? Is continuity *per se* a difficult concept? What are the cognitive difficulties underlying the understanding of continuity?

The study of this problem starts considering two definitions –or models– of continuity as they are found in textbooks. An informal/intuitive one called the ‘natural continuity’ based upon concepts, ideas and examples provided by ‘the everyday understanding of motion, flow, and wholeness’. And the ‘Cauchy-Weierstrass definition’ that ‘involves radically different cognitive content’ [...] ‘dealing exclusively with static, discrete, and atomistic elements’. Both concepts are of the same nature (in the sense that they are both embodied) but grounded on different and even contradictory cognitive primitives (also embodied in nature).

The ‘pedagogical’ problem initially considered is explained in the following terms:

“Students are introduced to natural continuity using concepts, ideas, and examples which draw on inferential patterns sustained by the natural human conceptual system. Then, they are introduced to another concept –Cauchy-Weierstrass continuity– that rests upon radically different cognitive contents (although not necessarily more complex). These contents draw on different inferential structures and different entailments that conflict with those from the previous idea. The problem is that students are never told that the new definition is actually a completely different human-embodied idea. Worse, they are told that the new definition captures the essence of the old idea, which, by virtue of being ‘intuitive’ and vague, is to be avoided.” (Ibid., p. 55)

The mathematical ‘piece of knowledge’ involved in the considered problem is the concept of continuity of a function, which is taken as *completely isolated from the rest of concepts of calculus*: the concept of function, of real number, of limit of a function at a point, etc. The only considered aspect of this piece of knowledge is its definition: a more intuitive versus a more formal one. There is no mention of problems that could give (or have given) utility to this concept. Neither are the mathematical techniques or ‘ways of doing’ that could be used to approach these

problems taken into consideration. And there is no mention of the propositions or general statements where the notion of continuous function could play a crucial role (for instance, when a function is given as a solution of a functional equation and we need to suppose it continuous).

The definition of a mathematical notion is taken here as the main factor to explain students' difficulties in working with this notion. It reveals that the general epistemological model of mathematics underlying the analysis is close to a 'conceptualist' one: mathematics is a system of concepts and doing (or learning) mathematics consists of building up concepts. This general model gives rise to specific or local models of previously defined mathematical concepts to show its dependence on embodied and social experience. It has the defect (as shown by Schirally and Sinclair, 2003) of considering only one dimension of mathematical practice (defining new objects) and does not provide a description of the dynamics of the construction, that is, the way mathematics is used as a tool to build up new knowledge and, in particular, to formulate and solve new problems.

4.2. *Praxeological analysis of the involved mathematical activity*

Using the epistemological model provided by the ATD (in terms of praxeologies institutionally conceived) we can show that mathematical practices actually developed in secondary schools do not really require a definition of *continuity* (neither 'natural', nor 'formal'). In effect, it is very unusual to find a type of problems which resolution needs to use this notion as a main tool at this level. Certainly students are asked to determine if a given function (or a given type of functions) is continuous at a point and, if not, the kind of discontinuity it has. But these are 'formal' problems, mathematically irrelevant, that do not lead anywhere. They are only proposed to 'justify' the inclusion of the notion of continuity in curricula and to provide some application cases to the computation of limits.

The 'transposition' in the classrooms of a praxeological environment that really integrates the definition of continuity as an essential tool (it being intuitive or formal) would require some kind of problematic questions that are very difficult to set out at this level. We can ask what kind of questions could 'give sense' to the concept of continuity, in the 'praxeological' sense of leading to the production of a new praxeology with its types of problems, techniques to approach them and technological-theoretical environment to explain and justify the delimitation of problems and the use of the techniques. It can also be shown that an answer to this question would require to go beyond the work with functions determined by their algebraic expression (such as solving functional equations, differential ones in particular) and to approach the problem of the construction of the set of real numbers.

In this situation, students' difficulties in the learning of a "piece of knowledge" that is praxeologically 'out of meaning' can be taken as a positive symptom of the educational system, instead of a problem in itself. The permanence of this notion in secondary schools may be explained by the supremacy of the 'scholarly' point of

view about mathematics that implicitly defines (and puts pressure on) what mathematical ‘concepts’ should be learned, even if it is impossible to implement them praxeologically at this level. For this to be possible, we need to find a question that could ‘take sense’ in the mathematical universe of students and which answer would require the building up of a mathematical praxeology that includes the notion of continuous function in its theoretical block as well as in its practical one.

The praxeological analysis also suggests that it is not always meaningful to talk about the teaching or learning of ‘a concept’, or to decide about the inclusion or the exclusion of ‘a concept’ in the curriculum. ‘Concepts’ and ‘definitions’ or ‘notions’ and ‘ideas’ correspond to particular interpretations of mathematics that have an indisputable usefulness in the production of mathematics, in what has been called the ‘scholarly regime of mathematics’ (that is, in mathematical activities developed in particular institutions that are dominant since considered as ‘reference’ ones). But they are not necessarily the best way of approaching didactical problems, for instance the problem of the curriculum of mathematics.

Furthermore, to elucidate didactic phenomena (including cognitive ones) it is essential to take into account empirical facts that arise in the *intermediate institutions* between individuals and scientific communities or between individuals and societies or cultures. It is essential to enlarge the empirical basis of research. The anthropological approach requires taking into consideration –and thus modelling– an empirical system that takes us out of the classroom, out of the educational system and impels us to question mathematical knowledge through the different mathematical practices that exist in our social institutions. This means, in a sense, to consider all the stages of the process of didactic transposition, the minimal unity of analysis of didactic phenomena.

Any research concerning educational problems uses models of the reality under study. In some cases, these models are close to the point of view of educational institutions, which implicitly define what learning and teaching are, what mathematics is, what elementary algebra is, what calculus is and why it is necessary to calculate the limit of a function. In this case institutional models appear as the “natural way of looking” at educational problems. They are rarely clarified, giving the impression that there is no need for specific theoretical approaches in mathematics education. These implicit assumptions, especially when widely shared, appear as the “common-sense vision” of problems and are quite impossible to discuss and contrast. In the anthropological approach here presented, specific theories and models allow researchers to protect themselves against the “common-sense definition” of educational problems, because educational institutions are considered part of the empirical reality we want to know and wish to change. They propose a vision of the educational world that does not intend to photographically represent it nor to obtain magic and global solutions of problems as complex as those related to the production and diffusion of mathematical knowledge.

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RELATING THEORIES TO PRACTICE IN THE TEACHING OF MATHEMATICS

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There is nothing as practical as a good theory. (Richard Skemp, 1989, p. 27.)

Abstract: *This paper reports the coming together of two major goals, the first to build a cognitive theory of mathematical development that has wide application at different stages of development and in different contexts, the second to address a particular practical problem in the classroom. This problem related to the teaching of vectors, which lies at the confluence of mathematics and physics and builds from practical contexts to theoretical mathematics. We seek to generate a coherent theory that is consonant with many aspects from the literature rather than aggregating disparate aspects of different theories. In the practical context we listened to the voices in the classroom, both teachers and students, seeking a practical solution that would make sense to the participants and be of direct value in both teaching and learning.*

Introduction

This paper is a contribution to a discussion on “Different theoretical perspectives in research: From Teaching problems to Research Problems”. Our purpose is to see how the development of a broad cognitive theory and a rich practical problem can be of mutual benefit. The specific problem considered is the teaching of vectors in the context of school physics and mathematics. The broader cognitive theory is the theory of three worlds of mathematics, which begins with the child’s perception and action on the world to carry out thought experiments to develop an increasingly sophisticated conceptual-embodied world, a focus on actions that are symbolised to give a proceptual-symbolic world of arithmetic and algebra and beyond, and a long-term focus on properties that, for some, leads to a formal-axiomatic world of definitions and proof (Tall, 2004). The specific problem is the teaching of vectors in school with its embodiments in physics and mathematics developing into the symbolism of vectors in two dimensions (Watson¹, Spyrou & Tall, 2002). Here we focus on the relationship between the worlds of embodiment and symbolism.

¹ Anna Poynter published under the name Anna Watson before her recent marriage.

The British culture is one of practical approaches to practical problems. The pragmatic solution to teaching vectors is to introduce them in practical situations in physics as forces, journeys, velocities, accelerations, and only later to study the mathematical theory in pure mathematics. The teaching of vectors has not gone well. It has followed the path of many other topics that students find difficult. The initial presentation has been made more and more practical and less and less dependent on mathematical theory. It shares a similar fate to other 'difficult' parts of mathematics, including fractions and algebra.

In the pragmatic culture of Britain, the teachers are professionals. They take their work seriously, work hard with long hours and relatively little time scheduled for analysis and reflection. Our experience (Poynter & Tall, 2005) of interviewing colleagues show that they are aware that students have difficulties, but their awareness relates more to an episodic memory of what didn't work last year rather than a theory that attempts to explain *why* it went wrong and what strategies might be appropriate to make it go right. Where there are problems, the response is to try a new strategy the following year in an attempt to improve matters.

As an example, consider the case of adding two vectors geometrically. The students are told that a vector depends only on its magnitude and direction and not on the point at which the vector starts. Therefore vectors can be shifted around to start at any point and so, to add two vectors, it is simply a matter of moving the second to start at the point where the first one ends, to give a combined journey along the two vectors. All that is necessary is to draw the arrow from the start point of the first vector to the end point of the second to give the third side of the triangle, which is the sum.

The problem is that many students don't seem to be able to cope with these instructions. Some 'forget' to draw the final side of the triangle to represent the result of the sum, others have difficulties when the vectors are in non-standard positions to start with, such as two vectors pointing into the same point, or two vectors that cross. Some find it difficult to cope when two vectors start at the same point and draw the 'result' of the two vectors AB and AC as the third side of the triangle, BC .

Here we have a specific teaching problem that requires a solution. What theories are available to solve it? The science education theory of 'alternative frameworks' (Driver, 1981) suggests that that the students may have their own individual ways of conceptualising the concept of vector. However, it does not offer a theory of *how* to build a new uniform framework for free vector in a mathematical sense. Our goal is to study this problem not only in its own right to be meaningful to students and fellow teachers, but also within the goal of developing a wider theoretical framework.

Some existing theories

The embodied theory of Lakoff and his colleagues offers a viewpoint that encourages us to consider how students *embody* a concept such as vector. However, this theory

takes a high-level view of mathematical concepts to perform a top-down *idea analysis* theorizing how such concepts have their origins in embodiment rather than a global view that integrates the genesis of the mathematical concepts with the actual conceptual development of the child. For instance, *Where Mathematics Comes From* (Lakoff & Núñez, 2000) includes references from mathematics education papers in its bibliography but makes no reference to them anywhere in the main text. We find the notion of ‘idea analysis’ formulated by Lakoff and Núñez to be a valuable technique, but prefer to use an analysis that relates to the cognitive development of the individual. For us, cognitive development builds from perception and action through reflection to higher theoretical conceptions. We use the term ‘embodiment’ first in the colloquial sense that a sophisticated concept may be ‘embodied’ physically (such as fractions represented as part of a physical whole or a vector as a physical transformation) after the manner of Skemp (1971) and later in the sense of conceptual mental embodiment using thought experiments. This sense relates to Bruner’s notions of enactive and iconic modes of operation as distinct from his symbolic mode, which we see in three distinct parts: language which underpins all increasingly sophisticated modes of thought, and the two increasingly sophisticated worlds of perceptual symbolism in arithmetic and algebra and the more advanced logical symbolism of axiomatic mathematics.

Focusing on the development from physical actions to mental conceptions, a relevant approach may be found in the APOS theory of Dubinsky (Dubinsky & MacDonald, 2001). Dubinsky theorizes that mathematical objects are constructed by reflective abstraction in a dialectic sequence **A-P-O-S**, beginning with **Actions** that are perceived as external, interiorised into internal **Processes**, encapsulated as mental **Objects** developing within a coherent mathematical **Schema**. The actions with which the theory begins may be physical or mental and, in the case of vector, we see transformations as actions on physical objects being routinized into thinkable processes and then encapsulated as mathematical objects in the form of free vectors. There is, however, a possible problem. Several papers in the literature show how students may routinize actions as processes but in several cases (including the notion of limit or of function) the further step to an object conception is less easily accomplished (e.g. Cottrill et al 1996, Dubinsky & Harel, 1992). This signals a possible problem in the shift from a procedural action to a conceptual mental object.

We considered Skemp’s (1976) theory of instrumental and relational understanding. It seemed evident that many students were learning instrumentally how to add vectors without any relational understanding. But what is the relational understanding that is necessary and how is it formulated? Likewise the theories of procedural and conceptual knowledge (Hiebert & Lefevre, 1986, Hiebert & Carpenter, 1992) suggest that the students may be learning procedurally and not conceptually. But here again, what is the conceptual structure and how are procedures and concepts related?

It is apparent that students learn based on their own experiences. They meet various practical examples of vectors, including vectors as journeys and vectors as forces. Many theories (e.g. Dienes 1960) suggest that students must experience *variance* in different examples and abstract the essential properties that are common while ignoring incidental properties that occur in some examples but do not generalise. In the case of vector, these incidental properties are coercive and lead to alternative frameworks that are difficult to shift.

We considered other frameworks, for example the framework of intuition and rigour that occurs in Skemp's (1971) distinction between intuitive and reflective thinking or in Fischbein's (1987, 1993) tripartite system of intuitive, algorithmic and formal thinking. Indeed the latter theory is strongly related to our own development of three worlds of mathematics except that the three categories exist as separate aspects, as they did in the first design of the English National Curriculum where Concepts and Skills were put under separate headings.

Our inspiration for putting these elements together in an integrated manner arose from several theories that include both a global development of successive modes of operation (such as Piaget's stage theory or the enactive-iconic-symbolic modes of Bruner) and also a local sequence of concept formation within each of these modes. In particular, the SOLO taxonomy of Biggs and Collis (1982) made a significant step forward involving not only successive development of different modes (*sensori-motor, ikonic, concrete-symbolic, formal and post-formal*) but also local cycles of concept formation within each mode which were termed *uni-structural, multi-structural, relational, extended abstract*.

Pegg (2002) took a further step by noting how the Biggs and Collis cycle of concept formation operates in a similar sequence to the compression of process to concept, linking to the theory of Gray & Tall (1994) in which action-schemas such as counting (uni-structural) are developed into more compressed procedures such as count-all, count-on, count-on-from-larger (multi-structural), to the overall process of addition that may be implemented by different routes (relational), and the concepts of number and sum seen as mentally manipulable concepts (extended abstract).

This opens up a vision of a cognitive development from embodied beginnings encompassing the SOLO sensori-motor and ikonic (a combination of Bruner's enactive and iconic modes) through successive encapsulations of actions as processes represented by symbols to symbolic manipulation of symbols as thinkable entities, relating the worlds of conceptual-embodiment and proceptual-symbolism.

Developing a general theory that also fits the problem

At this point, a single incident gave us a sudden insight into the relationship between embodiment and symbolic compression. The first-named author (Anna Poynter) was convinced that the problem arising from the complications of the examples of physics with their different meanings for journey, force, velocity, acceleration and so on, could be replaced by a much simpler framework in mathematics, if only (and this

is a big if) the students could focus on the fundamental mathematical ideas. The problem was how to give a meaning to the notion of ‘free vector’ in a *mathematical* way that was meaningful and applied to all the other contexts in an overall coherent way.

The breakthrough came from a single comment of a student called Joshua. The students were performing a physical activity in which a triangle was being pushed around on a table to emulate the notion of ‘action’ on an object. Joshua explained that different actions can have the same ‘*effect*’. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same.

This single example led to a major theoretical development. In performing an action on objects, initially the action focuses on what to *do*, but abstraction (to coin a phrase of John Mason, 1989) is performed by ‘a delicate shift of attention’, to the *effect* of that action. Instead of saying that two actions are equivalent in a mathematical sense, one can focus on the embodied idea of having the same effect. At a stroke, this deals with the difficult compression from action to process to object formulated in APOS theory, by focusing attention on shifting from embodied action to effect.

In the case of a translation of an object on a table, what matters is not the path taken, but the change from the initial position to the final position. The change can be seen by focusing on *any* point on the object and seeing where it starts and ends. All such movements may be represented by an arrow from start point to end point and all arrows have the same magnitude and direction. In this way *any* arrow with given magnitude and direction can represent the translation, and the addition of two vectors can be performed by placing two such arrows nose to tail and replacing them by the equivalent arrow from the starting point of the first arrow to the end of the second. The *embodied world* of action has a *graphical mode of representation* that is more than a static picture: it represents the mental act of carrying out the transformations so that the learner can focus not just on the actions but on their *effect*.

This theory of compressing action via process to mental object by concentrating on the embodied effect of an action is widely applicable. It is a practical idea that can prove of value in the classroom, as well as bringing together a range of established theories developed over the last half century by Piaget, Bruner, Dienes, Biggs & Collis, Fischbein, Skemp, Dubinsky, Lakoff & Núñez and many others. In the following sections we give a brief outline of our empirical evidence from Poynter (2004a) which are summarized on the web (Poynter, 2004b).

Empirical results

Poynter (2004a) compared the progress of two classes in the same school, Group A taught by the researcher using an embodied approach focusing on the effect of a translation, Group B taught in parallel using the standard text-book approach by a comparable teacher. The changes were monitored by a pre-test, post-test and delayed

Stage	Graphical	Symbolic
0	No response	No response
1	Journey in one dimension	A signed number
2	Arrow as a journey from A to B	Horizontal and vertical components
3	Shifts with same magnitude and direction	Column vector as relative shift
4	Free vector	Vector \mathbf{u} as a manipulable symbol

Figure 1: Fundamental cycle of concept construction of free vector

post-test, and a spectrum of students were selected for individual interviews. The tests studied the students' progress in developing through a cycle of concept construction in both graphic (embodied) and symbolic modes of representation.

In figure 1, two cycles of concept construction are involved. Stage 1 refers to the earlier cycle formulating the notion of a signed number in one dimension as journey or as a signed number. Stages 2, 3 and 4 are successive stages of encapsulation of the notion of free vector in two dimensions, starting from a graphical representation of an arrow as a journey represented symbolically as horizontal and vertical components, then focusing on the effect of the shift as shifts with the same magnitude and direction or as a column vector as a relative shift, then finally as a manipulable free vector that can be given a single symbol that can be operated upon. A similar cycle was formulated for the encapsulation of the process of adding two vectors to give the concept of sum, starting from addition of signed numbers in one dimension, then in two, where the arrows are seen, for example, as one journey following another then focusing on the effect to see the sum of two vectors as the single vector with the same effect and finally as free vectors added as mental entities.

Poynter (2004a) focused on several aspects of the desired change that could be tested. Here we consider three of them. It was hypothesised that students, who encapsulate the process of translation as a free vector, are able to focus on the *effect* of the action rather than the action itself. This should enable them to add together free vectors geometrically even if the vectors are in 'singular' (non-generic) positions, such as vectors that meet in a point or which cross over each other. It should enable them to use the concept of vector in other contexts, e.g. as journey or force. In the case of a journey, it should allow the student to recognise that the sum of free vectors is commutative. (As a journey, the equation $AB + BC = BC + AB$ does not make sense, because $AB + BC$ traces from A to B to C but, $BC + AB$ first represents a journey from B to C and requires a jump from C to B before continuing. As free vectors, $\mathbf{u} = AB$ and $\mathbf{v} = BC$, we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.)

It was hypothesised that experimental students would be more able to:

1. add vectors in singular (non-generic) cases
2. use the concept of vector in other contexts (eg as journey or as force)
3. use the commutative property for addition.

Students were asked to add two vectors in three different examples:

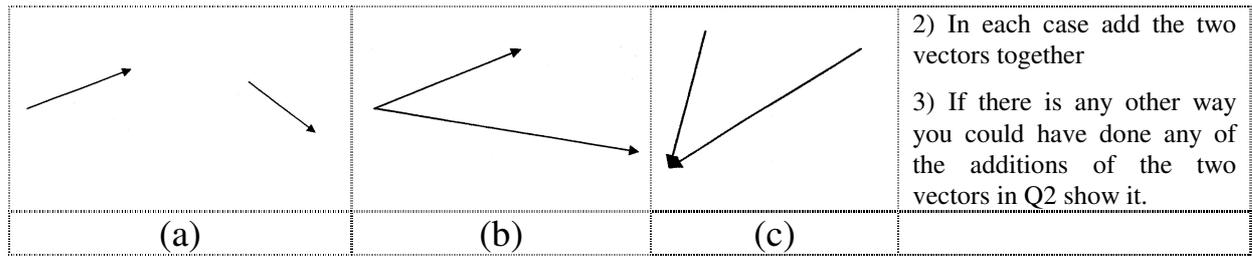


Figure 2: questions that could be considered singular

When we asked other teachers what they felt students would find difficult, we encountered differences between the responses of a colleague who taught physics and two others who taught mathematics. As mathematicians, we saw part (a) to be in a general position, because it only required the right-hand arrow to be pulled across to the end of the left-hand arrow to add as free vectors; (b) evoked the idea of a parallelogram of forces; (c) was considered singular because it was known to cause problems with some students embodying it as two fingers pressing together to give resultant zero.

All teachers considered part (c) would cause difficulties. However, they differed markedly in their interpretations of parts (a) and (b). The physics teacher considered that the students would see the sum of vectors either as a combination of journeys, one after another, or as a sum of forces. For her, (a) was problematic because it does not fit either model, but (b) would invoke a simple application of the parallelogram law. As an alternative some students might measure and add the separate horizontal and vertical components. The two mathematics teachers considered that students would be more likely to solve the problems by moving the vectors ‘nose to tail’ with the alternative possibility of measuring and adding components. One of them considered that students might see part (a) as journeys and connect across the gap, and in part (b) might use the triangle law in preference to the parallelogram law. The other sensed that (b) could cause a problem because ‘they have to disrupt a diagram’ to shift the vectors nose to tail—an implicit acknowledgement of the singular difficulty of the problem—and part (c) would again involve shifting vectors nose to tail although she acknowledged that some students might do this but not draw the resultant (which intimated again that they see the sum as a combination of journeys rather than of free vectors).

The performance on the three questions assigning an overall graphical level to each student is given in Table 1.

Graphical stage	Group A (Experimental) (N=17)			Group B (Control) (N=17)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
4	0	1	12	2	0	7
3	1	9	4	1	10	3
2	4	6	1	1	3	2
1	4	1	0	4	1	0
0	8	0	0	9	3	5

Table 1: Graphical responses to the singular questions

Using the t-test on the numbers of students in the stages reveals that there is a significant improvement in the experimental students from pre-test to delayed post-test ($p < 0.01$) but not in the control students.

Similar results testing the responses to questions in different contexts and questions involving the commutative law are shown in tables 2 and 3.

Graphical stage	Group A (Experimental) (N=17)			Group B (Control) (N=17)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
4	0	0	8	0	0	2
3	0	9	3	2	3	5
2	1	2	2	0	3	3
1	1	5	4	0	2	3
0	15	1	0	15	9	4

Table 2: Graphical responses to questions set in different contexts

The change is again statistically significant from pre-test to delayed post-test ($p < 0.01$) using a t-test.

Graphical stage	Group A (Experimental)			Group B (Control)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
TOTAL	0	7	12	4	6	5

Table 3: Responses using the commutative law of addition

In this case the change is from a significant difference in favour of Group B on the pre-test ($p < 0.05$ using a χ^2 -test) to a significant difference in favour of Group A ($p < 0.05$ using a χ^2 -test). Further details may be found on the web (Poynter, 2004a, b).

What is clearly important here is not the statistical significance, but the evident changes which can be *seen* not only to improve the situation for Group A from pre-test to post-test, but more importantly to *increase* the level of success by the delayed post-test. There is a clear difference in the long-term effect of the experimental teaching programme.

Broader theoretical aspects

The theory reveals a parallel between focusing on the effect of embodied actions and the compression of symbolism from procedure to process to object has the potential to be simple to describe and implement with teachers and students. The theory has proved to be a practical theory, in that the idea of focusing on the *effect* of an action in the case of vector has proved to be not only successful with students, as in the experiment described, but also in subsequent discussion with other teachers (Poynter 2004b). All that is necessary to have appropriate activities and to mentor the participants to focus on the effects of carefully designed actions.

This applies in a variety of areas, not only in representing vectors dually as transformations and as free vectors, but also in other areas where symbols represent a

process being encapsulated into a concept. For instance the process of counting is compressed to the concept of number by focusing on the *effect* of counting in terms of the last number spoken in the counting schema. Likewise, the process of sharing and the concept of fraction, in which, say, sharing something into 4 equal parts and taking 3 of them has the same effect as sharing into 8 equal parts and taking 6. This corresponds symbolically to having equivalent fractions ($\frac{3}{4}$ or $\frac{6}{8}$). Likewise different algebraic procedures having the same effect gives an alternative way of looking at the idea of equivalent algebraic expressions. Other processes in mathematics, such as the concept of function, also result from a focus on the *effect* of an input-output action, rather than on the particular sequence of actions to carry out the process, revealing the wide range of topics in mathematics that benefit from this theoretical analysis.

This research into a single classroom problem has therefore stimulated developments in the relationship between embodiment and (proceptual) symbolism as part of a wider general theory of the cognitive development of three worlds of mathematics (embodied, symbolic and formal), (Watson, Spyrou & Tall, 2003, Tall, 2004). This theory, in turn, also builds on earlier work that theorizes three distinct kinds of mathematical object: “One is an *embodied object*, as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. The second is the *symbolic procept* which acts seamlessly to switch from an often unconscious ‘process to carry out’ using an appropriate algorithm to a ‘mental concept to manipulate’. The third is an *axiomatic concept* in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory” (Gray & Tall, 2001).

In this way, looking at how a particular teaching problem benefits from different theories can be fruitful, not only in addressing the teaching problem in a way that makes practical sense to pupils and teachers, but also in analysing and synthesising aspects of a range of theories to produce a practical theory.

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CONCEPTUALISATION THROUGH SEMIOTIC TOOLS IN TEACHING/LEARNING SITUATIONS

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Abstract: *This paper addresses the role of mathematical signs in teaching/learning situations. During the heuristic phase of a situation we can observe signs being produced, interpreted and used in very unusual ways –from a mathematical point of view. Then the notion of ostensive (Bosch & Chevillard 1999) is not sufficient to analyse the students' work; the research of Steinbring (2005), or the three dimensions introduced by Godino (2004), though very interesting to appreciate interpretation within mathematics, do not seem satisfactory to give an account of the way signs are produced and (mis)interpreted. We use C.S. Peirce's theory of semiotics to understand this phenomenon: signification is not definitely deduced from (mathematical) signs because interpretation is a triadic process that requires an interpretant; though mathematical signs are always arguments (they involve a rule), they can be understood as icons, indices or arguments in the interpretation process.*

Keywords: mathematical signs, interpretation, Peirce's theory of semiotics.

I. The use of semiotic tools in didactical situations

Mathematics teaching (e.g. about functions) is often organised as follows: First, the teacher performs a standard task in the classroom with his/her students, using a variety of representatives of the target concept. Next, students are supposed to do a similar task, with other emblematic representations of the same concept. Students are expected to interpret the signs used in the situation (geometric figures, graphs, tables of numbers, formulae ...) in the same way as the teacher, that is, as representatives of mathematical concepts and of their properties. This presentation is supposed to be more “intuitive” than a formal one. In fact, it does not bring out the fundamental mathematical knowledge; in doing this work, students indeed cannot learn or imagine what are the properties of the objects – functions for instance; what is the use of a property in the mathematical organisation, i.e., why it is useful to study functions; how it is possible to distinguish a property from other connected properties; what is the opposite of a property, i.e. if the property “p” is known, how can the property “not p” be formulated? In other words, this way of teaching does not lead to real work on mathematical statements: it is a specificity of mathematical statements that they allow us to know what properties they determine, what mathematical objects satisfy these properties and which ones *do not* satisfy the properties. Moreover, if we know a property we can also know its opposite, which is not possible in an ostensive organisation like that. Students are thus going from one representative to another without knowing the use of them to solve problems.

As Schwarz and Dreyfus (1995) say, in mathematics “learning is reduced to mapping between several notation systems signifying the same abstract object”. In the same paper, the authors point to the fact that research about learning functions and graphs shows persistent difficulties in linking those different notation systems. As students have great difficulties in constructing graphs, tables, or formulae by themselves, teachers usually make them work on given notational systems in order to avoid tasks of construction. These authors insist on the ambiguity of all representatives of mathematical objects, and on the fact that teaching often does not explicitly address this ambiguity. Only algebraic ambiguities are dealt with because it is in the nature of algebraic work to see if two formulae represent the same function. In brief, ambiguities are usually treated as if they belonged to the didactical contract. For instance, depending on the context –and the level of teaching– it is considered obvious (or not obvious) that a table (two numbers and their images) is a representative of a linear function, or that a curve is the graph of a quadratic function or not. Schwarz & Dreyfus conclude that “ambiguity problems are avoided in standard curricula because students do not have the tools to cope with them” (Schwarz & Dreyfus 1995, p.263). We think that conceptualisation of an abstract concept like functions cannot avoid ambiguity, especially in its representation and the use of signs.

At the same time Duval (1993, 1996) studies the partiality and ambiguity of mathematical signs: every sign is partial to what it represents, and partiality leads to ambiguity. For instance a graph is the graph of a *class of functions* and not of a single one; on a graph you can see that two points are different but not that they are mingled. Graphs may also suggest false properties:

If you draw the graph of a plain function such that $\lim_{x \rightarrow \infty} f(x) = 0$ you can easily imagine that $\lim_{x \rightarrow \infty} f'(x) = 0$ too, because the curve seems to decrease regularly to 0.

You have to come back to the algebraic and analytical register to be able to argue and decide if the last property is true or not.

Duval concludes that we must consider the interactions among different representatives of a mathematical object as absolutely necessary for constructing a concept. We follow Duval in acknowledging this necessity, but two representatives of a same object being given, how can one be sure that they signify the same for students, or even that they signify *anything mathematical* for them?

We can also connect this view with Slavit's analysis (Slavit 1997) whose aim is to develop a *property-oriented view* of concepts:

"[A] property-oriented view is established through two types of experiences. First, the property-oriented view involves an ability to realize the equivalence of procedures that are performed in different notational systems. Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x -intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions. Here, students can

relate procedures across notational systems, but they are also beginning to realize that some of these procedures have analogues in other types of functions. For example, one can find zeroes of both linear and quadratic polynomials (as well as many other types of functions), and this invariance is what makes the property apparent." (Slavit 1997, p. 266-267).

This citation stresses that, to realise the invariance of properties, students have to do many comparisons between different functions in different notational systems. Not every task will help them in achieving this aim. This means that, while the choice of pertinent representatives and different classes of functions is necessary to reach this aim, it is not sufficient: the situation in which students are immersed is essential to produce the target knowledge. By "situation", we mean the type of problems students are led to solve. Immersing students in problems makes it possible to obtain a work on mathematical signs, mathematical statements and an activity of reasoning.

This approach is convergent with the one of Brousseau's Theory of Didactical Situations (Brousseau 1997). To make mathematical signs 'full of sense' –which means that signs have a chance to be related to a conceptual mathematics object– the TDS proposes the organisation of situations that would allow the students to engage with validation, that is, to work with mathematical formulation and mathematical statements.

As it is now well known, the work with the TDS has permitted to develop very interesting situations for almost all the main themes of mathematics at Primary school. In each case, the situation organises a 'material' milieu that allows experiments for pupils; and the milieu gives feedbacks. The material milieu is made of "material things" to act with (when we say "things" we mean that for the students they do not necessarily represent mathematical objects). The heuristic milieu includes procedures of verification which must lead students to formulate mathematical properties. Then the didactical situation allows the teacher to declare the intended knowledge. (Brousseau 1997, p.17, p.248; Bloch 2003a, p.12). We cannot say that, at this primary level, there are no mathematical signs: but at the first level of milieu at least, signs can be embodied in material things.

There exist not so many 'good' situations for secondary school and high school; at this level, the question would be to find collections of problems that allow to explore the fundamental meanings of a mathematical concept (for examples of such situations about functions, see Bloch 2003a). A main difficulty at this level is that the 'material' milieu is already constituted with abstract mathematical signs; but if the teacher sees them as mathematics, students sometimes see them only as sorts of conventions that the teacher introduces. The success of the learning process is then decided by the capacity the situation gets of confronting students with actions, leading them to interpret the signs in the milieu and formulate mathematical knowledge with their own signs in a first time; the situation provides feedbacks and this process organises the students' activity, including the work with/about a mathematical semiosis (Bloch 1999).

This approach of the TDS leads to build a milieu for students' work, a milieu where three phases are necessary¹: the first stage consists in acting upon the material milieu that provides feedbacks; the function of the second stage is to validate the results (calculations, writings, arguments...), and the last phase leads to institutionalisation.

Such situations lead students not only to "correct" mathematical writing that they are supposed to copy, but to produce their own formulations during the heuristic phase in stage 2. This typical feature of didactical situations includes a tolerance to "approximate" formulation and moreover, the utilisation of such signs during the formulation phase; it means that such situations call, not only on certified knowledge but also on students' "private knowing". This has been studied by Conne and Bloch (Conne 1992; Bloch 1999).

This private knowing can find its expression in a number of semiotic ways, including drawings, words and sentences, graphs, computation, 'false' writings... While looking at this process, we can see that the relationship between signs and objects is not yet well established. Moreover, the constitution of mathematical objects is not already done: it takes place in a dialectical interaction between the situation, the signs and the concepts (as Steinbring also says, see below). As in a number of recent researches, we recognise then the necessity of studying the part of semiotic dimension in the process of conceptualisation.

II. Theoretical frameworks about semiotics in mathematics

II.1 Research on semiotics through mathematics epistemology

Researches on semiosis in mathematics have been widely developed in the last ten years. Duval studies representation settings and their congruence in conversions (Duval 1996). The term of 'semiotic tools' or 'ostensive' to speak of mathematical signs has first been used by Chevallard (Bosch & Chevallard, 1999). Semiotic tools are signs that are also tools to do a mathematical work: it is a distinctive feature of mathematics that signs can be used as tools and that new actions on these tools produce new significations and even sometimes new concepts. This characteristic can be illustrated –ad absurdum!– by a classic joke (in French):

Let us prove that $cheval/oiseau = \pi$: $cheval/oiseau = cheva l/oiseau = vache l/oiseau$ because 'cheva' is commutative; 'vache' is a "bête à pis": $\beta\pi$ and 'oiseau' is a "bête à ailes": βl then $vache l/oiseau = \beta\pi l / \beta l = \pi$ (Cheval = horse; vache = cow; oiseau = bird and a cow is an animal with an udder ('pis' in French); a bird is an animal with wings ('ailes' in French).

Anyway let us notice that the semiotic tools that Chevallard introduces are related to 'correct' denotation of mathematical concepts: for us, this is the fundamental limitation of his theory, as of every other framework based on epistemology of mathematics only.

¹ A complete example about functions and graphs is given in Bloch, 2003a.

Radford (2002, 2003, 2004 for instance) relates semiotics and epistemology of mathematics. Godino (2002, 2004) works in a similar direction, as well as Steinbring: signs take place in the epistemological triangle whose other corner points are 'reference context' and 'concept'.

In this model we notice that:

"none of the corner point of the triangle is given explicitly in such a way that it could become a certain point of approach to a definite determination of the triangle; the three reference points "mathematical concept", mathematical sign/symbol and "object/reference context" form a balanced, reciprocally supported system". (Steinbring 2005, p.4)

This author also points out (op. cit. p. 11) the "exchangeability of the positions of sign/symbol and reference/context". This is an significant point to be noticed, and we agree with that, but we want to go farther and claim that a context *is a sign* as well: linking the TDS and a semiotic approach, we cannot but observe the fact that, for the students at work, situations play the role of signs of a certain knowledge, preferably the one that is the aim of the situation but in some case it can be different. In Steinbring's examples (Steinbring 2005 p. 15-23: a number wall and number squares), as the author focuses on verbal interactions and gestures, he also points out the way a student makes a circle round a number and crosses other numbers; we could add that the given structure itself –the number wall, or the number square– is a sign of the embodied knowledge, through the actions in the situation.

For Davis and McGowen (Davis and McGowen, 2002) mathematical objects are clearly embodied in (pre)mathematical signs. They also study the semantic dimensions of communication, considering "whether mathematics deals with a level of reference beyond the symbolic as it is understood in language". Referring to Peirce's semiotic levels as we intend to do, they observe in mathematics classrooms the same events that we describe in part III., that is, students considering a complex mathematical sign (the binomial theorem) as an index of something – "a conditioned response" – instead of an argument ("a richly connective symbolic interpretation").

Actually, even in very classical teaching situations we can see students that do not attribute the 'correct' signification to mathematical signs, or even any mathematical signification. Anyway, we can also be aware that the signification of a mathematical sign evolves according to different contexts: the signification of a sign is not (once and for all) definite, it depends on the situations students encounter. Signs are tools and their meaning is linked with the use of them in various mathematics theories. Moreover, for the same symbol it may depend on the level of mathematics: the sign of an integral does not mean the same thing in Riemann's integration theory as in Lebesgue's (even if there are connections or inclusions). Then the signification of a mathematical sign depends on the competence of an interpretant –a social entity that receives the responsibility of interpreting the symbol and doing 'something mathematical' with it. A student at the first year of University could not interpret an

integral sign as a Lebesgue's integral, as well as a 7 years-old pupil could not interpret a sign '=' as algebraic.

For these reasons we resort to Peirce's theory of semiotics to try to understand the use and the role of signs in mathematics education, especially in didactical situations.

II.2 Peirce's semiotics approach as a tool to understand an significant dimension of the students' work

At the end of the 19th century, the American philosopher C.S. Peirce proposed a dynamic theory of interpretation of the signs. As he was himself a mathematician, he applied his theory to mathematics. The fundamental characteristic of this theory is its triadic approach of interpretation: a sign is a triad whose components are, the sign (that represents) the object (that is represented) and the interpretant (that is a social entity able to interpret). This is a dynamic approach because this triad is a sign again and it represents an object to an interpretant, and so on... This dynamic component is very important because it provides the means to explain how the signification of symbols evolves:

"A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the **interpretant** of the first sign. The sign stands for something, its object. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of the representamen. [...]" (Peirce 1897, *C.P.* 2-228 - *Division of signs*)

This conception leads to different kinds of signs because the repeated process of interpretation leads to very sophisticated signs. Thus Peirce categorises three basic kinds of signs: icon, indexes, and arguments (Peirce 1978). An icon is a sign that represents a quality of something: for instance, a quality of colour (green) that can be perceived in a glance. An index gives an indication or a hint on the object, like an image of the Eiffel tower makes you think of the town of Paris, or may be a symbol of Paris. An argument is a sign that contains a rule.

"First, an analysis of the essence of a sign, (stretching that word to its widest limits, as anything which, being determined by an object, determines an interpretation to determination, through it, by the same object), leads to a proof that every sign is determined by its object, either first, by partaking in the characters of the object, when I call the sign an Icon; secondly, by being really and in its individual existence connected with the individual object, when I call the sign an Index; thirdly, by more or less approximate certainty that it will be interpreted as denoting the object, in consequence of a habit (which term I use as including a natural disposition), when I call the sign a Symbol".(Peirce 1906 - *C.P.* 4-531 - *Apology for pragmatism*).

We must see that the category of a sign can never be abstracted from the interpretant: a sign is an icon, for instance, in an interpretation process for somebody (at least virtual). Therefore a relationship between a sign and its object is never intrinsic², it depends on the quality of the interpretant. This paradigmatic signs organisation leads to a hierarchy between different kinds of signs, because the interpretant is distinct from the sign (the representamen) and the object. Peirce defines three categories for each instance of the sign, namely Representamen, Object and Interpretant. These categories came out later (Marty 1990) to a *lattice of signs*, a lattice with ten categories only (and not $3^3 = 27$) because a sign as an Icon, for instance, could never give an argument, as an argument can very well be interpreted only as an Icon by somebody who would not get the right interpretant³. This feature of Peirce's theory we find of course very relevant for mathematical signs interpretation. His theory is relevant to analyse the interpretation of mathematical signs, because all mathematical signs are arguments, even if of different levels, and because there will be some problems of misunderstanding if students do not interpret them in their suitable value.

Mathematical signs are symbols that give rise to arguments ('in consequence of a habit', here a mathematical habit of course) but of different levels: then '3' is the representamen of an argument because it always signifies that there are three elements somewhere (in a mathematics problem for instance; the relation between '3' and the number is a rule); but '123' (one hundred and twenty three) will be a more complex argument because the rule must include the decimal numeration, which is not the case in '3'. This complexity is what authorizes various interpretation of the same symbols, according to the mathematical competence of the interpretant.

This particularity of semiotic tools in mathematics will be in agreement with the relevant elements of the construction of a situation, as we said that in an adidactical situation we must foresee the 'wrong' expression of mathematical properties, or the misunderstanding of some ostensives. We are here lacking the place to analyse a fundamental situation with Peirce's theory: we shall limit our demonstration to some examples of various levels in mathematical signs interpretation.

III. Using the different modelisations to analyse experimental facts

We give now two examples of an interpretation process of mathematical signs in a classroom. In some times and places of the teaching/learning institutions, there exist a huge gap between what the teacher proposes and what the student is able to imagine. This is especially the case in two very different contexts: education with special needs, and scientific courses in the first years of University. These places provide therefore good opportunities for studying the misfortunes of signs interpretation. We shall see that Peirce's semiotics paradigm allows to go beyond the 'right' academic mathematical reading of the way students use the ostensives and to give sense to their

² Even in mathematics!

³ For more details, see Marty, 1990, 2003, Muller, 2004.

own atypical interpretation. This provides a help to understand the students' difficulties, and to be able to modify the situation –the context– they are trying on.

Example 1: In a class for students with special needs (14/15 years-old) the teacher was working on proportionality; students got a numeric table and were required to say if the situation was of proportionality or not. Notice that these students have heard of proportionality since four years at least; this is not a moment of 'first encounter' but a reinvestment. According to Chevallard, such a table is a numerical ostensive; as the students do not yet master the table as a computation means, we can say that the work takes place as a task, and not a technique or a technology.

During their work we could see that for some students, the numeric table was really an argument as suitable: they were able to say that in such a table, doing some computation you could find the rule –the proportionality coefficient– and some information about the numbers and their images, and even build images of new numbers from the given numbers.

For some other students, the table was obviously an index, that is, they were aware that the table contained indications, and told them something about proportionality; but they were not able to find a applicable indication in it. And moreover, some others just saw the table as an icon: the thing the teacher draws on the blackboard each time she speaks of proportionality.

We can see that an analysis in terms of reference, or context, could not be sufficient here, since some students remain very far of a mathematical interpretation of the table. From a classic mathematical analysis of this event, the teacher can do nothing but say that these students are unsuccessful in doing the prescribed task, and of course try another task –but for these students it has been done a lot of times yet! It becomes then a problem in a double perspective:

1) In this context –students with special needs– the teacher has to do something to declare some success in the learning, especially about ancient knowledge such as proportionality. If she states a global failure of the didactical project for a major part of the class any time she gives them a task to perform, the didactical relation will become very difficult. In a more personal point of view, the teacher also has to be able to explain herself why and how students fail, in order to envisage how going on.

2) If we notice that a large part of the students do not understand a proportionality table as a mathematical sign, then the situation we organise for such students cannot start from this table (even if this is suggested in the curriculum of the academic year) since this table does not mean anything in terms of proportionality or even any argument for them. Something has to be done before. Here we can use the Godino's and Steinbring's model, and say that the reference context must be adjusted –in terms of the TDS we will say that a more relevant situation must be organised: within the

frame of the TDS, it is the game in a situation that allows formulation and an adequate conceptualisation of what the mathematical signs can mean⁴.

Example 2: Students in a first year of mathematics University course have to study a function: $f(x) = |x| \sqrt{|x|} / x$ if $x \neq 0$, and $f(0) = 0$

The question is to determine whether f is continuous. Students first do not see what kind of function it is: they are confused by the signs of square roots and absolute values. In Chevallard's terms we could say that they do not have any technique at their disposal to deal with this expression. The lack of means to cope with this algebraic task involves an impossibility of devolution of the real problem.

Using our semiotic frame we can say that some students interpret these signs only as an index of complexity of the function; none of them interprets the sign as an icon of function, since they are advanced students in a calculus course. From their reactions we can see nevertheless that for other students, these writings are only *icons of complexity*, since these students are unable to undertake a calculation or a transformation of f although they are aware of this writing being an ostensive of a function. For the best students, these signs are indexes of the fact that there is something to do to obtain a better expression of f . Then students do not see that the only question is about the continuity in zero: they say that f is definite in zero so it must be continuous in zero (this interpretation obeys the rules of the didactical contract of mathematics studies at Upper Secondary School). They interpret the signs $f(0) = 0$ as a whole: it says everything about the function in zero, it is a global argument of definition and continuity. Finally, after a long time of misunderstanding between teacher and students, the teacher draws a graph to show them that there could be a problem in zero (change of setting). This graph can be interpreted by the students as an index of the difficulty in zero, so devolution of the problem: 'Is f continue in zero?' can take place in the classroom and lead to a formulation with an argument of what continuity really is.

These two examples show the interest of a semiotic analyse of the students' work: it helps to identify the difficulties students can encounter while trying to perform a task. These observations also provide a major indication that there exists more than one stage –linked with the signs' interpretation– in the process of conceptualisation. It is then not always sufficient to study the mathematical reference of a sign, since students in a first phase do not reach this stage of interpretation. The use of a pertinent and dynamic semiotic theory is also necessary to build new situations aiming a mathematical concept, since situations are using representatives from adequate settings.

⁴ For situations about proportionality, see Brousseau 1997, p. 177.

Conclusion

We think that this semiotic approach is useful to study the work students can perform in their mathematics studies. This analyse can use Peirce's theory, because this theory is adequate to explore mathematics representation, even before it concerns academic mathematical objects. The use of this theory helps to a pertinent investigation of the potential work students can perform in a situation. At any level of mathematics, it is essential to have a clear vision of the use students can do of representatives when they learn mathematics, and to understand students' level of interpretation. Moreover, a semiotic investigation is useful to build alternate routes to reification of mathematical concepts, as Slavit said. Our perspective is to lean on the Theory of Didactical Situations to elaborate adequate situations, even when complex mathematical concepts are at stake. This includes an unavoidable analysis of the signs that can be produced, provided and interpreted during the situation process. The already done analysis (Bloch 1999, Bloch 2003a, Bloch 2003b) make us confident that the stages in an adidactical situation are to a large extent compatible with the different levels of interpretation in the production of meaning.

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CROSSING THE BORDER INTEGRATING DIFFERENT PARADIGMS AND PERSPECTIVES

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Abstract: *Traditional interest theory has started within the paradigm of stability orientation including more and more aspects of change over time while maintaining the individual view. This paper presents a research project in which a concept of interest supporting situations is constructed. Investigations on these situations lead to an interest theory of mathematics education. The development process of this theory begins within the paradigm of change orientation and reconstructs patterns of change from empirical data. It describes how follow-up studies could be used to connect the social and the individual perspective and the two paradigms into one interest theory of mathematics education.*

Keywords: research paradigm, research perspective, paradigm of change, paradigm of stability, interest theory of mathematics education, motivation, social perspective, individual perspective.

Researchers often carry out their work within a network of specific assumptions which are not usually questioned. This network of assumptions which is taken for granted is called a paradigm. Within social research we can distinguish between two different paradigms: stability orientation and change orientation (Ulich 1979). Research within the scope of the paradigm of stability orientation investigates features which persons, situations, groups already have or bring with them. Whereas research within the paradigm of change orientation investigates how specific features emerge, change, develop or how they are generated. In the context of fostering students' interests it is necessary to know what kinds of interest in mathematics students bring with them but also how interest in mathematics comes into being, how it grows, develops and further develops. If a teacher starts with the idea that students bring their interests with them into the class then fostering interest in mathematics means adapting the lesson to their preexisting interests. If, on the other hand, interest is seen as something one can initiate and change depending on the situational conditions during the lesson, the teacher thinks about aspects of the mathematical area and ways of organizing the learning process that might create a process of learning with interest. Thus, for mathematics lessons it seems to be important to do the one thing without forgetting about the other. This, however, demands an interest theory for mathematics education which describes

- what kinds of interest in mathematics might exist and explains how preexisting interest is respected and allowed and

- how interest in mathematics emerges, how it may be caught, held, supported and stabilized and how interest development is hindered or blocked.

Therefore, research within mathematics education needs investigations on stable features which mean identifying and structuring them and, at the same time, it is necessary to know how these features are generated, stabilized and destabilized.

I am now going to present my path of study and ongoing work about interest in mathematics (Bikner-Ahsbals 2001, 2002, 2003a, 2003b, 2004), the role of both paradigms and the social and individual perspectives. This paper will show that, the more we try to understand the change of a person's features the more we need to consider situational aspects. At the same time, individuals are never able to perceive a whole situation, which has an impact on them. Therefore we have to investigate the social situation as a unit, too. However, if we do this we cut off the perspective of the individuals. This means we have to investigate the social situation and the participating individuals separately, the features of the individuals are investigated in the direction of change and the change of the social situation is investigated in the direction of stability.

1. From stability orientation to change orientation

Traditional interest research is research which is carried within the paradigm of stability. The assumption of this research is that a person already has an interest. In this tradition interest is either a disposition of a person or an object of preference. In the eighties this duality was overcome by defining interest as a motivational relation towards an object. In more detail, interest was now understood as a relational construct between a person and an object which is shown through acting epistemically. These actions show cognitive, emotional and value aspects (H. Schiefele et. al. 1979, H. Schiefele 2000, Schiefele 1996, Krapp 1992, 1998, 2003a, 2003b, 2003c, 2003d). After connecting this concept of interest with self determination theory, researchers were able to investigate conditions of interest change (Krapp 1992, 2003a). Interest was now seen as a stable but changeable feature of an individual's personality (Deci 1992, 1993, 1998). Nowadays, psychological research distinguishes between two kinds of interest, first, personal interest is a relative stable kind of interest which students bring with them into the class and, secondly, situational interest is a more changeable kind of interest which is created through situational conditions.

The pedagogically interesting question remains how interest comes into being, how it grows and develops and how it can be supported within classes.

The question how interest emerges and how it can be supported by the teacher was already posed by Deci and Ryan in the eighties. They came to the conclusion that interest development is influenced to great extent by the social context. Based on their observations they postulated three basic psychological needs that accompany interest support, these are the need to experience competence, autonomy and social relatedness (Deci 1992, 1998). Meanwhile research on vocational education has confirmed

the importance of supporting the experience of competence, autonomy and social relatedness in order to foster interest development (Prenzel & Drechsel 1996; Prenzel et. al. 1998; Lewalter et. al. 1998). How these basic needs have to be interpreted within mathematics education remains an open question.

In 1993 Mitchell presented an investigation of situational interest occurring within mathematics lessons. He distinguished between catch-interest and hold-interest arguing that interest which is caught might not necessarily be held. Interest is caught by cognitively or socially stimulating situations such as working with computers or group work. Interest that is caught can be held if the students experience themselves as being involved in the activity (involvement) and if they experience this as meaningful (meaningfulness) (Mitchell 1993). Interest in mathematics that is caught can disappear if the teacher evokes expectations which the tasks cannot fulfil. This occurs if the teacher overmotivates the students (Bikner-Ahsbahr 1999). Today, researchers believe that personal interest develops through the repeating experience of situational interest (Krapp 1998, 2003d).

Thus, psychological interest research is rooted within the paradigm of stability orientation. It has developed towards the paradigm of change orientation without giving up the consideration of stability. During this process of development more and more situational aspects have been taken up while maintaining the individual perspective.

Krapp postulates a conscious and an unconscious level of regulating interest through actions. Cognitive aspects are regarded to regulate interest based actions on a conscious level. Emotional aspects like the experience of the basic needs are assumed to regulate interest based actions on a more unconscious level (Krapp 2003d).

Therefore, students will not be able to inform the researcher about all the aspects of a lesson which have had an impact on their interest. We have to take into account that the social environment influences interest development unknown to the individual. Especially the aspects on a micro-social level seem to influence the students on a more unconscious level. Therefore, interest research in mathematics education has to include both, the individual and the social view of interest development.

Research on interest development from a social point of view means identifying interest supporting situations and investigating how they are arranged. In order to investigate interest supporting situations we have to know how interest supporting situations are arranged anyway, but exactly this is the aim of research on aspects of interest supporting situations. If we question interest supporting aspects of lesson arrangements we investigate the change of interest. In order to be able to investigate aspects of interest supporting situations, we have to assume that these aspects are stable characteristics of social situations anyway, which collectively support the interest of a group of students. Thus, we would investigate changes of individuals by reconstructing stable aspects of the social situation. Arrangements of lessons on a micro level can be seen as a changing situation. Therefore, investigating the arrangements of situations means investigating change. This leads to something more stable, that is,

the pattern of interest support but at the same time investigating interest support means observing interest change. We quickly see, that there are too many changing and stable aspects to observe. All of them are somehow interwoven. How can a researcher escape this network of mutually connected aspects?

The insight is that research according to these paradigms and the individual and the social perspectives has to be carried out separately and can then be brought together. The question now is how to reduce complexity?

Concerning one aspect, psychological interest research has shown how to develop theories in the direction of the other paradigm. Personal and situational interest are two different interest concepts which tendentiously belong to different paradigms. Situational interest is a wider interest concept which is situated in the transition from noninterest to personal interest while maintaining the individual perspective. The missing point now is, an interest concept, which allows the investigation of interest supporting situations the other way round, that is, a concept that allows the identification of interest supporting situations and their investigation, which begins with questions of change and leads towards stability while the social view is maintained.

2. Characterization of interest-dense situations and its sensitizing background

Distinguishing between personal and situational interest, psychological interest research has taken a step from the paradigm of stability to change orientation. In my project “interest in mathematics between subject and situation” I adopted a pragmatic research stance and separated the social from the individual perspective (Bikner-Ahsbals, 2003b). Results of psychological interest research about situations that do and do not foster interest, practice of mathematics education and the analyses of data served as a sensitizing background for the construction of a new interest concept: This interest concept characterizes situations with *a potential for interest support*, so called interest-dense situations. In this section I describe this process and stress the connection to the two paradigms and the two perspectives.

Psychological interest research states that contest situations are experienced as controlled and, therefore, these situations do not foster interest development (Deci 1992, 1998). Why is this so?

Microanalyses of situations in the mathematics classes of my project show that during contest situations students expect the teacher to act as a referee. As a fair referee the teacher has to strictly follow the rules. Therefore, he has to treat all the students in the class in exactly the same manner. During learning situations the opposite is the case. A good teacher has to treat the students individually according to their ability and the level of the individual learning process. This type of teacher behaviour causes conflicts in contest situations. In the project class the teacher interpreted the rules for students with special needs differently from the way he interpreted the rules for other students. This led to conflicting situations. The students demanded the teacher to be fair. Thus, fairness was a global requirement that controlled the contest situations col-

lectively and so none of the students experienced competence, autonomy and social relatedness. Thus, interest supporting situations cannot be controlled. They support the experience of social belonging on the one hand and the individual learning processes on the other in order to allow the experience of competence and autonomy. Thus, interest supporting situations foster interest growth, hence the change of interest.

What has to be extracted in order to build up a collective interest concept which characterizes interest supporting situations?

Let us have a look at everyday mathematics lessons in more detail. A teacher calls a class "interested in mathematics" if the students advance the process of constructing mathematical meanings, if the students become involved in the activity as a group and if this involvement is shown by contributions of one student after the other. The students need not be interested individually, but together they act as if they were interested. Although they get involved individually the process may be regarded as a process of social interaction: The students' utterances are reactions to the contributions before and, at the same time, they initiate the next utterance. Triggered by this flow of interactions the students act and react individually but appropriate to the social situation.

How could a collective interest concept be characterized?

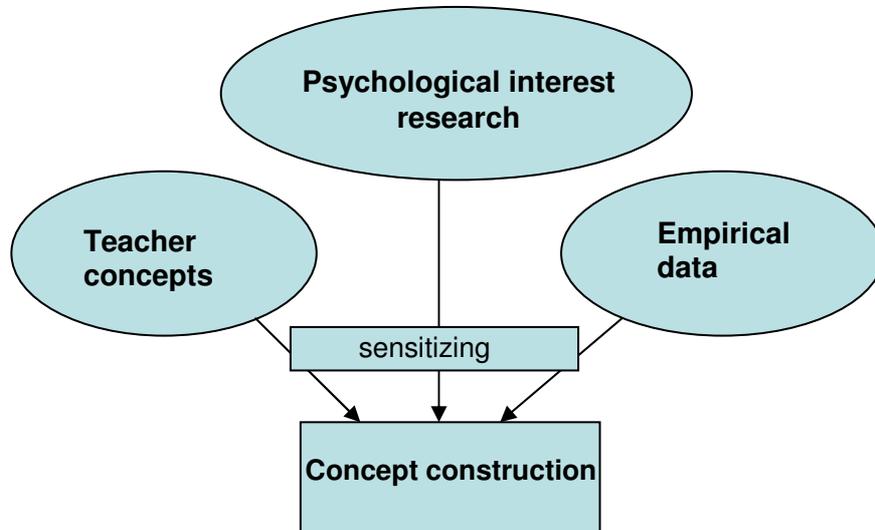


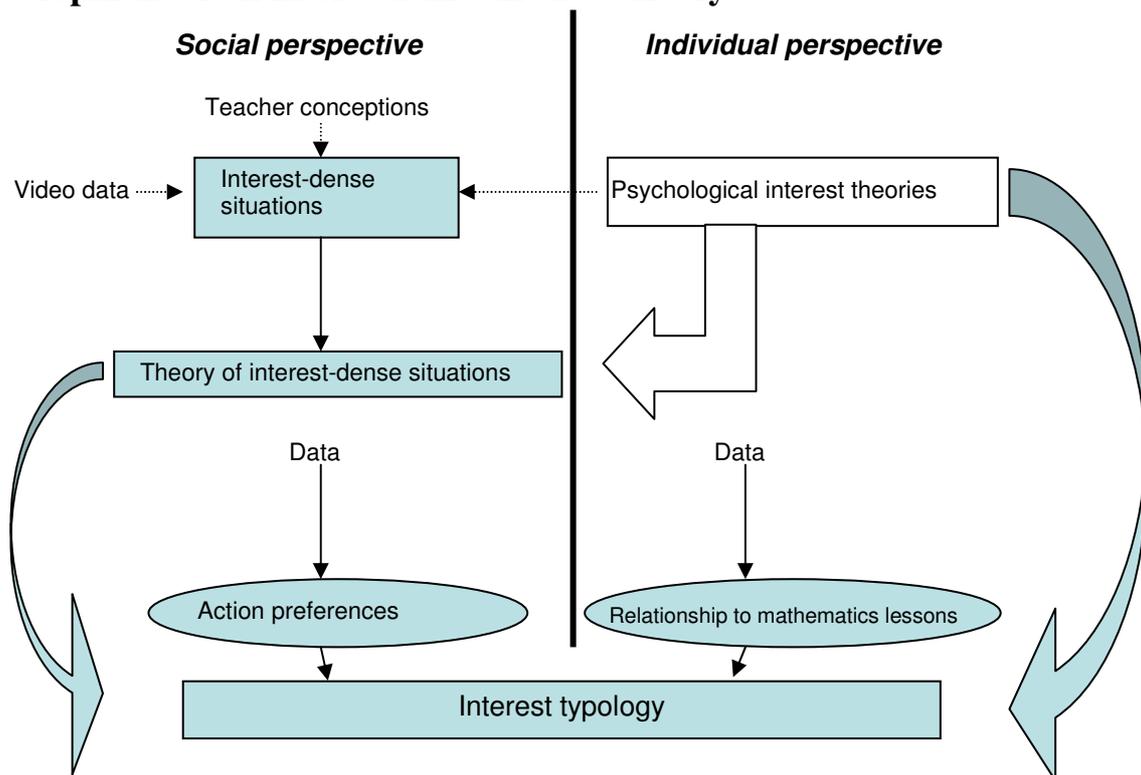
Fig. 1 Concept construction and its sensitizing background

The basic feature of interest is the orientation towards the growth of knowledge. Thus, interest supporting situations are epistemic situations. In these situations one student after the other gets involved, but not in a controlled way like in contest situations. The students use the opportunity to get involved according to their individual thought processes and preferences regarded as reactions to the contributions before and their initiations of the next utterances (*collective involvement*). Experiencing competence means, that one student after the other constructs further and farther

reaching mathematical meanings. This can be seen as a flow of social interactions (*dynamic of the epistemic process*). Finally, in these situations students are concerned with mathematics and not with pleasing the teacher, getting good marks or winning a contest (*mathematical valence*). These three features describe a *collective* kind of interest which I call *situated collective* interest. Situations in which situated collective interest emerge are now called *interest-dense* (Bikner-Ahsbabs 2000). These interest-dense situations have a *potential to support* the experience of competence, autonomy and social belonging, hence, individual interest development.

The concept of interest-dense situations is a new kind of interest concept which is not derived from known concepts, rather its construction is based on teaching practice, on theories and results of psychological interest research and on microanalyses of collected data of situations that do not foster interest (fig. 1). It is a social interest concept created to observe the emergence and the support of individual interest. The question now is not whether every single student is interested. The question is how this flow of social interactions within interest-dense situations is generated, stabilized or hindered.

3. Development of an interactionist interest theory



sensitizing →

leads by data analyses to →

is evaluation basis of ↑

is connected to ↪

Fig. 2 developing a theory about learning with interest

We still do not know whether interest-dense situations support interest development. From the perspective of individual interest theories they should, but this has yet to be proved empirically. However, the concept of situated collective interest enables us to identify situations with a *potential of interest support*. The question is now how these situations are arranged. This means how they are generated, how they are stabilized and how they are hindered. This cannot be answered from the individual perspective. The method of interaction analysis is used to reconstruct individual interpretations, which are regarded as reactions to the previous utterances and as roots for the next ones, hence, as the result of a process of social interactions. Focussing on the social interactions, psychological theories are cut off as far as possible within data analyses. This way a theory of interest-dense situations is developed. It regards interest from a social point of view. The empirical data consists of video data on fraction lessons of a sixth grade class taken over half a school year.

Interest-dense situations are characterized from the perspective of social interactions, epistemic processes and constructing value assignments in a collective way. All interest-dense situations and a set of non-interest-dense comparison situations are analyzed from these three perspectives. The results are integrated into one theory. This theory distinguishes between ideal types of situations with *different kinds of potentials of interest support*. Finally, psychological interest theories are used to evaluate this theory of interest-dense situations (fig. 2).

4. Integrating the individual perspective

The theory of interest-dense situations is at this point in time still an interactionist theory about situations with a well grounded potential to foster interest in mathematics. This raises the questions about how interest development relates to students' participation within interest-dense situations and whether interest-dense situations can indeed be regarded as interest supportive. This task could be carried out in follow-up studies.

What can be researched in the future?

First, the theory about interest-dense situations has to be broadened in order to include individual aspects and, then, linked with psychological interest theories. This could be carried out by analyses of individual data which were collected together with the video data of the investigated sixth grade class. Apart from personal data these individual data consist of

- data about the preferences for mathematics lessons in comparison with the other school subjects,
- data about a correspondence between the pupils in the class and the students of a university seminar on teaching fractions and
- video data of interviews with the students of the class about their preferences for arrangement aspects of mathematics lessons.

Interest-dense situations are mildly directed by the teacher. Therefore, we can expect that the participating students' preferences will become obvious during these situations. These preferences develop alongside the experiences during mathematics lessons. They relate to the overall preference for mathematics as a school subject. Action preferences could be worked out in a follow-up study through comparison analyses. In this comparison analysis, students' actions within interest-dense and non-interest-dense situations would have to be compared. The previous collected video data would provide an empirical basis for the analyses.

The correspondence between the university students and the pupils has documented how the pupils have experienced the lessons. This is the background information for the interviews where relationships between pupils and aspects of mathematics lessons are to be reconstructed. These relationships are to be taken as crystallizations of experiences during mathematics lessons, either the observed lessons or the ones before. Action preferences and the reconstructed relationships to aspects of the lessons could lead to the construction of a typology of interest in mathematics lesson. This typology could link the theory about interest-dense situations and psychological interest theories (fig.2).

5. Reflections

Without a doubt, change orientation and stability orientation cannot be investigated at the same time but both paradigms lead to complementary results.

Therefore, one starts with one paradigm and, over time, becomes more open towards the other.

Without a doubt, the social and the individual perspectives cannot be investigated at the same time but both perspectives lead to complementary results.

Therefore, one separates the research perspectives at the beginning and combines them in the end.

However, the question how far both paradigms and both perspectives can be integrated within one theory remains.

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CONCEPTUALIZATION OF THE LIMIT BY MEANS OF THE DISCRETE CONTINUOUS INTERPLAY: DIFFERENT THEORETICAL APPROACHES

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Abstract: *We investigate the contributions of three theoretical frameworks to a given research process and the complementary role played by each. First, we briefly describe the essence of each theory and then follow the analysis of their specific influence on the research process. The research process is on the conceptualization of the notion of limit by means of the discrete continuous interplay.*

1 The theoretical frameworks

1.1 The process-object model

This model deals with the dynamic process view and the static object view in relation to mathematical concepts. The dual character of mathematical concepts that have both a procedural and a structural aspect was dealt with many researchers. Sfard (1991) observed that abstract notions could be conceived *operationally* as *processes* and *structurally* as *objects*. Dubinsky (1991) postulated a theory of how concepts start as *processes* that are *encapsulated as mental objects* that are then available for higher-level abstract thought.

Gray and Tall (1994) introduced the notion of *procept*, referring to the manner in which we cope with symbols representing both mathematical *processes* and mathematical *concepts*. Their theory focuses on the relationship between mathematical processes, objects and symbols that dually evoke both.

Sfard (1991) talks about the transition from processes to abstract objects in enhancing our sense of understanding mathematics. She defined *reification* as a sudden ability to see something familiar in a totally new light, an instantaneous change in which a process solidifies into an object, into a static structure. Sfard added that *the new entity is detached from the process which produced it* and begins to draw its meaning from the fact of its being a member of a new category.

1.2 The instrumentation theory

Researchers have reflected on issues of ‘instrumentation’ and the dialectics between conceptual and technical work in Mathematics. See for example Artigue (2002), Guin and Trouche (1999), Lagrange (2000).

The instrumental approach is a specific approach built upon the instrumentation theory developed by Verillon and Rabardel (1995) in cognitive ergonomics and the anthropological theory developed by Chevallard (1992). The term ‘instrumentation’ is explained in Artigue (2002): The “instrument” is differentiated from the object, material or symbolic, on which it is based and for which the term “artifact” is used. An instrument is a mixed entity, in part an artifact, and in part cognitive schemes

which make it an instrument. For a given individual, the artifact becomes an instrument through a process, called instrumental genesis. This process leads to the development or appropriation of schemes of instrumented action that progressively take shape as techniques that permit an effective response to given tasks.

Artigue adds that it is necessary to identify the new potentials offered by instrumented work, but she also stresses the importance of identifying the constraints induced by the instrument. The instrumentation theory focuses on the mathematical needs for instrumentation, on the status of instrumented techniques as well as on the unexpected complexity of instrumental genesis.

1.3 The theory of abstraction and consolidation

Hershkowitz, Schwarz and Dreyfus (2001) have proposed a model of dynamically nested epistemic actions for processes of abstraction in context. The new abstraction is the product of three epistemic actions: Recognizing, Building-with and Constructing. The authors of the theory explain that Constructing is the main step of abstraction. It consists of assembling knowledge artifacts to produce a new mental structure with which students become acquainted. Recognizing a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. Building-with consists of combining existing artifacts in order to meet a goal such as solving a problem or justifying a statement.

In Hershkowitz, Schwarz and Dreyfus (2001) we read that the genesis of an abstraction passes through three stages: (a) the need for a new structure; (b) the construction of a new abstract entity and (c) the consolidation of the abstract entity through repeated recognition of the new structure and building-with it in further activities. Hershkowitz (2004) pointed out that knowledge might be constructed that remains available only for a short while. In a later stage the student may not recognize it as an already existing structure - no consolidation of this short-term construction has occurred. Thus the constructing stage of abstraction does not imply consolidation and a mental structure that has not been consolidated is likely to be fragile. Dreyfus and Tsamir (2004), Hershkowitz (2004), Monaghan and Ozmantar (2004) consider the process of consolidation with respect to the theory of abstraction. Dreyfus and Tsamir (2004) identify three modes of thinking that take place in the course of consolidation: building-with, reflecting on the building-with, and reflecting. Monaghan and Ozmantar (2004) note that using and reflecting on the new abstractions help the learner to establish interconnections between his established mathematical knowledge and the new abstractions.

2 The theoretical approaches and their influence on the research study

A description of the research study can be found in Kidron (2003). In this paper the research study will be described only through the demonstration of the role played by each theoretical framework and its influence on the research process.

2.1 The process- object model and its influence on the research study

Previous researches that used the process-object theoretical framework describe the cognitive difficulties that accompany the limit concept. See for example Cornu

(1991), Tall (1992). These researches, and others that are described in this section, which highlight students' dynamic process view in relation to concepts such as limit and infinite sums, were influential in defining the aim of the present research study. Tall (2000) theorized that the concept of limit is accompanied by cognitive difficulties because it conflicts with students' previous experience of symbols as procepts. In arithmetic, symbols have built-in computational processes to 'give an answer'. In school algebra, the symbols are algebraic expressions that are *potential procepts*. The operation (of evaluation) can only be carried out when the variables are given numerical values. Tall stressed that in the Calculus, the situation changes with limit processes that are *potentially infinite* and so give rise to a limit concept approached by an infinite process which usually has no finite procedure of computation.

Tall (1992) emphasized that the ideas of *limits* and *infinity*, which are often considered together, *relate to different and conflicting paradigms*. He illustrated this argument in analyzing the students' answer that $1+1/2+1/4+1/8+\dots$ is $2-1/\infty$ 'because there is no end to the sum of segments' (Fischbein, Tirosh, and Hess, 1979). Tall explained that here it is the potential infinity of the limiting process that leads to the belief that any property common to all terms of a sequence also holds of the limit. In this case, the suggested limit is typical of all the terms: just less than 2.

In previous studies concerning the way students conceive rational and irrational numbers the infinite decimals were viewed as potentially infinite processes rather than as number concepts. In Sfard's words, *the new entity was not detached from the process which produced it*.

Monaghan (1986) observed that students' mental images of both repeating and non-repeating decimals often represent "*improper numbers which go on for ever*".

Kidron & Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximations ("*three digits after the decimal point are sufficient, otherwise it is not practical*"), or as a dynamic creature which is in an unending (a potentially infinite) process: in each next stage we improve the precision with one more digit after the decimal point. Let us summarize what we learnt from the previous researches with the process object theoretical framework: a. the students viewed the limit concept as a potential infinite process b. the students expressed their belief that any property common to all terms of a sequence also holds of the limit. This natural way in which the limit concept is viewed might be an obstacle to the conceptual understanding of the limit notion in the definition of the derivative function $f'(x)$ as $\lim_{h \rightarrow 0} (f(x+h)-f(x)) / h$. The derivative might be viewed as a potentially infinite process of $(f(x+h)-f(x)) / h$ approaching $f'(x)$ for decreasing h . As a result of the belief that any property common to all terms of a sequence also holds of the limit, the limit might be viewed as an element of the potentially infinite process. In other words, $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$ might be conceived as $\Delta y / \Delta x$ for a small Δx . How small? If we choose $\Delta x = 0.016$ instead of 0.017, what will be the difference? There is a belief that gradual causes have gradual effects and that small changes in a cause should produce small changes in its effect (Stewart, 2001). This intuition might explain the

misconception that a change of, say, 0.001 in Δx will not produce a big change in its effect.

The cognitive difficulties relating to the understanding of the definition of the derivative as a limit are reflected in the historical evolution of the concept. Kleiner (2001) suggested that before introducing rigorous definitions we have to demonstrate the need for higher standards of rigor. This could be done by introducing counterexamples to plausible and widely held notions. I was interested in a counterexample that will demonstrate that one cannot replace the limit " $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$ " by $\Delta y / \Delta x$ for Δx very small and that omitting the limit will change significantly the nature of the concept. The counterexample was found in the field of dynamical systems. A dynamical system is any process that evolves in time. The mathematical model is a differential equation $dy/dt = y' = f(t, y)$ and we encounter again the derivative $y' = \lim_{\Delta t \rightarrow 0} \Delta y / \Delta t$. In a dynamical process that changes with time, time is a continuous variable. Using a numerical method to solve the differential equation, there is a discretization of the variable "time". Our aim is that the students will realize that in some differential equations the passage to a discrete time model might totally change the nature of the solution. We also aim to help students realize that gradual causes do not necessarily have gradual effects, and that a difference of 0.001 in Δt might produce a significant effect.

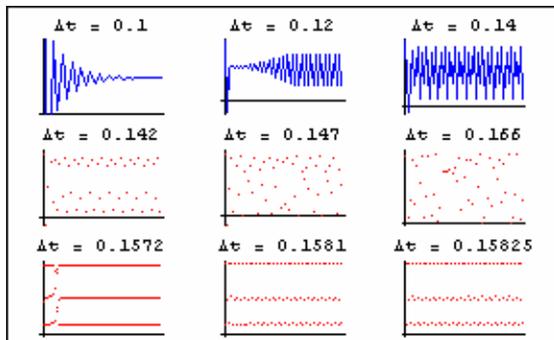
In the following counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. Moreover, using the analytical solution, the students use the concept of the derivative $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$. Using the discrete approximation by means of the numerical method the students use $\Delta y / \Delta x$ for small Δx . We will see that the two solutions, the analytical and the numerical, are totally different. The aim of the research study is to analyze the effect of this specific discrete-continuous interplay on the students' conceptual understanding of the limit in the definition of the derivative. Due to previous process object related researches I was conscious of the cognitive difficulties concerning the limit. This comes to light in the specific discrete-continuous interplay that motivated the design of the learning experiment.

2.1.1 The design of the learning experiment: Do gradual causes have gradual effects?

First year college students in a differential equations' course (N=33) were the participants in the research. The exercise sessions were held in PC laboratories equipped with MatLab software. The Mathematica software was also used during the lectures for demonstrations.

The students were given the following task: a point (t_0, y_0) and the derivative of the function $dy/dt=f(t, y)$ are given. Plot the function $y(t)$. The students were asked to find the next point (t_1, y_1) by means of $(y_1 - y_0)/(t_1 - t_0) = f(t_0, y_0)$. As t increases by the small constant step $t_1 - t_0 = \Delta t$, the students realized that they are moving along the tangent line in the direction of the slope $f(t_0, y_0)$. The students generalized and wrote the algorithm: $y_{n+1} = y_n + \Delta t f(t_n, y_n)$ for Euler's method. They were asked how to better approach the solution. They proposed to choose a smaller step Δt .

The logistic equation $dy/dt = r y(t) (1-y(t))$, $y(0) = y_0$ was introduced as a model for the dynamics of the growth of a population. An analytical solution exists for all values of the parameter r . The numerical solution is totally different for different values of Δt as we can see in the graphical representations of the Euler's numerical solution of the logistic equation with $r = 18$ and $y(0) = 1.3$. In the first plot, the



solution tends to 1 and looks like the analytical solution. In the second, third and fourth plot, the process becomes a periodic oscillation between two, four and eight levels. In the fourth plot, we did not join the points, in order that this period doubling will be clearer. In the fifth and sixth plot, the logistic mapping becomes chaotic. We slightly decrease Δt in the seventh plot. For

the first 40 iterations, the logistic map appears chaotic. Then, period 3 appears. As we increase Δt very gradually we get, in the eighth plot, period 6 and, in the ninth plot period 12 and **the belief that gradual causes have gradual effects is false!**

Students' reactions were observed by means of questionnaires.

Some reactions will be described in detail in relation to the influence of the two other theories on the analysis of the students' answers.

2.2 The instrumentation theory and its influence on the research study

Analyzing the influence of the process-object theory the emphasis is laid upon the cognitive difficulties that accompany the conceptualization of the limit. Analyzing the influence of the instrumentation theory the emphasis is on the way we take advantage of the discrete continuous interplay. It is obvious that in our research studies on the conceptualization of the limit in the years 1983, 1986 we did not have the same technological means. But numerical methods are not new. The founders of the mathematical theory developed numerical discrete algorithms to solve problems that relate to dynamical continuous processes. The old masters pointed out the importance of analyzing the error that arises from applying numerical processes. With the help of the CAS, we were able to apply Euler's algorithm to the logistic equation and to invite the students to analyze the accuracy of the discrete method. I wanted to examine the students' reactions when they realize that the approximate solution to the logistic equation by means of discrete numerical methods is so different from the analytical solution. My aim was that the students would realize that the source of the error resides in the fact that in the numerical methods the derivative is considered as $\Delta y / \Delta x$ for Δx very small, contrary to the analytical method in which the derivative is considered as $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$. Thus, the "instrument" plays a very important role. It enables us to "see" the significant difference between the discrete and the continuous methods. We can make advantage from the new potentials offered by the instrumented work, for example, by means of the discretisation processes. But, Artigue (2002) warns us that the learner needs more specific knowledge about the way the artifact implements these discretisation

processes. Thus, it is important to be aware of the complexity of the instrumentation process when we come to analyze the students' reactions in the research study.

By means of error analysis, I planned to help the students to better understand the continuous methods and the concept of limit. But, working with a CAS, there are other unexpected effects that are directly linked with the "instrument" and the way it influences the students' thinking. In addition to the error due to the discretization process, to the fact that an algorithm that belongs to a numerical method was used to solve the logistic equation in place of the analytical method, there are other sources of error that are directly related to the "tool". For example, an error could be a result of a round off in the computations that becomes more pronounced because of the "cumulative effect" of the iterative numerical method. The students' attention might be distracted by the round-off error especially if in previous experience with the computer they encountered such kind of round off error. This happened to a student, Hadas, which attributed the error to the round-off effect:

Hadas I remember from an exercise in the calculus course that the solution with Matlab was 0 but the solution using the symbolic form was 0.5. When we tried to understand why this happened we realized that MatLab computes only 15 digits after the decimal point.

Hadas referred to an episode in the calculus course in which the students were given the function $f(x) = (1 - \cos(x^6))/x^{12}$ and they had to explain why some graphs of f might give false information about $\lim_{x \rightarrow 0} f(x)$. The limit is $\frac{1}{2}$ but both Mathematica and MatLab give the answer 0 when we evaluate the function for $x = 0.01$. Working the exercise in the PC lab, the students understood that the computer with its limited precision gives the incorrect result that $1 - \cos(x^6)$ is 0 for even moderately small values of x . We have here a system of "double reference" (Lagrange, Artigue): mathematical meanings and meanings specific to the instrument. Therefore, there is a need to analyze the different sources of errors, and to help students differentiate between the error due to mathematical meanings, like the fact that the limit was omitted in Euler's algorithm, and the error due to the characteristics of the tool.

2.3 The abstraction and consolidation theory and its influence on the research study

Only a small percentage (30%) of the students wrote in their answers to the questionnaires that the source of the error resides in the fact that in the numerical method $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$ is replaced by $\Delta y / \Delta x$ for Δx very small. The students looked at Euler's algorithm and even those who previously defined correctly the derivative as a limit, did not succeed to identify the source of error in the algorithm.

Thus, for most students it might not be sufficient simply to introduce counterexamples to widely held notions. Taking also into account the complexity of the instrumentation process, I understood that I had to be more cautious in analyzing the students' answers to the questionnaire. Thus, it was decided to use new lenses to analyze the results in the present study and to analyze the knowledge construction by the individual. I decided to focus on process aspects of construction of the structure

knowledge rather than on outcomes (Hershkowitz, 2004). By analyzing each student's answers to the different questionnaires I aimed to examine the evolution of the student's thinking before and after being confronted with the counterexample.

In the following, I will analyze the answers of a student named Nurit to the questionnaires in terms of the theory of abstraction and consolidation. We will follow Nurit's *construction* of Euler's numerical method within which occurs the *consolidation* of the derivative concept as a limit. To the first question: "In Euler's method, if we attribute a very small value to the step Δt , for example - 0.02, can we be sure that we have a good approximation to the solution?" Nurit answered that *the smaller Δt , the better is the approximation*. In this first stage Nurit identified the limit as a process and that a small Δt may be not small enough.

Nurit was also asked to express her opinion about the following statement: "If in Euler's method, using a step size $\Delta t = 0.017$ we get a solution very far from the real solution, then a step size $\Delta t = 0.016$ will not produce a big improvement, maybe some digits after the decimal point and no more". This question was presented to the student before the introduction of the logistic equation. Nurit's first reaction was

It seems to me that if with $\Delta t = 0.017$ we didn't get a good solution, then $\Delta t = 0.016$ will not produce a big improvement.

Then she changed her mind:

That was my first impression but a second look at the expression for y_{k+1} in Euler's algorithm led me to the conclusion that the method is iterative, that is, on y_k we apply the algorithm in order to find y_{k+1} etc. etc. and after many iterations even a slightly smaller step size will produce a big improvement.

In this second stage, Nurit overcame the intuition that gradual causes have gradual effects by reflecting on the accumulating effect in the numerical solution. After being introduced to the logistic equation with the two different solutions, the analytical and the numerical, Nurit was asked to characterize the source of the error in Euler's method. At that point, she remembered an exercise on the sensitivity of some differential equations and realized that in the continuous approach too, small changes in a cause can produce large changes in its effect:

We have seen in an exercise that a change in the initial condition of a differential equation might cause a large change in the solution.

In this third stage, Nurit realized that small changes in a cause can produce large changes in its effect also without the accumulating effect and not only in iterative processes. But at that stage, the reason for the small change in the cause in the case of the numerical solution to the logistic equation was not clear to Nurit. Trying to identify the source of the error Nurit's first reaction was that the error is due to the round - off effect and the fact that the error accumulates. Then she changed her mind:

The source of the error in Euler's method is the way the derivative is defined $(y_{k+1} - y_k)/\Delta t = f(t_k, y_k)$ and by means of this definition we find y_{k+1} in Euler's algorithm. But we know that this definition is not precise. We have to add the condition that $\Delta t \rightarrow 0$ so we will know that

we are not dealing with the secant to the graph of the function but with the slope of the tangent. Because of the numerical method Δt was chosen as a small number $\Delta t=0.1$; $\Delta t= 0.12\dots$ but not small enough and in fact the derivative is defined for $\Delta t \rightarrow 0$.

Nurit reconstructed her knowledge about the definition of the derivative by means of interconnections with existing knowledge (the sensitivity of some differential equations) and intuitive ideas (gradual causes have gradual effects). Her process of construction led her to differentiate between the error due to mathematical meanings, namely, the fact that the limit was omitted in Euler's algorithm and the error due to meanings specific to the tool like the round off error.

Nurit *consolidated* her conceptual understanding of the limit concept in the definition of the derivative. She did it within her *constructing* of the error analysis in applying Euler's method to solve the logistic equation. Analyzing her lengthy process of error analysis we distinguish different phases. In the later phases her attention is no longer distracted by the accumulating effect of the numerical method, nor by the round off effect induced by the tool. She is ready to seek 'the reason for a small change in a cause' not only in error due to meanings specific to the instrument. This led her to seek for an error due to mathematical meanings. At the end of the consolidation process, Nurit is confident with her reconstruction of the limit concept and is also resistant to challenges

Now, it could be that there is also a round off error in the numerical method but a round off error by itself could not have a so big influence on the graph of the solution so that we will have a periodic oscillation between two levels instead of a solution that tends to 1. The error is due to the way the derivative is defined in the numerical method.

We recall that confidence is one of the psychological constructs that Dreyfus and Tsamir (2004) associated with the progressive consolidation of an abstraction. Other students in their process of construction of Euler's method did not consolidate the conceptual understanding of the derivative. For example:

Sarit In Euler's algorithm each step depends on the previous step; if we make an error in a certain step the error accumulates and is more pronounced in the next steps. I cannot say what is exactly the error but the accumulating effect is the problem.

Later Sarit related the error to the round - off error.

We should not see in Sarit answer the manifestation of some kind of cognitive inability. She and other students have to be faced with the necessity of developing schemes that will help them to differentiate between error due to mathematical meanings and error due to meanings specific to the instrument (Artigue, 2002).

3. The complementary role of the three theories

Due to the procept theory I was aware of the cognitive difficulties that accompany the limit concept. This helped me to clearly define the knowledge construct in the

research and to prepare the design of the learning experiment. The process object model was used in analyzing some students' reactions. Students' expressions like the following "*the smaller Δt , the better is the approximation*" reflect the way the limit is conceptualized as a potentially infinite process. We can observe the complementary role of the theories in the analysis of the students' reactions. The new lenses offered by the Abstraction and Consolidation theory enabled me to analyze the knowledge construction by the individual. It facilitated my understanding that it is within the process of construction of Euler's method, that the consolidation of the definition of the derivative as a limit took place. But I had to analyze the way students achieved this process of construction in the light of the instrumentation theory: The roles of the abstraction & consolidation theory and the instrumentation theory intertwined.

I will conclude with the question:

What is specific to the subject of the research study that demands the contribution of more than one theoretical approach on the research process?

Trying to answer this question we observe the dual character of the limit as a process and as a concept (**the procept theory**) but we may also consider the discrete continuous interplay that is the basis of the definition of the derivative as it is expressed in Berlinski (1995): "*In making possible the definition of the derivative, the concept of a limit unifies in a fragile and unlikely synthesis two diverse aspects of experience, the discrete and the continuous*".

In the research study we used the discrete – continuous interplay to help students conceptualize the notion of limit. To achieve this goal, the students were asked to compare continuous and approximate discrete methods in solving the same problem. The investigation of this specific usage of the discrete – continuous interplay in the research process demands the contribution of **the instrumentation theory**.

The students learned the notions of limit and derivative in the calculus course. In the research study with its specific usage of the discrete – continuous interplay, the students had to reconstruct the limit concept by recognizing it in the different context of a course in differential equations. This kind of reconstruction is the consolidation phase in **the abstraction and consolidation theory**.

This specific research study demands the contribution of more than one theoretical approach to the research process. The question whether one can take advantage of the combination of different theories in other research processes in mathematics education is an issue open for further investigation.

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THEORIES AND EMPIRICAL RESEARCHES: TOWARDS A COMMON FRAMEWORK

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Abstract: *Some examples of relationships between empirical and theoretical researches are introduced. A new frame is defined, which is based on the notion of second order variables and on that of space of action, production and communication (APC space). This allows a unifying approach to some didactical phenomena. The APC space is illustrated through a concrete example. The frame is then compared with the anthropological approach of Y. Chevallard and used to discuss the notion of ostensive/non ostensive objects.*

Introduction

The paper gives some examples of how empirical studies may contribute to the development and evolution of theories and, conversely how specific research paradigms may influence empirical research. The global result of such interactions consists in a deeper understanding of the didactical phenomena within an enriched and unifying theoretical frame. Specifically, we shall consider the following problem, which concerns the gap between theory and practice in Math Education:

Data from empirical research very often are difficult to discuss and interpret within a single theoretical frame. They appear contradictory and only partially in accord with theories.

In the literature the problem has been discussed by many people, e.g. by Steiner (1985) and Arzarello & Bartolini Bussi (1998). Steiner has elaborated the notion of *complementarity* of theories to interpret empirical data, which do not fit into a single frame. Arzarello & Bartolini Bussi have introduced so-called *second order variables* to elaborate further the notion of complementarity. In this study we will enlarge the notion of second order variable and will introduce the construct of a *space of action, production and communication* (APC-space in short). The APC-space requires specific and different, complementary, magnifying lenses of observation to interpret didactical phenomena, according to the complementary approach of Steiner. However it represents a concrete unifying frame at the cognitive level, that is at a level where the empirical research and the theoretical frames fit together. We will show how our approach fits and possibly enlarge also theoretical frames elaborated according to different philosophies, e.g. the anthropological approach by Y. Chevallard (Chevallard, 1992).

The paper is divided in 4 parts. In §1 we will summarize the notion of second order variable and illustrate it with an important example, the *didactical times*. In §2 we shall introduce the APC-space and illustrate sketchily one of its components, namely *gestures*. In §3 we shall sketch the dialectic *ostensive/non ostensive objects*, recalling how it is discussed within the frame of the anthropological approach and shall show how it can be interpreted within our frame. A short Conclusion ends the paper.

1. Second order variables

Didactical phenomena concern complex systems. We use the distinction between *first and second order variables* to discuss them. This is a typical theoretical construct that must be investigated through suitable observation tools. The terminology is borrowed from logic: first order variables are theoretical constructs which, within a certain context, are considered as simple enough to be irreducible to simpler ones, while second order variables are more complex relationships between two (or more) first order ones. We have argued elsewhere (Arzarello & Bartolini Bussi, 1998) that a relevant research study for innovation in mathematics classroom must deal with second order variables. It is important to say that a second order analysis does not simply combine the first order components like in a jigsaw, but these variables are related to one another within a system. The idea comes from Vygotsky (1992): he advocates for the necessity of studying single components of a phenomenon, which still have the same characteristics of the global phenomena, without reducing the phenomena to too fine components which have lost the global features. In the same way, in research in Mathematical Education it is worth studying conceptual hierarchies for mathematics and for social interaction within a context in which they interact.

Examples of first order variables are gestures, speech, writing, signs, and so forth, in relation to students' and teacher's activities in the classroom. Within a certain grain of analysis they constitute important atomic components in learning processes. But it is only considering the mutual interactions among them that we have a more realistic picture and a deeper analysis of such activities. Namely a second order analysis is necessary. An example of such an analysis for those component has been developed by L. Radford, who has introduced a second order construct to give count of the relationships among the first order variables above, namely the *semiotic means of objectification*:

This led us to envisage a broader context large enough to conceive of tools, things, gestures, speech, writing, signs, and so forth, in relation to the individuals' activities and their intentional goals. In this broader context, we called semiotic means of objectification the whole arsenal of intentional resources that individuals mobilize in the pursuit of their activities and emphasized their social nature: The semiotic means of objectification appear embedded in socio–psycho–semiotic meaning-making processes framed by cultural modes of knowing that encourage and legitimize

particular forms of sign and tool use whereas discarding others. (Radford, 2003, p.44).

Other examples of first order variables are the *ostensive* and *non-ostensive* objects introduced by Bosch & Chevallard (1999). They will be discussed in § 3.

Further examples of first order variables are the different didactical times existing in the classroom phenomena, typically the *inner time* and the *physical time*. Let us discuss these two and the reason why only a second order analysis can deepen their meaning in learning processes: for more details see Arzarello et al. (2002) and also Lemke (2000). Time reveals as a very complex second order variable that must be analysed through different tools. In fact, the temporal development of didactical phenomena in the classroom takes place with different velocities and some particular phenomena may be fully understood only if what happens in the micro-time is connected to what happens in the macro-time. The temporal development of didactical phenomena can be grasped only through an analysis, which involves both first and second order variables. They need to be studied according to both a fine and a global analysis: so they require the use of a number of different research methods, always complementary, sometimes contradictory. On the one hand there are long-term processes, typically connected to innovative “problematiques”, based on the analysis of the evolution of students’ processes, of teachers’ beliefs, etc. in macro-situations. On the other hand, there are short-term processes, based on the fine analysis of micro-situations. Moreover in the class there is the physical time, i.e. the linear sequence of moments measured by the clock, as described by Varela (1999). Yet, Varela himself calls our attention to the existence of an ‘inner time’, that gives rise to human temporality, centred on the present and manifesting itself as a threefold unity of the just-past and the about-to-occur. This inner time is mostly individual and unconscious. However its features may be inferred from external traces (linguistic expressions, metaphors, gestures, glances).

In Arzarello et al. (2002) it is shown that: (a) Both kinds of time (i.e. physical and inner time) are relevant in the research in Mathematics Education, when focus is on the processes of teaching and learning mathematics. (b) A further finer specification of both is needed, that requires the introduction of several theoretical constructs related to human temporality and that puts forward a lot of methodological problems concerning the relationships between them. (c) Last but not least, the system of time variables shows deep connections with mathematical and linguistic components.

Despite their importance, as far as we know, little attention has been given in the literature on mathematics education to the variables concerning time, maybe diverted by the belief that time is not so important, because the final, accepted product of mathematical activity has usually a de-timed structure. On the contrary, when one speaks of time in the classroom he/she is using a word which refers to a typical second order variable.

2. The APC-space

In the previous paragraph we have argued why didactical phenomena can be described suitably only looking (also) at second order variables. The cognitive space of action, production and communication (APC space in short), introduced in Arzarello (in press), is the space where such second order phenomena concretely live.

As we recalled above, Vygotsky (1992) warned about the hope of classical psychology of getting a knowledge of something by studying only the components separately and suggested an useful metaphor: it makes no sense to study separately hydrogen and oxygen in order to study water since they do not have the properties of water. APC space is a concrete realisation of this metaphor for math education. In fact, only considering its components in interaction one can get a realistic picture of what happens in the class of mathematics. This construct is the result of both theoretical and empirical research. It is an environment for cognition, which may be built up, developed and shared in the classroom. It is a typical second order construct, namely an integrated and dynamic set, which acts as a whole. Its main components are: the body, the physical world, the cultural environment. When students learn mathematics all these components (and possibly others, e.g. emotional ones) are active and interact. The APC-space is built up in the classroom through the interactions among pupils, the mediation of the teacher and possibly through interactions with artefacts. A finer analysis of its three components allows to define its ingredients, namely culture, sensory-motor experiences, embodied templates, languages, signs, representations, and so on. The three letters A, P, C illustrate its dynamic features, namely the fact that in learning mathematics there are three main components: students' acting and interacting (in the situation, with mates, with the teacher, with oneself, with tools), their productions (e.g. the answer to a question, other questions, and so on), the communication aspect (e.g. when the discovered solution is communicated to a mate, to the teacher, using suitable representations). These aspects of the didactical phenomena have been pointed out for many years by many people, e.g. by G. Brousseau (1997). The APC space contains as an integrated whole its components, which all are essential and always active in all forms of learning, even the most abstract ones. As such, the APC-space allows to study properly the perceptuo-motor features in processes of knowing. In fact any such a process involves action and perception: learning is often based on doing, touching, moving and seeing. Such features do not only characterise the first phase of cognitive development, but are also involved in the most advanced learning processes.

This shifting in the approach to learning has been pointed out by many studies in the field of neurosciences in these last years (see Lawson, 2003, and the quotation at the end of this paper) and is summarized by R. Nemirovsky (2003, p. 1-108) as follows:

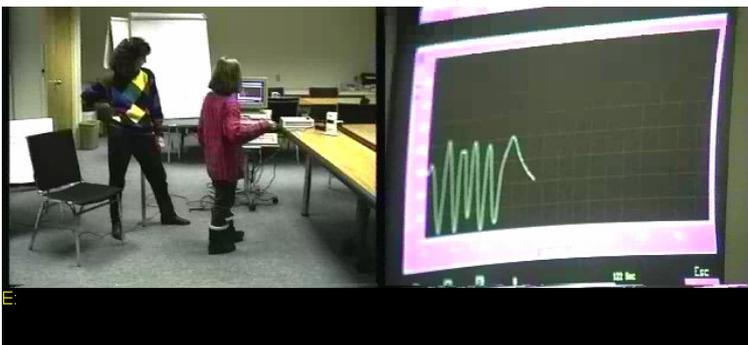
- a) Mathematical abstractions grow to a large extent out of bodily activities having the potential to refer to things and events as well as to be self-referential.

- b) While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the subject itself. [Moreover,] that of which we think emerges from and in these activities themselves.
- c) [As a consequence,] the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context.

This approach challenges the traditional one mainly based on the transmission of contents through formal language. This point is discussed widely in Antinucci (2000), who contrasts this approach to what he calls the symbolic-reconstructive one. The first approach, which is present from the beginning of the cognitive development of the child, works on symbols (linguistic, mathematical, logical) and reconstructs “objects”, their meanings and mental representations, in the mind. It is a sophisticated way of knowing and requires awareness of the procedures and the appropriation of the symbols used and their meanings. In this regard, we note that "traditional" teaching in mathematics, which is usually characterised as “transmissive”, is based, almost exclusively, on a symbolic-reconstructive approach. According to the “transmissive” model, the teacher tries to put the student in contact with mathematical objects by means of the use of techniques which require high competencies in order to reconstruct, in the student’s mind, the properties which characterise them. We observe, moreover, that this type of approach, disconnected from the construction of a rich experiential base, can create obstacles to learning. For example, interpreting or explaining a mathematical concept, without having created proper experiential conditions, generally produces resistance and confusion in learners. The three components of the APC-space allow to consider both the symbolic-reconstructive and the perceptuo-motor way of learning within an unifying environment where all such processes develop. It is a second order space, in the sense that such components do interact in a systemic and intrinsic way. An example taken from a case study developed by the research team of R. Nemirovsky (Nemirovsky, R. et al., 1998) will illustrate this (other examples are in Arzarello & Robutti, 2004, and in Arzarello, in press). In the study a young girl, Eleanor, makes experiences with a motion sensor that allows to measure her distance from a fixed place. Distances Vs/ time are recorded in a Cartesian graphic which appears in real time on the screen of a computer. Eleanor can see the screen and move the device as she likes. Doing that, she enters into some of the many different meanings of a function. Of course only the video can show completely what is happening, insofar not only Eleanor’s words but also her gesture and body motions are important for understanding the multivariate way in which she is able to build some of the meanings of a function. Namely to grasp what is happening you need a second order approach. In Arzarello (in press) it is shown that Eleanor’s learning can be described properly using the APC-space: its components and ingredients are all

active and interacting while Eleanor is learning. Considering only one or the other means to make it impossible to see the concrete learning process while it is happening. To have an idea of this one can look at the following excerpt from that protocol (E = Eleanor; T = Teacher):

12. E: Let's see...I wonder if you could get it to go straight up? [she follows the graph very fast with her forefinger]
13. E: Not like diagonal. Probably you couldn't because if it would go straight up it would have to just be the same time, because it's moving along [she makes with her hand an horizontal movement on the screen across the graph]
no matter what you do
14. T: Right, it's...moving along in time?
15. E: Yeah. So you'd have to kind of stop the time and go like that.
[with her taut arm, E points the forefinger to the screen and produces the form of the graph on the screen]
And go like this. [E moves back and hints the movement she had done previously]
Because, because it's moving along that way or this way the same time.
16. E: It's going that way. So it kind of goes like, instead of just going like this...
[she makes a vertical movement on the screen with her forefinger]
... it kind of goes like that probably this. [she makes a slow oblique movement on the screen with her forefinger]
17. T: Do you think you can make a steeper line than this? Maybe you can't make it go straight up but maybe you can make it a little bit ...
18. E: May be, maybe if you do it faster"
19. T: OK, shall we try that?



20. E: No, I'm not going to worry about like...
[first E runs twice back and forth, then she stops and continues moving only the arm back and forth twice...]
...and if you just go slowly
[then she runs again but very slowly...see figure]

The excerpt illustrates how Eleanor can experience the concept of function as a model of her motion: interacting with the device through her motion and discussing with her teacher she has realized the relationships between the geometric properties of the graph and the properties of her motion. For example the inclination of the line as an index of her speed (#18, #20). The discussion shows that she is able to enter into concepts in a deep way. See for example the discussion from #12 to #16, where Eleanor elaborates essentially the idea of slope as speed and as distance over time; she can do it because she is pushed to interpret the vertical lines in the diagram by the questions asked by the teacher. See also the lines #18 up to #20, where she tests her conjectures about the

relationship between slope and speed, disregarding the inessential variable of the form of the graph and concentrating only on its slope. What we wish to stress is that Eleanor has been able to enter into some of the various representational facets of a function in a multivariate way, with her body, through cultural and physical tools. All this may seem very far from the usual definition of function that we find in the books of mathematics. But it is exactly this experience to allow Eleanore to enter into (some of) the mathematical aspects of function's concept. The APC space gives reason of the complex way according to which this can happen, namely puts forward all the components through which learning develops. Eleanor's learning may happen since all these components are active as a whole. In other words, learning processes are second order processes and APC-space models them.

The APC-space requires complex and different tools of analysis, in order to isolate the second order components which constitute it: for example the analysis of the structure of pupils oral and written productions or the fine analysis of the different didactical times within which the didactical phenomena develop, as sketched in §1. Another example is given by the analysis of gestures in students who solve mathematical problems. Our team in Turin has collected many examples of this kind: see Arzarello & Robutti (2004), Arzarello (2004), Ferrara (2004), Sabena (2004). Gestures are a typical ingredient of the APC-space: as the analysis of Goldin-Meadow (2003) shows, gestures are typically intermingled with speech and, together with language, help constitute thought. This happens for all scientific learning, as it is shown by the work of Wolff-Michael Roth, who analyses gestures in students who learn science: see Roth (in press) and the quoted literature. As such, they enter into a second order variable, that all people, especially students and teachers, do 'hear', possibly in an unconscious way. More specifically, gestures may be a thinking tool; may have explorative, anticipative and organising functions; moreover they may have social features. They belong to the peripersonal space of people who are interacting and may contribute to the dialectic of the social construction of knowledge, provided the teacher encourages gesturing in the class. Students' gestures may surrogate or integrate the role of instruments in the conceptualisation process. In fact they can produce virtual objects in students' peripersonal space, which they 'manipulate' and with which they carry out mental experiments. Generally, gesture is not considered as an essential part of the mathematical activity. But it is not so: this has been pointed out also by Bosch & Chevallard (1999) within another frame (i.e. that of the so called anthropological approach), as we shall see in the next paragraph.

3. The ostensive objects enter the APC-space

A typical didactical problem faced by theoretical and empirical research consists in the dramatic loss of meaning that mathematical formulas present in many students, who conceive them in a purely syntactic way. This problem has been afforded by Bosch and Chevallard (1999, B&C in short), who introduce the dialectic between "ostensifs" and

“non ostensifs”. According to our terminology, which of course is not that of the authors, their study is a typical second order approach. It contrasts the objects that can be perceived and manipulated, like sounds, graphemes, gestures with those that have an abstract status, like ideas, intuitions, concepts. B&C underline that the mathematical activities can develop only through a plurality of “registres ostensifs”:

“...[le] registre de l’oralité, registre de la trace (qui inclut graphismes et écritures), registre de la gestualité, enfin registre de ce que nous nommerons, faute de mieux, la matérialité quelconque, où prendront place ces objets ostensifs qui ne relèvent d’aucun des registres précédemment énumérés.” (p. 96) (¹).

Such registers are intertwined: B&C underline the multiplicity of relationships between written ostensive and oral objects. They point out the “individual micro genesis of techniques for solving specific problems” (p. 104), that is a process that starting from ostensive objects (in discursive, gesture, graphic, written form) ends with stable techniques. B&C call such a reduction “chirographique” (from the Greek χειρ, hand): it consists in the “transfer of gesture and material objects to the oral and graphic registers” (p. 105). For example they analyse gesture and speech, which accompany the accomplishment of matrix product. In the end, these ostensive objects are integrated in new mathematical objects, represented through the algebraic formalism, where each trace of gesture and oral activity is eliminated. This is at the root of a paradox. On one hand the genuine mathematical job seems to consist in “pure computation and pure syntax”, namely in typical first order variables (according to our frame) that live in the timeless world of pure mathematics. The other ostensive aspects, which are embedded in the stream of time and involve second order relationships, do not seem to acquire a clear mathematical status. On the other hand, it is exactly this private, second order component of the mathematical activity (e.g. gesture, speech, and so on) that seems able to give meaning to the official mathematical formalism. B&C thus describe “a model of mathematical activity that integrates the ostensive objects as basic components of the mathematical knowledge” (p. 119). Within our frame, this approach is a typical theoretical second order analysis, where the ostensive, non/ostensive components are the first order variables, while their relationships are the second order ones. B&C say explicitly that it is necessary to consider both components. In our language they are typical ingredients of an APC-space; namely, they are perceptuo-motor, second order ingredients, which are necessary to give sense to mathematical ideas and signs (B&C call this “de-mathematisation of the activity”, p. 107). Such ingredients do live together in the genesis of algebraic signs, as described in Radford (2003, p. 64) with great care:

“For as long as a sign system, S_1 , is still heavily dependent on other sign systems, S_0, S_2, \dots , from which S_1 arises, iconizing, pointing or other indexical devices

¹ “...the *oral* register, the *trace* register (which includes all graphic stuff and writing products), the *gesture* register, and lastly the register of what we can call the *generic materiality*, missing a better word, namely the register where live those ostensive objects that do not belong to any of the registers above”

play a fundamental role in ensuring the connection between the emergent system S1 and the source systems S01, S02, and so on. The aforementioned semiotic connection between the emergent and the source sign systems, I suggest, is what happened during the sprouting of algebraic language in the classroom. The link relating the algebraic letter symbols to the students' actions serves as the semiotic means of objectification underpinning the students' production of signs. This link makes indexes meaningful. Delete the action and the sign will lose its semiotic power and become an unrecognisable hieroglyphic-like mark."

APC space allows to consider the indexical components in the processes of productions/use of signs and meanings, i.e. in the productions/use of *ostensifs* and *non-ostensifs* in the language of B&C. In fact such indexical features are typical second order components that are the base for the integration of the ostensifs within mathematical knowing processes. Such an approach underlines further the necessity of considering all the components (body, physical world, culture) of such an integration as an environment where the processes of conceptualisation and generalisation happen. As such it enlarges the frame of B&C.: it allows to consider physical and cultural objects in a wide sense, e.g. the interactions with the instruments, as discussed in Arzarello (in press).

Conclusion

In this paper we have sketched a common theoretical frame, the APC-space. It provides a framework within which the complex relationships among the variables that feature the didactical phenomena can be efficiently described. It is based on the notion of second order variables and contains many cognitive aspects, like gesture, glances and so on, which generally do not enter into the official list of mathematical objects. On the contrary, the meaning of the mathematical concepts in the classroom is especially rooted on these perceptuo-motor aspects. The results discussed here illustrates the reciprocal influences between empirical research, theoretical paradigms and back. It has been the possibility of analysing carefully the videos of what happens in the classroom through the new technologies that has allowed us to see phenomena that happen on the spot and that a video looked at in the usual format could not show (e.g. gestures which last less than one second, or the synchronization of the order of a few hundredths of second among gestures, speech and glances). The idea of APC-space has sprouted out from the necessity of giving reason of phenomena which need a different grain of observation but must be considered together. On the other hand, the elaboration of a theoretical construct has pushed our empirical research towards the designing of teaching situations where the different components of APC-space could be accessible to the students, e.g. the approach to functions through models of moving objects and devices (see Arzarello & Robutti, 2004). As we have underlined above, this approach represents a deep shift in the definition of the meaning of concepts, which in these years comes from neurosciences. Hence I conclude with the following quotation from a neuroscientist, which can be seen as a comment about this

shift“Representational content, and thus –a fortiori– conceptual content, cannot be fully explained without considering it as the result of the ongoing modelling process of an organism as currently integrated with the object to be represented, by intending it. This integration process between the representing organism and the represented object is articulated in a multiple fashion, for example, by intending to explore it by moving the eyes, intending to hold it in the focus of attention, by intending to grasp it, and ultimately by thinking about it.” (Gallese, 2003, p.1236)

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COMPARISON OF DIFFERENT THEORETICAL FRAMEWORKS IN DIDACTIC ANALYSES OF VIDEOTAPED CLASSROOM OBSERVATIONS

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Abstract: *The aim of the research described in this submission was to compare different theoretical frameworks among the most used in France, in order to analyse the students' mathematical activity. For this research, we have chosen to consider two different situations: the first one has been elaborated with a researcher in the framework of an engineering design relying on a precise didactic theory; a pre-service teacher has built the second one. This study shows the potentialities of the didactic tools, at our disposal today, to analyse research situations or ordinary situations. Furthermore, our research poses the question of the connection between theoretical frames and of their complementarities.*

Keywords: Comparison of theoretical frameworks, students' mathematical activity, teacher practice.

I- Introduction

We present here a research project (Artigue, Lenfant, Roditi, 2003) whose aim was to question the potentialities of three different theoretical frameworks among the most used in France: the tool-object dialectic and interplay between settings¹ (Douady, 1986), the theory of didactic situations² (Brousseau, 1997), the ergonomic and didactic approach³ (Robert, Rogalski, 2002), in order to answer the following questions set up in the frame of a national project: when does a student do mathematics? How do mathematical or didactic organisations influence the student's mathematical work? How do they favour it or, on the contrary, thwart it?

¹ R.Douady emphasised the difference between the status of "tool" of a mathematical notion (when it's used to solve a problem) and its status of "object" (when it's presented in a general way). Mathematical notions appear generally in their dimension of tool before becoming an object. The framework developed by R.Douady takes this characteristic into account. The tool-object dialectic is made up of five phases which structure the development of new concepts from former knowledge. She defines a "setting" as a set of objects of a mathematical field, of the relationships between these objects and of their different formulations. The interplays between settings have an important role in the implementation of the tool-object dialectic because the interpretation of a problem in an other setting can allow to advance the solution. Thus they appear as privileged levers to cause the elaboration of new knowledge.

² The theory of didactic situations has two objectives: on the one hand the study of the consistency of the objects and their properties necessary for the elaboration of didactic situations, on the other hand the scientific confrontation between the adaptation of these models and their characteristics with the contingency.

³ This framework relies on researches emerging from the ergonomic psychology and the didactics of mathematics. Its objective is to analyse and to understand teachers' practices.

Indeed each theoretical framework influences the ways in which the "reality" is questioned and studied, and therefore the knowledge about teaching and learning mathematics which can be developed. Each theoretical framework also conditions actions which can be considered in order to try to improve teaching and learning, on the basis of this knowledge. So it seems fundamental to understand how these theoretical frameworks shape the didactic work and to examine if the different viewpoints they offer can be seen as complementary perspectives which can be at least partially linked together, or have to be considered as perspectives which are mutually exclusive.

In this article, we present the main choices made in this research project and we synthesise the results that we consider the most interesting ones within the perspective expressed above.

II- Our choices

For this research, we have chosen to consider videotaped classroom observations. We began to work on two videos filmed in the classrooms of experienced teachers. The two videos dealt with the same mathematical topic: the study of the sign of polynomial functions with grade-10-students. The objective of the videotaped sessions was to introduce a precise object of the French curriculum: the table of sign⁴, and to establish this table as a particularly efficient way to condense information about the sign of a polynomial function and to make it visually accessible. These sessions had been developed in a precise theoretical framework: the tool-object dialectic. So it was particularly interesting to confront the analysis made during the conception of the sessions and analyses carried out in other theoretical frameworks. In our research work, we thus develop a new analysis relying on the theory of didactic situations, and compared its results with those provided by the initial one.

The mathematical and didactic organisations of these sessions didn't correspond to an ordinary classroom session: there was first a one-hour work in small groups, and then a collective synthesis having about the same duration. So, in a second phase of our research project, we decided to integrate a new set of data to this corpus: a video filmed in the classroom of a pre-service teacher. This time, the focus of the session was the first meeting with the general notion of function, in the context of a problem of variations posed in a geometrical context. The didactic organisation was absolutely different from that of the first sessions. In order to understand this organisation, its coherence, in order to understand the students' mathematical work in its relation with the teacher's work, we were interested in introducing another theoretical framework.

⁴ It's a table used to determine the sign of a polynomial function, like in the following example :
Study of the sign of the function $f(x) = -x(x + 1)(3 - x)$

x	-1	0	3		
-x	+	+	0	-	-
x + 1	-	0	+	+	+
3 - x	+	+	+	0	-
f(x)	-	0	+	0	+

Two theoretical frames were thus used for this second corpus: the theory of didactic situations and the ergonomic and didactic approach.

III– The first situation

III.1– Presentation

This situation is the first in an engineering design for grade-10 students developed by R. Douady, included in a research project about the learning of polynomial functions. This engineering design uses the interplay between different settings: the numeric, algebraic and functional settings (with the meaning given to this term by R. Douady (Douady, 1986)), as a lever for the development of mathematical knowledge. The objectives of the first part of the engineering design are:

- to study the relationships between the sign of a polynomial function $P(x)$ according to the values of x and the number, values and multiplicities of the real zeros of this polynomial,
- to establish that the table of sign is an efficient way to condense information about the sign of the function P and to make it visually and quickly accessible.

In the first session, different factorised expressions⁵ are proposed to students working in groups. Each group has to answer the same questions. Two phases are planned:

- Phase 1: (example) "For different values of x , calculate numeric values of the following expression: $f(x) = (x - 2)(2x - 3)(x + 5)(4x + 1)(1 - x)$. Are the results always positive? Are they always negative? Are they sometimes positive, sometimes negative? Calculate!"
- Phase 2: "Find a way, which allows you to determine quickly and surely the sign of an expression when your teacher gives you a numerical value for x ".

The aim of the first phase is to make students note that the sign of the different expressions is not constant, and thus give meaning to the problem set in the second phase. In this second phase, as already expressed, the table of sign is expected to appear as an efficient way to solve the task at stake, and take thus the status of an "implicit tool".

III.2– A first analysis relying on the tool-object dialectic

For this a priori analysis, we have used the tools provided by the tool-object dialectic and the interplay between settings. This framework led us to investigate the strategies that a grade-10 student could develop to solve the problem, and to question the respective efficiency of these strategies. It also led us to discuss the hypothesis articulated just above about the table of sign. We focus here on the second phase of the session. For this phase, three settings can be considered: a numerical setting, an algebraic setting and a functional setting. Let us specify the main possible strategies in each of these:

⁵ In order to choose these expressions, different didactic variables have been considered: the number of factors or the values of the roots (sign, nature), for example.

- In the numerical setting, students can determine the sign for different values of x and try to notice regularities. This strategy is in continuity with the work of the first phase, but it does not allow to solve the problem. Indeed, we can hypothesize that students will only take simple values for x , and especially small integer values. This can only lead to partial or erroneous results as all the polynomials proposed have zeros which escape this category.
- In the algebraic framework, students can determine the sign of each factor of the expression by solving the corresponding inequality and then use the rule of signs for a product to conclude. This strategy is underlying the table of sign, but this does not ensure that the students will build such a table. It is more likely that they use representations on the number line (the favoured representation used in junior high school in France to summarise the information about the solutions of inequalities). Students can also notice that the sign of their expression only changes when one goes through a zero of the expression, look for the zeros by solving the equations associated to the different factors, and then either simply alternate the signs from one interval to the next one, or calculate the value of the expression for a given x in each interval.
- In the functional framework, several strategies mobilising the graphic register can be considered: representation of straight lines associated with each factor of the expression and interpretation on the one hand, construction point by point of a graphic representation of the function on the other hand. The first strategy can be expected because, in grade 9, students have solved systems of two linear inequations with graphic techniques. But this strategy will not necessarily succeed because the expressions have a lot of factors, which makes the interpretation of the resulting graphic representation very complex. The second seems less probable because the observed students are not experienced with the work on functions, beyond the linear case, and are not allowed to use graphic calculators during this session.

In conclusion an algebraic resolution should be a source of winning strategy and it should allow the necessary transition between the initial pointwise vision and the global vision by intervals. Then the table of sign could be introduced by the teacher as a convenient way of representation to summarise at the same time the phases of the algebraic procedure and its result. But one cannot ensure that all groups of students will leave spontaneously the numerical setting, even if unsuccessful. Indeed a lot of researches have shown that students do not easily change the setting in which they initially approach a problem.

The theoretical framework of the tool-object dialectic also led us to investigate which knowledge could be built in the interaction with this problem. The first phase should lead the students to the conviction that the sign of the expressions is not constant. We can also hope they will note that the expressions keep a constant sign on some intervals. More elaborate knowledge can appear if the students work in the algebraic setting or in the functional one: relationships between the boundaries of these

intervals and the zeros of the expression, for example. But, as we said before, this is not sure.

III.3– The effective realisation

We give below some elements of the effective realisation in one of the two classrooms. The first point we would like to point out is that the three settings mentioned above spontaneously appeared, as evidenced by the presentations made during the collective synthesis:

- A first group, working on the expression $x(2x - 3)(1 - x)(x + 1)$, stayed in a numerical setting. This allowed these students to get a very partial conclusion: for $x > 1.5$, the expression is negative.
- A second group, working on the expression $x(7 - 3x)(5x - 3)$, developed a strategy in the algebraic setting based on the resolution of the equations. First they presented their results like that :

$x < 0 \rightarrow +$	$x = 0 \rightarrow 0$
$3/5 > x > 0 \rightarrow -$	$x = 3/5 \rightarrow 0$
$3/5 < x < 7/3 \rightarrow +$	$x = 7/3 \rightarrow 0 \dots$

Then, using the graphic semiotic register, they put the found values on the real number line and coded the sign of the expression on the intervals associated to these values. An analysis of their production shows that they worked in a pragmatic way, taking several values to check but without reasoning algebraically on the sign of the different factors.

- A third group, working on the expression $(4x + 1)(3x - 6)(7 - 3x)$, worked in the functional setting. They tried to represent graphically the associated function. For that, they first chose four numbers randomly. They thus obtained three points above the x -axis and one below and joined these points with a regular curve. This drew their attention to the zeros of the function and they calculated these by solving the equations associated to the three factors. In fact their graphic was not correct and their work stopped here without leading to a clear conclusion.

More globally, the data collected show the pertinence of the *a priori* analysis carried out with the tools of the tool-object dialectic and the efficiency of these tools to anticipate the possible mathematical work of students faced with this problem.

In a second phase, we explored the possible connections with an analysis referring to the theory of didactic situations (TDS in the following).

III.4– The connection with the theory of didactic situations

We exploited this theory in order to study the potential work of students in the phase of research on the one hand and in order to analyse the role of the teacher on the other hand.

As regards the first point, using this framework led us to evaluate the capacity of the situation to reveal the table of sign as an optimal answer to the problem in an a-

didactic functioning. If we use the language of the theory of didactic situations, the first phase of the situation has a function of devolution: thanks to it, the determination of the sign of the polynomial function becomes a mathematical problem for the student. This phase can also be used to check that the knowledge necessary to a productive interaction with the 'milieu' is available. In the second phase, the pointwise strategy mobilised in the first phase acts as a 'basic strategy'. But, as already pointed out, this cannot ensure complete success. In this particular situation, we see that the teacher plays an active role in the 'medium' thanks to the possibility she has of choosing the numbers which serve to check the results obtained by the students. Except for this fact, we are in the classical case for the TDS. There is a basic strategy; this is not appropriate and a new strategy has to be developed. Does the interaction with the 'milieu' allow it? This is a fundamental question in the TDS. The analysis to carry out for answering this question is very close to the first analysis we have presented above: identification of possible strategies, evaluation of their cost and their efficiency in the perspective of an a-didactic functioning. And it leads to the same conclusion.

The teacher plays thus a crucial role in this situation for linking what can be achieved by the students in the research phase to the official knowledge aimed at. What is offered by the two theoretical frames we have used for approaching this dimension of the observed session? We have to confess that we did not find explicit and specific tools in the tool-object dialectic. As regard the TDS, we found some help in more recent developments of the theory than those evoked so far, and especially the vertical structuration of the 'medium' (Brousseau, 1997), (Margolinas, 1998, 2002). This is not easy to explain in a few words. In it, a situation appears as a complex of imbricated structures, each situation becoming the 'milieu' for the upper one, in a kind of reflective process. At the bottom of the structure, there is the 'objective-situation' (named 'S minus 3' and noted S_{-3}), in which the 'material-milieu' takes place. In our case for instance, it contains the expression at stake and the numerical results obtained in the first phase of the session. In the next step of the structure, the reference situation (S_{-2}), the student (E_{-2}) interact with S_{-3} in order to try to solve the problem. The 'milieu' of S_{-2} become enriched with new objects. This situation is itself included in the 'learning situation' (S_{-1}) whose 'milieu' contains the results obtained in S_{-2} . The student E_{-1} is modelled there in a reflective position, emitting conjectures and seeking to validate them. Finally the didactic situation situates at the level S_0 . At this level, the knowledge worked out in the groups, takes a public status and is connected to the official knowledge. In our particular session, S_0 can be associated with the phase of collective synthesis⁶.

We used this framework in order to study the work of the teachers during the research phase of the session and the collective synthesis. It allowed us to identify and describe some interesting phenomena and regularities. For instance, this analysis revealed the following regularity in the practice of one teacher during the collective

⁶ The structure goes on with positive levels S_1 and S_2 . For limiting the complexity, we do not evoke these here.

synthesis. When a group was asked to present their work, the teacher systematically put it at the S_2 level, and then tried to exploit the milieu M_1 resulting from this situation in order to make the group go a little further than in the research phase. She ended every such episode with a short institutionalisation phase at the S_0 level. For example, she helped the first group, which had used a pointwise numeric strategy, to move from this vision to a more global vision bringing into play intervals. Especially adapted to each group, this strategy allowed the progressive and coherent construction of a collective knowledge in the classroom.

This first experience of comparison between different theoretical frames had very positive results. Connecting the two approaches, seeing their respective strengths was rather easy. Both were well adapted to the analysis of the corpus we dealt with, and they gave coherent visions of it. Both had evident potential for understanding the mathematical potential and limits of the problem at stake, for identifying its key didactic variables. Moreover, the TDS had helped us to give account for the collective phase in a rather accurate way, to understand better the complexity of the teacher work in this phase, and the ways the expertise of our experienced teachers is expressed in it.

IV– The second situation

IV.1–Presentation

This situation was built by a pre-service teacher for grade-10 students. It was presented to the students as an activity introducing the notion of function (a definition of this notion had been briefly introduced at the end of the previous session, and no more). The activity relates to right-angle triangles ABC whose hypotenuse AB is fixed and is 6 cm long. These triangles are called "glenatris". The lengths of the two sides CA and CB are respectively designated by x cm and y cm. Three phases are planned:

- in the first phase, the students have to study the possible places for C and examine the particular case of an isosceles triangle,
- the second phase is about the study of the function f which expresses how y varies according to x. In this part, the students have to determine this function, to build point by point its graphic representation and to study graphically its variations.
- in the third phase, the students meet another function g which expresses how the triangle area varies according to x. They have to find graphically the maximum of this new function, then relate it to the particular case studied in phase 1, in order to develop a geometrical proof.

This situation was not built in reference to a precise theoretical framework. But we tried to exploit the theoretical frameworks already used with the first corpus in order to analyse a priori its mathematical potential.

IV.2– The first analysis

The mathematical potential of this situation can be explored in different ways, according to the freedom one takes with the teacher project. Initially, we considered the global context of this work: the glenatri object and the question of the variation of its area. It quickly appeared that, within this perspective, the functional modelling proposed by the teacher had little interest. Indeed, taking as variable one side of the right angle breaks the geometrical symmetry of the problem and complicates unnecessarily the resolution. This led us, quite naturally, to reject the project of the pre-service teacher or, at least, to consider that it required major changes.

In a second phase, respecting more the choices of the teacher, we decided to fix another context: the glenatri object and the functional relationships which it makes likely to consider when the independent variable is one of the sides of the right angle. We were thus interested by the potential offered by this particular context in order to make the notion of functional relationship emerge as an optimal tool in an a-didactic functioning. This new study showed that the context was a priori well appropriate, thanks to the possible interplay it offered between the geometric, numerical and algebraic settings. But the scenarios we elaborated in order to support these analyses were very distant from the one built by the pre-service teacher. They did not respect the logic of her construction and were not helpful for understanding the mathematical activity of her students during the session.

In a third phase, we decided to use the TDS to analyse the situation built by the pre-service teacher while respecting both its mathematical and didactic organisations. Such an analysis is, of course, possible and the didactic literature has offered us some very interesting examples in the last years (see for instance, Dorier & al., 2002). This approach led us to model this session by a succession of situations, nearly one per question asked to the students. For each situation, we had to identify a 'milieu' and to specify what could be produced by the interactions with this 'milieu'. This kind of analysis is not easy at all because the situations are not independent but overlapping. For this session, our attempts resulted in an extremely complex construction, certainly interesting from a theoretical point of view but not really convincing as regards its practical interest. And, once more, our construction tended to move us away from the coherence that seemed to underlie the functioning of the pre-service teacher. It made, above all, visible the weaknesses of her construction. Nevertheless, in the video, we could see a class which was working and doing mathematics, a class in which there were stakes of knowledge. In order to account for these characteristics, we decided thus to exploit a complementary approach: the ergonomic and didactic approach.

IV.3– Another analysis

In the ergonomic and didactic approach, the teacher is seen as a coherent professional submitted to constraints of different nature but having nevertheless a space of freedom he or she invests according to his or her specific characteristics (Robert, Rogalski, 2002). Teachers' practices are thus analysed according to three dimensions: the study of the contents worked in class and of the respective allocation of

mathematical work between students and the teacher, the study of the forms of work of the students, and the study of the interactions between teacher and students. This analysis is complemented by a study of the institutional and social constraints influencing teachers' practices and by a study of the personal characteristics of the teacher. In the following, we synthesise the results of the analysis carried out within this framework.

This analysis of the pre-service teacher's preparation shows that she wants to organise a real space of mathematical work for her students, within a succession of phases of personal research and of collective synthesis. The study of the effective realisation highlights that the times of research are indeed important especially during the first two parts of the situation. In the third one, they are shorter, and this is certainly due to an increasing pressure of time.

Furthermore the analysis of some episodes of the session shows that this teacher tries to take her students into account, to make them take part, to give them a real place during the different moments of the problem solving process (individual research, formulation of answers, justification). But, when the students are in difficulty, she has also a tendency to take their task under her responsibility, while using strategies which allow her to associate them. In fact, time constraints do not allow teachers to start again continually the debate in the class and the teacher must progress in his project. The strategies of this pre-service teacher contribute to this progression, but their installation often contributes to make much easier the students' task.

This tendency to make easier students' tasks appears more particularly during the geometrical questions of the first part of the situation and during the treatment of new specific tasks about functions. As regards the geometrical questions, the study of the mathematical field shows that the knowledge at stake is not clearly related to the objectives of the situation. Making the task easier thus allows the teacher to progress in her project, even if some prior geometrical knowledge is not mobilizable by the majority of the students. As regards the more specific questions about functions, it proves that this pre-service teacher has not built tools to deal with possible difficulties of her students faced with these new tasks. In fact, in her meticulous work of preparation, she anticipates very precisely some algebraic difficulties she has met before and for which she has built ways to anticipate, and tools to deal with. On the other hand, even if she knows that some more specific questions about functions can be difficult for the students, she does not have equivalent knowledge to anticipate the difficulties which can appear here (about the notion of variable or functional dependence for example) and to deal with them. By taking some precautions during the session, she visibly tries to limit the complexity of the questions.

V– Conclusion

In this research project, we have questioned the potentialities and limits of different theoretical frameworks to analyse the mathematical activity of students; we have also questioned their possible complementarities. We have worked with two different

situations. In fact their stakes are different: in the first case, a technique is aimed; in the second case, the teacher wants to organise the entry into the functional world. The contexts for the conception of these two situations are also distinct: the first one has been elaborated with a researcher in the framework of an engineering design relying on a precise didactic theory; a pre-service teacher has built the second one. Finally the teachers are different: experienced teachers on the one hand and a beginner on the other hand.

It seems to us that this study shows the potentialities of the didactic tools, at our disposal today, to analyse research situations or ordinary situations. It also confirms our first conviction: the complex reality we study cannot not be exhausted in only one of the theoretical frameworks existing now. Each of these opens some rationalities while masking us others. Our research also poses the question of the connection between theoretical frames and of their complementarities. Connections were easy for the first corpus, less evident for the second one.

The question of relationships between theoretical frameworks is, in our opinion, a crucial one, with important consequences at the level of action on didactic systems. Our corpus clearly shows that the theoretical choices we make influence the vision we have of the situations we observe and study, and of the idea we develop about their possible improvement. Some frames tended in our case to suggest at least a global reconstruction of the situation, others seemed more compatible with local changes. In addition, the choices carried out a priori to develop a scenario of teaching, only partially condition the students' activity: in class, the diverse forms of mediation and interactions decided on the spot strongly influence the nature of this activity. Moreover current research tends to show that teachers cannot adopt any kind of scenario. So, within the context of an initial or continuous training, balances have to be found between global reconstructions, not always possible, and local reconstructions, not always sufficient, if we want to promote a mathematical activity of greater quality among the students.

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INTUITIVE VS. ANALYTICAL THINKING: FOUR THEORETICAL FRAMEWORKS

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Abstract: *Research in mathematics education often consists of interpreting students' performance on mathematical tasks, in particular their misconceptions, or non-normative responses. In such situations it is natural to compare students' intuitive vs. analytical ways of thinking, bearing in mind that these terms need to be specified more precisely. In this paper, data on one such task is used to compare four theoretical frameworks for interpreting the same data, all dealing in some way with the intuitive/analytical distinction. The first two frameworks come from research in mathematics education, the third from cognitive psychology, and the fourth from evolutionary psychology. The insights gained by the various frameworks are not meant to be seen as conflicting; rather, they illuminate the same phenomenon from different perspectives, and they look for explanatory mechanisms on different levels.*

Keywords: intuitive vs. analytical thinking, dual-process theory, evolutionary psychology, Wason card selection task, group theory, Lagrange's theorem.

A. The Task and the Data

Background. The data is drawn from the performance of university students on a group theory task, but no previous knowledge of group theory will be assumed in this discussion. I thus start by presenting the task to the readers in a completely self-contained way. This is achieved by explaining the relevant group theoretical terms only in the context of this particular example rather than in their full generality. A few generalizations and subtleties are mentioned in the footnotes and can be safely ignored.

The entire task takes place within the group Z_6 , consisting of the set $\{0,1,2,3,4,5\}$ and the operation of addition modulo 6, denoted by $+_6$. For example, $2 +_6 3 = 5$, $3 +_6 3 = 0$, $3 +_6 4 = 1$, and, in general, $a +_6 b$ is defined as the remainder of the usual sum $a + b$ on division by 6.

Z_6 is a *group* in the sense that it contains 0 and is *closed* under addition mod 6: if a and b are in Z_6 , then so is $a +_6 b$.¹ Similarly, we define Z_3 to be the group consisting of the set $\{0,1,2\}$ and the operation $+_3$ of addition modulo 3. A *subgroup* of Z_6 is a subset of $\{0,1,2,3,4,5\}$ which is in itself a group under the operation defined in Z_6 .

¹ In the general definition of a group there are more requirements, namely associativity and the existence of inverses. However, we do not need to worry about them here because, in general, associativity for addition mod n can be shown to be inherited from the associativity of the usual addition of integers; and the existence of inverses can be shown, in the finite case, to follow automatically from the other properties.

For example, it can be checked that the subset $\{0,2,4\}$ is a subgroup of Z_6 , since it contains 0 and is closed under $+_6$.

All the groups in this discussion are *finite*, in the sense that they have a finite number of elements; this number is called the *order* of the group. Thus the order of Z_6 is 6 and the order of Z_3 is 3. Finally, an important theorem of group theory, called *Lagrange's theorem*, states that *if H is a subgroup of Z_6 , then the order H divides 6*. Thus, for example, the order of H cannot be 4 or 5 but 3 is possible, and indeed, we have seen above an example of a subgroup of Z_6 with 3 elements². For what follows, it is relevant to mention that the *converse* of Lagrange's theorem is *not* true in general: It is possible to give an example of a group G of order 12 which does not contain a subgroup of order 6 (cf. e.g., Gallian, 1990, Example 13, p. 151).

The task and data. (Hazzan & Leron, 1996)

The following task was given to 113 computer science majors in a top-notch Israeli university, who had previously completed courses in calculus and in linear algebra (an abstract approach), and were now in the midst of an abstract algebra course:

A student wrote in an exam, " Z_3 is a subgroup of Z_6 ".
In your opinion, is this statement true, partially true, or false?
Please explain your answer.

An incorrect answer was given by 73 students, 20 of whom invoked Lagrange's theorem, in essentially the following manner:

Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6.

Mathematical remark 1. The correct answer is that Z_3 is *not* a subgroup of Z_6 . The reason is that Z_3 is not closed under the operation $+_6$ (for example, $2 +_6 2 = 4$, and 4 is not in Z_3). The question is tricky because Z_3 is a *subset* of Z_6 and is a group (relative to $+_3$), but it is not a *subgroup* (since it is not a group relative to $+_6$). There is a sophisticated sense in which the statement " Z_3 is a subgroup of Z_6 " is partially true, namely, that Z_3 is *isomorphic* to the subgroup $\{0, 2, 4\}$ of Z_6 . We would of course be thrilled to receive this answer, but none of our 113 subjects had chosen to so thrill us.

Mathematical remark 2. As can be seen from the previous remark, our solution does not use Lagrange's theorem. It is relevant to mention that in spite of superficial resemblance, there is no way Lagrange's theorem could even *help* on this task, since "H is a subgroup" is the hypothesis of that theorem, not its conclusion. What the students seem to be using is an incorrect version of an incorrect theorem (namely, the *converse* of Lagrange's theorem).³

² More generally, Lagrange's theorem applies to any two finite groups H and G : *If H is a subgroup of G , then the order of H divides the order of G .*

³ Hazzan & Leron (1996) discuss the data on two more tasks, which shows that this misuse of Lagrange's theorem is deeper and more persistent than might appear merely from the data presented here.

B. Four Theoretical Frameworks for Interpreting the Data

I will now present four theoretical frameworks for interpreting the particular response: Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6. The frameworks are: identifying 'bugs' in students' mathematics, 'coping' perspective, dual-process theory from cognitive psychology, and evolutionary psychology. Due to space limitations, all the theoretical frameworks are presented in outline only, but contain references to fuller expositions.

Framework 1: Analyzing students' errors by identifying "bugs" in their subject matter knowledge or in their logical reasoning (Hazzan & Leron, 1996).

"Examining this amazing answer seriously, turns out to yield some interesting observations on students' ways of using theorems in problem-solving situations. [...]. Specifically, students tend to:

- use theorems as "slogans", as a way of answering test questions while avoiding the need for understanding or for making other kinds of excessive mental effort;
- in particular, use Lagrange's theorem or some version of its converse in situations where such use is quite irrelevant to the problem at hand;
- use a theorem and its converse indistinguishably." (p. 23)

Framework 2: Analyzing students' errors from a "coping perspective" (Hazzan & Leron, 1996; Leron & Hazzan, 1997, pp. 284ff; Vinner, 1997, 2000).

This framework introduces two innovations relative to the first framework. First, it attributes the above response partly to "pre-logical" factors in the student, such as loss of meaning, utter confusion, "groping in the dark", and the constant pressure to supply some answer –*any* answer!– while trying to meet the expectations of the authority figure involved in the interaction (teacher or researcher). We propose that these forces operate in the student's world even before starting to apply mathematical knowledge and logical thinking. Secondly, in order to give *a vivid description of our view of the student's mind* under such pressures, we have introduced the tool of *virtual monologue* (or virtual interview), using the student's own voice in the first person. We feel that the narrative mode (Bruner, 1985) better enables us to give as it were an "inside view" of the student's mind. Hazzan & Leron (1996) and Leron & Hazzan (1997) give detailed analyses of the Lagrange's theorem data, both from a cognitive perspective, and –using a virtual monologue and a virtual interview– from a coping perspective. The analysis itself is too long to bring here; suffice it to say at this stage that it already contains precursors of dual-process theory (our third framework below), which we would import from cognitive psychology seven years later. Here is one relevant quotation, with some dual-process terms –to be explained below– inserted:

"It is possible that these phenomena occur mainly with a certain type of theorem: perhaps one which has a name, or one which is particularly

memorable for other reasons, e.g., especially simple formulation involving natural numbers. If, as in the case of Lagrange's theorem, the theorem can be memorized as a "slogan", then it can easily be retrieved from memory [*accessibility*] under the hypnotic effect of a magic incantation. However, using a theorem as a magic incantation may increase the tendency to use it carelessly [*System 1 thinking*], with no regard to the situation or to the details of its applicability [*System 2 thinking*]." (Hazzan & Leron, 1996, p 26).

Vinner (1997) uses the terms "pseudo-conceptual and pseudo-analytical thought processes" to present a similar analysis of other mathematical tasks. He also presents a related analysis of the use of proofs as rituals (Vinner, 2000). A fine-grained comparison of his and our analyses would be interesting, but is beyond the scope of this abstract (cf. Leron & Hazzan, in print).

Framework 3: Dual-process theory and the Heuristics-and-biases research program in cognitive psychology (led by Kahneman and Tversky over the last 30 years; cf. e.g., Gilovich, Griffin, & Kahneman, 2002; Kahneman, 2002; Stanovich & West, 2000; Stanovich & West, 2003. For a brief overview, cf. Leron & Hazzan, in print).

Dual-process theory. The ancient distinction between intuitive and analytical modes of thinking has achieved a new level of specificity and rigor in what cognitive psychologists call *dual-process theory*. In fact there are several such theories but since the differences are not significant for our context, we will ignore the nuances and will adopt the generic framework presented in Gilovich, Griffin, & Kahneman, 2002 and in Kahneman, 2002. To the best of my knowledge, the first application of this theory to mathematics education research has been Leron & Hazzan (in print); the present exposition and analysis is an abridged version of the one given in that paper.

According to dual-process theory, our cognition and behavior operate in parallel in two quite different modes, called *System 1* (S1) and *System 2* (S2), roughly corresponding to our commonsense notions of intuitive and analytical (or reasoning) modes of thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins (S2 being evolutionarily more recent and, in fact, largely reflecting *cultural* evolution). The distinction between perception and cognition is ancient and well known, but the introduction of S1, which sits halfway between perception and (analytical) cognition is relatively new, and has important consequences for how empirical findings in cognitive psychology are interpreted, including the wide ranging *rationality debate* and the application to mathematics education research.

Like perception, S1 processes are characterized as being fast, automatic, effortless, unconscious and inflexible (hard to change or overcome); unlike perceptions, S1 processes can be language-mediated and relate to events not in the here-and-now (i.e., events in far-away locations and in the past or future). S2 processes are slow,

conscious, effortful, computationally expensive, and relatively flexible. The two systems differ mainly on the dimension of *accessibility*: how fast and how easily things come to mind. In most situations, S1 and S2 work in concert to produce adaptive responses, but in some cases (such as the ones concocted in the Heuristics-and-biases research), S1 generates quick automatic *non-normative* responses, while S2 may or may not intervene in its role as monitor and critic to correct or override S1's response. The precise relation of this framework to the concepts of intuition, cognition and meta-cognition as used in the mathematics education research literature is elaborated in Leron & Hazzan (in print).

Many of the non-normative answers people give in psychological experiments –and in mathematics education tasks, for that matter– can be explained by the quick and automatic responses of S1, and the frequent failure of S2 to intervene in its role as critic of S1.

Here is a striking example (Kahneman, 2002) for the tendency of the fast-reacting S1 to “hijack” the subject's attention and lead to a non-normative answer.

“A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?”

Almost everyone reports an initial tendency to answer ‘10 cents’ because the sum \$1.10 separates naturally into \$1 and 10 cents, and 10 cents is about the right magnitude. Frederick found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students, and 56% (164/293) of students at the University of Michigan gave the wrong answer.” (p. 451)

According to dual process theory, this situation is analogous to that of the famous optical illusions known from cognitive psychology. The salient features of the problem cause S1 to jump immediately with the answer of 10 cents, since the numbers one dollar and 10 cents are salient, and since the orders of magnitude are roughly appropriate. Many people accept S1's conclusions uncritically, thus in a sense “behave irrationally”. For others, S1 also immediately jumped with this answer, but in the next stage, their S2 interfered critically and made the necessary adjustments to give the correct answer (5 cents). Significantly, the way S1 worked here, namely coming up with a very quick decision based on salient features of the problem and of rough sense of what's appropriate in the given situation, usually gives good results under natural conditions, such as searching for food or avoiding predators. Hence the insistence of Gigerenzer (e.g., Gigerenzer & Todd, 1999) that this is a case of *ecological rationality* being fooled by a tricky task, rather than a case of irrationality. The various debates arising from different interpretations of the Heuristics-and-biases research program form a fascinating topic which is, however, beyond the scope of this paper.

It is important to note that skills can migrate between the two systems. When a person becomes an expert in some skill, perhaps after a prolonged training, this skill may become S1 for this person. For example, driving is an effortful S2 behavior for

beginners, requiring deep concentration and full attention; for experienced drivers, in contrast, driving becomes an S1 skill which they can perform automatically while engaged in a deep intellectual or emotional conversation. Conversely, many S1 skills (such as walking straight or talking in a familiar but non-native language), with advancing age, or when just being tired or drunk, suddenly require conscious effort to perform successfully.

Dual-process analysis of students' misuse of Lagrange's theorem. Applying a dual-process perspective, Leron & Hazzan (in print) proposed that students' misuse of Lagrange's theorem reflects a combined S1-S2 failure⁴. The analysis closely resembles Kahneman's analysis of the bat-and-ball data, except for the somewhat surprising demonstration that S1 can hijack cognitive behavior even in advanced mathematical settings, where the name of the game is explicitly reasoning and analytical thinking (i.e., S2 mode).

As usual, the S1 response is invoked by what is most immediately accessible to the students in the situation, which also looks roughly appropriate to the task at hand. Specifically, the students know that using a theorem in such situations is expected; they also know more-or-less immediately and effortlessly that Lagrange's theorem says something about subgroups and divisibility of their orders (it is the *details and logic* of what the theorem says that requires the effortful and pedantic intervention of S2); finally, the appearance of the two numbers 3 and 6 as orders of the groups Z_3 and Z_6 and the fact that 3 divides 6, immediately and automatically cues Lagrange's theorem, yielding the answer, " Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6". This is a striking example for an answer that is entirely appropriate by the "logic" of S1, but is extremely inappropriate by the logic of S2.

In addition to S1's inappropriate reaction, S2 too fails in its role as critic of S1, since there is nothing in the task situation to alert the monitoring function of S2. The missing judgment –mainly that Lagrange's theorem cannot be used to establish the existence of a subgroup but only its absence– clearly require S2 processes. It is important to note that some of the students may well have the knowledge required to produce the right answer, had they only stopped to think more (that is, invoke S2). The problem is, rather, that they have no reason to suspect that the answer is wrong, thus the "permissive System 2" (Kahneman, 2002) remains dormant:

"[An] evaluation of the heuristic attribute comes immediately to mind, and [...] its associative relationship with the target attribute is sufficiently close to pass the monitoring of a permissive System 2." (p. 469)

Just as in the bat-and-ball situation, the final (erroneous) response is a combination of S1's quick and effortless reaction, together with S2's failure to take a corrective action in its role as critic and monitor of S1. Since the operation of S1 is so easy and that of S2 so hard, students will not make the extra effort unless something in the situation alerts them to such a need. It is a feasible (and eminently researchable)

⁴ In that paper, the well-known students-and-professors phenomenon is also analyzed in a similar spirit.

hypothesis, that at least for some of the students, a small cue (about the situation or about their answer, not even about the mathematics) would be enough to set them on the path for a correct answer. They may already have all the necessary (S2) knowledge to solve this problem correctly, but a cue is needed to mobilize this knowledge. This shows, incidentally, that the dual system framework leads not only to new explanations, but also (like all good theories) to interesting new research *questions*.

Framework 4: Evolutionary psychology (Cosmides & Tooby, 1992,1997; Pinker, 1997,2002; Plotkin, 2004)

Evolutionary psychology. This framework is the hardest to introduce in the small space available, since it harbors many subtleties and it runs against deep-rooted biases and emotional obstacles. I will only bring here a brief summary adapted from Leron (submitted).

I take from the young discipline of Evolutionary Psychology (EP) the scientific view of human nature as a collection of universal, reliably-developing, cognitive and behavioral abilities –such as walking on two feet, face recognition, and the use of language– that are spontaneously acquired and effortlessly used by all people under normal development (Cosmides & Tooby, 1992, 1997; Pinker, 1997, 2002, Ridley, 2003). I also take from EP the evolutionary origins of human nature, hence the frequent mismatch between the ancient ecology to which it is adapted and the demands of modern civilization. To the extent that we do manage to learn many modern skills (such as writing or driving, or some math), this is because of our mind’s ability to “co-opt” ancient cognitive mechanisms for new purposes (Bjorklund & Pellegrini, 2002; Geary, 2002). But this is easier for some skills than for others, and nowhere are these differences manifest more than in the learning of mathematics. The ease of learning in such cases is determined by the *accessibility* of the co-opted cognitive mechanisms.

I emphasize that what is part of human nature need not be innate: we are not born walking or talking. What seems to be innate is the motivation and the ability to engage the species-typical physical and social environment in such a way that the required skill will develop (Geary, 2002). This is the ubiquitous mechanism that Ridley (2003) has called “Nature *via* Nurture”. I also emphasize that what is *not* part of human nature, or even what goes *against* human nature, need not be unlearnable. Individuals in all cultures have always accomplished prodigious feats such as juggling 10 balls while riding a bicycle, playing a Beethoven piano sonata, or proving an abstract mathematical theorem (such as Lagrange’s) in a formal language. However, research on people’s reasoning, and on mathematical thinking in particular, usually deals with what most people are able to accomplish under normal conditions. Under such conditions, many people will produce non-normative answers if the task requires reasoning that goes against human nature. In terms of mathematical education, this means that learning such skills will require a particularly high

motivation and perseverance – conditions that are hard to achieve for a long time and for many people in the standard classroom.

Finally, it is in order to note here that EP is a hotly debated discipline. Much of the criticism leveled at EP is ideologically or emotionally motivated, but see, e.g., Over (2003) or Fodor (2000) for a sample of scientifically respectable alternative views.

A word about the relation of human nature to dual-process theory (Framework 3 above): Human nature consists by definition of a more-or-less fixed collection of traits and behaviors that all human beings in all cultures acquire spontaneously and automatically under normal developmental conditions. System 1, in my view, contains all the traits and behaviors that comprise human nature but, on top of that, also all the traits and behaviors that became S1 for a particular culture or a particular person because of specific (non-universal) developmental conditions. For example, speaking English is not part of human nature but is an S1 skill for whole cultures; and reasoning (correctly) with Lagrange's theorem may be an S1 skill for group-theory specialists.

Students' misuse of Lagrange's theorem: an EP perspective. Cosmides and Tooby (1992, 1997) have used the *Wason card selection task* (Wason, 1966; Wason & Johnson-Laird, 1972) to uncover what they refer to as people's evolved reasoning "algorithms". In a typical example of the Wason task, subjects are shown four cards, say

A

6

T

3

, and are told that each card has a letter on one side and a number on the other. The subjects are then presented with the rule, "if a card has a vowel on one side, then it has an even number on the other side", and are asked the following question: *What card(s) do you need to turn over to see if any of them violate this rule?* The notorious result is that about 90% of the subjects, including science majors in college, give an incorrect answer. Many similar experiments have been carried out, using rules of the same logical form "if P then Q", but varying the content of P and Q. The error rate has varied somewhat depending on the particular context, but mostly remained high (over 50%).

The motivation behind the original Wason experiment was partly to see if people will naturally behave in accordance with the Popperian paradigm that science advances through *refutation* of held beliefs (rather than their confirmation). The normative response to the Wason task depends on the question: What will *refute* the given rule? The answer is that the rule is violated if and only if a card has a vowel on one side but an *odd* number on the other. Thus, according to mathematical logic, the cards you need to turn are A (to see if it has an odd number on the other side) and 3 (to see if it has a vowel on the other side)⁵.

Cosmides and Tooby (1992, 1997) have presented their subjects with many versions of the task, all having the usual logical form "if P then Q", but varying widely in the contents of P and Q and in the background story. While the classical results of the Wason task show that most people perform very poorly on it, Cosmides and Tooby

⁵ Most subjects choose the

A

 card and sometimes also

6

, but rarely

3

.

demonstrated that their subjects performed significantly better on tasks involving conditions of *social exchange*. In social exchange situations, the individual receives some benefit and is expected to pay some cost. On theoretical grounds, and from what is known about the evolution of cooperation, certain kinds of social skills are expected to have conferred evolutionary advantages on those who excelled in them, and thus would be naturally selected during evolutionary history. In the Wason task, social exchange situations are represented by statements of the form “if you get the benefit, then you pay the cost” (e.g., if you give me your watch, then I give you \$20). A *cheater* is someone who takes the benefit but do not pay the cost. Cosmides and Tooby explain that when the Wason task concerns social exchange, a correct answer amounts to detecting a cheater. Since subjects performed correctly and effortlessly in such situations, and since evolutionary theory clearly shows that cooperation cannot evolve in a community if cheaters are not detected and punished, Cosmides and Tooby have concluded that our mind contains evolved “cheater detection algorithms”.

Significantly for the Lagrange’s theorem task discussed here, Cosmides and Tooby also tested their subjects on the “switched social contract” (mathematically, the converse statement “if Q then P”), in which the correct answer by the logic of social exchange is different from that of mathematical logic (Cosmides and Tooby, 1992, pp. 187-193; Leron, submitted). As predicted, their subjects overwhelmingly chose the former over the latter: *When conflict arises, the logic of social exchange overrides mathematical logic.*

I note that there are many competing theories to explain the content effects of the Wason task, and the Cosmides and Tooby theory is used here mainly as illustration. For our purposes, we can summarize their approach as follows. In non-social-exchange situations, people mostly find it hard to relate to the Wason task in any meaningful way. In a social exchange situation, in contrast, people find the situation meaningful, but will mostly interpret this statement in a symmetrical way, rather than a directional way as required by mathematical logic, as if it were an “if and only if” statement.

This theory adds a new level of support, prediction and explanation to the many findings that students are prone to confusing between mathematical propositions and their converse, in particular, to our Lagrange’s theorem data presented above. Importantly, in the EP view, people fail not because of a weakness in our cognitive apparatus, but because of its *strength*: our impressive skill in negotiating social exchange situations. Unfortunately for mathematics education, this otherwise adaptive skill, may sometime clash with the requirements of modern mathematical thinking. It is a fascinating theoretical and empirical research issue, to map out the topics and skills where human nature helps the learning of mathematics and where it may get in the way. (Some first steps in this direction have been taken in Leron, submitted).

C. Conclusion

As in the old Buddhist fable about the six blind men trying to “see” an elephant, complex phenomena can often be described from several perspectives, which in turn lead to several different explanations. Notable examples are the global vs. the local (or molecular) theories of heat in physics, or proximal vs. ultimate explanations in psychology (Cosmides & Tooby, 1997). The different perspectives and the corresponding different explanations usually answer different questions and are useful under different circumstances. The more perspectives and the more explanations, the deeper the understanding of the phenomenon under study. It is my hope, therefore, that the four complementary perspectives offered here, will help us gain deeper understanding of students’ performance on advanced mathematical tasks.

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DIDACTIC EFFECTIVENESS OF EQUIVALENT DEFINITIONS OF A MATHEMATICAL NOTION THE CASE OF THE ABSOLUTE VALUE

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Abstract: *Quite often a mathematical object may be introduced by a set of equivalent definitions. One fundamental question consists of determining the “didactic effectiveness” of the techniques associated with these definitions for solving one kind of problem; this effectiveness is evaluated by taking into account the epistemic, cognitive and instructional dimensions of the study processes. So as to provide an example of this process, in this article we study the didactic effectiveness of techniques associated with different definitions of the absolute value notion (AVN). The teaching and learning of the AVN are problematic; this is proved by the amount and heterogeneity of the research papers that have been published. We propose a “global” study from an ontological and semiotic point of view (Godino, 2002; Wilhelmi, Godino and Lacasta, 2004).*

1. Mathematical equivalence vs. Didactic equivalence of definitions

One of the goals for the teaching of mathematics should be to channel everyday thinking habits towards a more technical-scientific form of thinking at an earlier stage, as a means for overcoming the conflicts between the (formal) structure of mathematics and the cognitive progress. The process of definition of mathematical objects represents “more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition” (Vinner, 1991, p.65). This fact justifies the great number of papers in the didactics of mathematics for which the subject matter is mathematical definition (Linchevsky, Vinner & Karsenty, 1992; Mariotti & Fischbein, 1997; De Villiers, 1998; Winicki-Landman & Leikin, 2000; etc.). We are interested in justifying the fact that the mathematical equivalence of two definitions of the same object does not imply their epistemic, cognitive or instructional equivalence, that is to say, the didactic equivalence.

From the viewpoint of the didactics of mathematics, one fundamental question consists of determining the *didactic effectiveness* of problem-solving techniques associated with a mathematical definition; this effectiveness is assessed by taking into account the *epistemic* (field of applicability of the techniques and mathematical objects involved), *cognitive* (effectiveness and cost in the use of the techniques by the individuals) and *instructional* (amount of material resources and time required for its

teaching) dimensions. Hence, with the expression *didactic effectiveness* we refer to the articulation of these partial types of effectiveness in an educational project.

In relation to a mathematical notion it is necessary to: 1) determine mathematically equivalent definitions of the said notion; 2) describe the relations that are established between these definitions; 3) construct an explicit reference for the notion defined that envisages the complexity of objects and meanings that constitute the equivalent definitions associated with that notion in the different contexts of use; and 4) assess the didactic effectiveness of the techniques associated with the different mathematical definitions. A study of this kind may be performed for any kind of mathematical notion; however, the specific didactic decisions are consubstantial to each mathematical notion. In this article we aim to identify mathematically equivalent definitions of the notion of absolute value and discuss its equivalence or its diversity from a cognitive and instructional viewpoint. To do so, we answer the following questions:

- Is there a technique that minimises the cognitive and instructional cost of use of resources, that maximises the effectiveness of the individuals in the specific field of problems and that facilitates adaptation to new problems?
- Is it possible to classify the techniques according to their *scope or generality* (field of applicability), their *mutual implication* (one technique may be obtained deductively from another one) or their *role within the institutional practices* (social, cultural, conventional)?

So as to answer these questions it is necessary, in the first place, to determine the nature of the notion of absolute value and accept the complexity of objects and meanings that explicitly refer to it. In section 2, a set of research problems are described, the purpose of which is the understanding of the difficulties for the teaching and the learning of the AVN. From these investigations we deduce the ontological and semiotic complexity of the AVN, but none of them deals with the problem that arises when trying to integrate the meanings attributed to this notion in the different contexts of use. In section 4, we clarify a way to structure the models and meanings associated with the AVN and we describe its “overall” meaning. Beforehand, in section 3, we introduce the different definitions of the AVN and, backed by the calculation of the solutions of a linear equation with an absolute value, we indicate how these definitions condition mathematical practices.

2. Nature of the notion of absolute value

The teaching and learning of the AVN are problematic. This is proved by the amount and heterogeneity of the research papers that have been published. Gagatsis and Thomaidis (1994), after showing a succinct anthropology of the knowledge about “absolute value”, determine the processes for adapting that knowledge in Greek schools and interpret the students’ errors in terms of *epistemological obstacles* (linked to the historical study) and *didactic obstacles* (related with the processes of transposition). More recently, Gagatsis (2003, p.61) reasons from empirical data that

the “obstacles encountered in the historical development of the concept of absolute value are evident in the development of students’ conceptions”.

From a professional point of view, Arcidiacono (1983) justifies a instruction of the AVN based on the graphic analysis on the Cartesian plane of piece-wise linear functions and Horak (1994) establishes that graphic calculators represent a more effective instrument than pencil and paper for performing this teaching. On the other hand, Chiarugi, Fracassina & Furinghetti (1990) carried out a study on the cognitive dimension of different groups of students faced with solving problems that involve the AVN. The study determines the need for research that will allow the errors and *misconceptions* to be overcome. On her part, Perrin-Glorian (1995) establishes certain guidelines for the institutionalisation of knowledge about the AVN in arithmetical and algebraic contexts; so she argues that the central function of the teacher’s didactic decisions in the construction of the AVN, that must take into account the students’ cognitive restrictions and must highlight the instrumental role of the AVN.

All these research papers implicitly consider that the nature of the AVN is transparent. From an ontological and semiotic point of view of mathematical cognition and instruction (Godino, 2002; Godino, Batanero and Roa, in print) it is necessary to theorise the notion of meaning in didactics. This theorising is done using the notion of semiotic function and an associated mathematical ontology. They start off with the elements of the technological discourse (notions, propositions, etc.) and it is concluded that its nature is inseparable from the pertinent systems of practices and contexts of use.

Godino (2002) identifies the “system of practices” with the contents that an institution assigns to a mathematical object. The description of the meaning of reference for an object is presented as a list of objects classified into six categories: problems, actions, language, notions, properties and arguments. Wilhelmi, Godino and Lacasta (2004) argue in what way this description of the system of practices is insufficient for the description of the institutional meanings of reference and, in order to overcome those deficiencies, the theoretical notions of *model* and of *holistic meaning* of a mathematical notion are introduced. These notions will allow us to structure the different definitions of the AVN and the description of the meaning of the AVN as a “whole”, in a coherent complex whilst drawing some conclusions of a macro and micro didactic nature.

3. Definitions for the notion of absolute value

In this section, we introduce some definitions of the AVN associated with different contexts of use and we briefly indicate how these definitions, as objects emerging from the different subsystems of practices, condition the operational and discursive rules. In the arithmetical context, the AVN represents a rule that “leaves the positive numbers unchanged and turns the negative numbers into positive ones”.

“The absolute value of x , denoted by $|x|$, is defined as follows:

$$|x| = x \text{ if } x > 0; \quad |x| = -x \text{ if } x < 0; \quad |0| = 0$$

Thus, the absolute value of a positive number or zero is equal to the number itself. The absolute value of a negative number is the corresponding positive number, since the negative of a negative number is positive.” (Leithold, 1968, p.10).

The absolute value provides the set of real numbers with a metric; the distance of a real number x to the origin 0 is defined by the relation: $d(x; 0)=|x|$.

“Intuitively, the absolute value of a represents the distance between 0 and a , but in fact we will *define* the idea of ‘distance’ in terms of the ‘absolute value’, which in turn was defined in terms of the ordering.” (Ross, 1980, p.16).

In the geometrical context, the NVA may be understood in terms of vectors as the module for a one-dimensional vector. What is more, this fact may be generalised as a property that is derived from the “ordered” and “complete” nature of \mathbf{R} (Aliprantis & Burkinshaw, 1998, p.66–67).

The classic definition of absolute value, as a basic notion for the foundations of mathematical analysis, is sometimes reformulated in terms of the maximum function: $|x| = \max\{x; -x\}$. In this same context, the AVN is often introduced using a piecewise function in \mathbf{Q} and, by extension, in \mathbf{R} .

“For any rational number q : $|q| = \begin{cases} q & \text{si } q \geq 0 \\ -q & \text{si } q < 0 \end{cases}$ [...] We extend the definition of ‘absolute value’ from \mathbf{Q} to \mathbf{R} [...] $|x|$ equal x if $x \geq 0$, and $-x$ if $x < 0$.” (Truss, 1997, pp.70–102).

Finally, it is easy to demonstrate that: $|x| = +\sqrt{x^2}$ (Mollin, 1998, p.47).

The aforementioned definitions are mathematically equivalent, but their use conditions mathematical activity: they do not involve the same mathematical objects in the resolution of a same problem. For example, let it be the linear equation with absolute value $|x - 2| = 1$, its solution in an arithmetical context involves a reasoning of the kind: “the absolute value of a number is 1, then this number is 1 or -1 ; What number, when subtracting 2 from it, gives 1?, What number, when subtracting 2 from it, gives -1 ?”. The formalisation of this method may be done in the following way:

$$|x - 2| = 1 \Rightarrow \begin{cases} x - 2 = 1 \Rightarrow x = 3 \\ x - 2 = -1 \Rightarrow x = 1 \end{cases}$$

However, the analytical demonstration, according to the compound function definition, is performed in the following way:

$$\begin{aligned} |x - 2| = 1 &\Rightarrow \sqrt{(x - 2)^2} = 1 \Rightarrow (x - 2)^2 = 1 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow \\ &\Rightarrow x = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} = \begin{cases} x = 3 \\ x = 1 \end{cases} \end{aligned}$$

Next, in section 4.1, we shall show the onto-semiotic complexity of the AVN, that is deduced from the diversity of contexts of use, from the definitions associated with them and the operational and discursive practices that these definitions condition and, in section 4.2, backed by the theoretical notion of *holistic meaning* (Wilhelmi,

Godino and Lacasta, 2004), we shall organise the models of absolute value, whilst showing the relations that are established between them.

4. Onto-semiotic complexity of the absolute value

4.1. Structure of definitions, models and meanings associated with the notion of absolute value

The professional mathematician identifies the same formal structure in the variety of objects and (operational and discursive) practices; a structure that he/she considers to be “the mathematical object”. This formal structure represents the implicit reference in the resolution of types of problems associated with the variety of systems of practices and objects emerging in the different contexts of use. Figure 1 shows schematically the diversity of objects associated with the AVN.

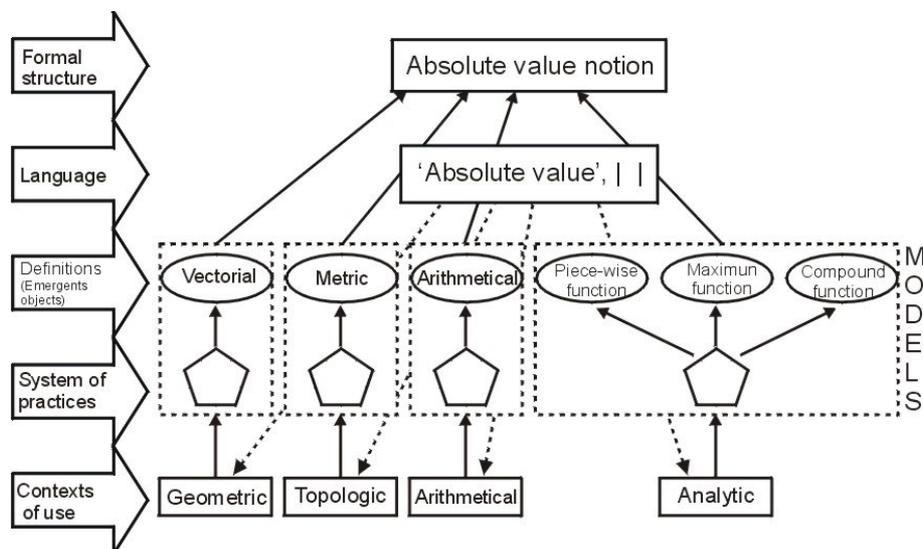


Figure 1. Structure for the models and meanings associated with the absolute value.

Each definition represents an object emerging from a system of practices in a given context of use. No definition may be privileged *a priori*. Each “emergent object - system of practices” binomial determines a *model* of the AVN. The model is then a coherent form for structuring the different contexts of use, the mathematical practices relating to them and the objects emerging from such practices; so forming a network or *local epistemic configuration* (associated with a specific context of use).

4.2. Holistic meaning of the notion of absolute value

From the strictly *formal and official* viewpoint (Brown, 1998), it is accepted that the definition of a mathematical object constitutes its meaning. The description of the system of models and meanings associated with a notion is obtained from the statement and demonstration of a theorem for characterisation: privilege of one of the definitions and justification of the equivalence of the rest of the definitions.

The empirical data provided by Leikin & Winicki-Landman (2000) allow to state that the equivalence of mathematical definitions cannot be assessed just from the epistemological viewpoint, it is necessary take into account the *cognitive* (What

strategies for action generate each one of the definitions?), *instructional* (What definition is the most suitable within a given project for teaching?) and *didactic* (What relationship is established between the personal meaning learnt and the institutional meaning intended?) dimensions. The *holistic meaning* (Wilhelmi, Godino & Lacasta, 2004) comes from the coordination of the meaning attributed to the models associated with the notion of equality and the tensions, filiations and contradictions that are established between them.

5. Cognitive effectiveness of the arithmetic models and “piece-wise function” of the absolute value

As we mentioned earlier, from the viewpoint of the didactics of mathematics, a fundamental question consists of determining the *didactic effectiveness* of a mathematical process for problem-solving. In this section we aim to analyse the *cognitive* dimension (effectiveness and cost in the use of the techniques by individuals) of the problem-solving techniques associated with the “arithmetical” definitions and “piece-wise function”. To do so, we use an experimental study with a group of 55 students (trainee teachers) solving a set of elemental exercises that require the AVN (Table 1).

<p>1. Complete, if you can, the following equalities:</p> $ -2 = \quad 2 = \quad 0 = \quad \sqrt{-2} =$ $ \sqrt{2} = \quad -\sqrt{2} = \quad 2 - \sqrt{2} = \quad \sqrt{2} - 2 =$
<p>2. State, if you can, the numbers that would have to be inserted to replace the dots so the following expressions will be correct:</p> $ \dots - 2 = 1; \quad \dots + 2 = 1; \quad \dots - 2 = 0; \quad (\dots)^2 - 4 = 0;$ $ (\dots)^2 + 4 = 0; \quad (\dots)^2 - 1 = 1; \quad (\dots)^2 - 3 = 1$
<p>3. Represent in a graphic way the function $f(x) = x+1$.</p>
<p>4. Let a be a real number. Complete, if you can, the following equalities:</p> $ -a = \quad a = \quad a - 2 =$ $ -a - 2 = \quad 2 - a = \quad a + 2 =$

Table 1. Questionnaire.

5.1. Predominant model and effectiveness in problem-solving

Generically, we affirm that a person understands the AVN if he/she is capable of distinguishing its different associated models, structuring the said models in a complex and coherent group and meeting the operative and discursive needs in relation to the AVN in the different contexts of use.

Formally, a definition may be reduced to axioms; however, in a process of study, the definition represents a formalization of a *pertinent* notion (it allows a consistent interpretation of a problem) or *operative* (it conditions a useful action). The only means for distinguishing the meaning attributed by an individual to an object is by means of a situation or a set of problems that may be solved by using different models capable of generating pertinent and useful actions, that, however, comply with different “economic” laws.

The experimental work performed has allowed us to classify the students according to the model of absolute value associated with the operative and discursive practices in relation to the problems proposed (that determines a certain level of effectiveness). So as to be able to classify the students, it is necessary to interrelate a collection of tasks and determine (with a level of approximation) the tasks that allow the performance of other tasks to be assured.

5.2. Analysis of a questionnaire

The main purpose of the experimentation is to empirically support the thesis according to which the models “arithmetical” and “piece-wise function” associated with the AVN are extremely similar (see Section 4.2). The analysis of the institutional meanings determines selection criteria of the variables for the *implicative* study (Gras, 1996). The system of variables is shown in Table 2.

Variable	Description	No. of answers
v1	$ \pm\sqrt{2} = \sqrt{2}$ (without numerical approximation)	47
v2	$ \pm\sqrt{2} \approx 1,41$ (with numerical approximation)	20
v3	$ \sqrt{-2} = \text{no } \exists$ (does not make sense in \mathbf{R})	16
v4	$ 2 - \sqrt{2} \approx 0,59$ (with numerical approximation)	30
v5	$ 2 - \sqrt{2} = 2 - \sqrt{2}$ (without numerical approximation)	17
v6	$ \sqrt{2} - 2 = 2 - \sqrt{2} \approx 0,59$ (with or without numerical approx.)	30
v7	Two solutions in $ \dots - 2 = 1$ or in $ \dots + 2 = 1$	21
v8	Determination of the two solutions of $ (\dots)^2 - 4 = 0$	26
v9	$ (\dots)^2 + 4 = 0$ has no solution	24
v10	Solution of $ (\dots)^2 - 1 = 1: 0$	29
v11	Solution of $ (\dots)^2 - 1 = 1: 0$ and $\sqrt{2}$, $-\sqrt{2}$ or $\pm\sqrt{2}$	9
v12	At least two solutions for $ (\dots)^2 - 3 = 1$	25
v13	They construct the graph and give the formula correctly	28
v14	They construct the graph and give the formula incorrectly	15
v15	At least 4 correct sections from exercise 4	14
v16	Mean in the course ≥ 14 (out of 20)	17

Table 2. Small set of variables.

The aim is to find whether, in the sample, the fact of having answered a question correctly statistically implies the response to another question. In particular, it is admissible to expect that any individual who is capable of performing a task that is more complex than another (and that generalises it in a certain way), then he/she will also be capable of performing the second one. However, this is not always so; in many circumstances it is necessary to compare certain hypotheses for implementing a hierarchy for performing tasks. Below, we comment on some of these implications:

- *Implication at 99%.* A group of students is stable in solving the equations: they perform the search for roots in an equation (linear and quadratic) with absolute value in a routine manner.
- *Implication at 95%.*

- $v_2 \rightarrow v_4$: Mostly, the students who have given an approximation for $|\sqrt{2}|$, also establish that $|2 - \sqrt{2}| \approx 0,59$. One possible interpretation: the arithmetical model of absolute value is understood as a rule that operates on the “numbers”, that is to say, numbers “in decimal format”.
- $v_3 \rightarrow v_9$: The didactic contract assumed by the vast majority of the students establishes the existence of a solution for any problem; their function is find it (the sentence “when you can” is skipped by these students). Hence, the relation “ $v_3 \rightarrow v_9$ ” distinguishes a group of students that separates action and meaning.
- $v_{16} \rightarrow v_6$ and $v_{16} \rightarrow v_{13}$. The students who have a “good” behaviour in the course mostly operate the absolute value ($|1\sqrt{2} - 2| = 2 - \sqrt{2}$) “symbolically” and understand the $f(x) = |x + 1|$ function analytically and graphically.

A wider question that may be posed is whether the fact of having answered a set of questions correctly implies (in a preferential manner) the right answer in another set of questions. The *hierarchical analysis* (Gras, 1996) allows the implicative relationships between the kinds of questions to be described in a more “dynamic” way and, therefore, constitutes a response to the question posed. Based on the experimental data, it is established that the most significant classes are: $v_7 \rightarrow v_{12} \rightarrow v_8 \rightarrow v_6$ and $v_{15} \rightarrow v_{16} \rightarrow v_{13}$. What individuals contribute to the formation of each one of the classes? The students who most contribute to both classes are those who perform the tasks symbolically and are capable of applying the model piece-wise function systematically and effectively.

6. Macro and micro didactic implications

The cognitive difficulties (Chiarugi, Fracassina & Furinghetti, 1990) and the incapacity of the educational institution to draw up a pertinent curriculum for the introduction and development of the AVN (Perrin-Glorian, 1995; Gagatsis and Thomaidis, 1994) has led to merely technical teaching based on the arithmetical model (as a rule that “removes the minus sign”). The arithmetical model of the AVN proves to be a didactic obstacle that restricts, in many cases, the personal meaning to a mere game of symbols. This obstacle is shown in different ways; for example: $|a| = a$ and $|-a| = a$, for any $a \in \mathbf{R}$; $|1\sqrt{2} - 2| = \sqrt{2} + 2$, etc.

Macrodidactic implications

The introduction of the absolute value in the arithmetical context represents an unfortunate decision in modern-day school institutions: it means the inclusion in the curriculum of the notion “absolute value” for merely cultural reasons. However, the curricular structure is not ready at the present to properly cope with the study of the notion in an exclusively arithmetical context. It would be advisable to “temporarily” remove the notion. This would be temporary, either until a pertinent didactic transposition, or until the students start to study the theory of functions, central in relation to the notion of absolute value (Arcidiacono, 1983; Horak, 1994).

This “drastic” didactic decision means, on the one hand, the acceptance by the

educational institution of the existence of a didactic that is not pertinent in relation to the notion of absolute value and, on the other hand, its incapacity to produce a *viable* (admissible cost of material and time resources), *reproducible* (institutional stability in relation to the availability of resources) and *reliable* (the personal meanings learnt are representative of the institutional meanings intended) “*de-transposition*” (Antibi and Brousseau, 2000). Gagatsis (2003, p.61) gets a similar conclusion: “There are also a number of obstacles with didactic origin relating to the ‘strange’ didactic transposition or the restrictions of the educational system [...] There is a problem of legitimization of the content to be taught.”

Microdidactic implications

From the point of view of learning, the models associated with mathematical notion are ordered according to their hierarchy. The structuring of the models is carried out in terms of the “field” of the latter in the curriculum. The dominant model must clearly and specifically participate in the first encounter with the notion. For the AVN, the model “piece-wise function”, using the graphic representation of the function in the Cartesian plane and using the discursive practices pertaining to the theory of functions.

Hence, it is necessary to establish a didactic engineering for developing the “absolute value” object (understood as a system). This engineering will have to articulate the epistemological analysis with the methodological and time restrictions within each specific institution. In relation to the AVN, the objective consists of establishing a system of practices that will make the explicit interaction of the arithmetical model with the rest of the models possible and, most particularly, with the analytical model.

7. Synthesis and conclusions

The notion of holistic meaning of a mathematical notion makes it possible to describe the latter as an epistemic configuration that takes into consideration both the praxis and discursive elements of mathematical activity. Furthermore, it provides an instrument for controlling and assessing the systems of practices implemented and an observable response (and, in a certain way, quantifiable) for the analysis of personal meanings. More precisely speaking:

- The notion of holistic meaning (network of models) represents the structuring of the knowledge targeted and may be used to determine the degree of representation of a system of practices implemented in relation to the institutional meaning intended.
- The notions of model and holistic meaning provide a response to the questions: What is a mathematical notion? What is understanding this notion?; in particular, What is the AVN? What does understanding the AVN mean?

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WORKING IN A DEVELOPMENTAL RESEARCH PARADIGM: THE ROLE OF DIDACTICIAN/RESEARCHER WORKING WITH TEACHERS TO PROMOTE INQUIRY PRACTICES IN DEVELOPING MATHEMATICS LEARNING AND TEACHING

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Abstract: *This paper presents and explores the use of a developmental research paradigm and its necessity to the growth of knowledge about improving mathematics learning and teaching. It reports on a project whose chief aim is to create and study inquiry communities between mathematics teachers and didacticians. Its principal focus is the roles of didacticians as they interact with teachers to develop working notions of inquiry and community for developments in practice. Its analytical stance is dialogical, tracing meanings and ideas through the words of individuals in meetings to plan the work of the project. We show that meanings develop as individual perspectives are presented, considered and modified, enabling community understandings to grow and facilitating individual interpretation in practice.*

Keywords: Community of inquiry, developmental research paradigm, dialogic inquiry, role of didacticians.

Introduction: research focus

A research project, Learning Communities in Mathematics, is underway in Norway¹ to explore the development of inquiry communities in mathematics learning, teaching and teaching development. The project is designed to enhance mathematics learning in classrooms through development of teaching. The project creates and studies learning communities between teachers and didacticians as partners in development and research to design and explore classroom activity in mathematics involving an inquiry approach.

Research here both *studies* developmental processes and *is a part of the processes studied*. We agree with Chaiklin who writes:

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Social science research has the potential to illuminate and clarify the practices we are studying as well as the possibility to be *incorporated into the very practices being investigated*. (Chaiklin, 1996, p. 394. Our emphasis)

We therefore regard our research as *developmental*, and consider ourselves to be working within a developmental paradigm whose “dialect” contrasts with dialects of *confirmation* or *description*, which have, respectively, “grammars” of *randomized testing* or *ethnographic description* (Kelly, 2003, p. 3).

The operative grammar, which draws upon models from design and engineering, is generative and transformative. It is directed primarily at understanding learning and teaching processes when the researcher is active as an educator. (Kelly, 2003, p. 3)²

In this paper we focus on a major issue that has arisen in early stages of the project. This concerns the roles of didacticians in working with teachers to develop an inquiry approach according to theoretical principles in the project. In the early stages of interactivity, didacticians have to find ways of drawing teachers into understandings of inquiry and inquiry approaches so that teachers can explore possibilities related to their own school practices. Interaction has to respect and build on teachers’ professional autonomy in their work with pupils. As research questions, we ask, what is the nature of the didacticians’ role(s)? How are such roles conceptualised, and what issues do conceptualisation and subsequently implementation raise for didacticians and for the project? We are also interested in issues raised for teachers, but data to address this question will be gathered at a later stage. Here we draw on data from meetings in which prospective issues are discussed, and from very early interactions with teachers.

Theoretical Perspectives

Our focus on teaching development through building and studying communities of inquiry draws on Wells (1999) perspective of dialogic inquiry as “a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p. 122). A community of inquiry can be regarded as a context for teaching practice, for research practice and for a transition from the research to the teaching practice. Theory is implemented and developed, from the didacticians’ background concepts of inquiry, by a discourse about inquiry within an inquiry community of didacticians and teachers. We see “inquiry” as a unifying factor between research and the learning and teaching development on which research has focused. We develop inquiry approaches to our practice and together use inquiry approaches to develop practice. Thus, we see inquiry in three mutually embedded forms or layers:

² Here, Kelly is talking about a “Design Research” paradigm (DR), but we believe DR to be part of a broader paradigm which we regard as *Developmental* (Jaworski, 2004b).

- *Inquiry in mathematics*: Pupils in schools learning mathematics through exploration in tasks and problems in classrooms;
- *Inquiry in teaching mathematics*: Teachers using inquiry to explore their design and implementation of tasks, problems and activity in relation to pupils' learning in classrooms;
- *Inquiry in research which results in developing the teaching of mathematics*: Teachers and didacticians researching the processes of using inquiry in mathematics and in the teaching and learning of mathematics.

In each of these layers we have people as individuals and people as groups inquiring into mathematics, mathematics teaching or into the contribution of research to teaching development. We are all deeply embedded in social and cultural worlds (including political, economic, religious and systemic factors). Knowing can be seen both as situated in the context, community and practices in which we engage and as distributed within a community of practice (Cole & Engeström, 1993). Wenger (1998) has emphasized learning as a “process of becoming” in a community of practice (p. 218). We see inquiry as an important element of agency within a process of becoming, and prefer to talk of a *community of inquiry* in which both teachers and didacticians engage in inquiry. A feature of a community of inquiry that distinguishes it from a community of practice, according to Wells (1999) is

the importance attached to *meta-knowing* through reflecting on what is being or has been constructed and on the tools and practices involved in the process’ (page 124, our emphasis).

Inquiry can be conceptualized as both a *tool* and a *way of being* (Jaworski, 2004a). The project aims to use inquiry *as a tool* to develop inquiry *as a way of being* in developing teaching and studying related classroom activity and learning of pupils. Inquiry (as a tool) can be seen to stimulate accommodation of meanings central to individual growth and is also a way of acting together (a way of being) that is inclusive of the distributed ways of knowing in a community. As part of a community of inquiry, individuals are encouraged to look critically at their own practices and to modify these through their own learning-in-practice. It is within this theoretical frame that teachers and didacticians collaborate for mutual learning.

Potter and Wetherall (1987) suggest that there is always a tension between individual self expression and social determinism. We see the concept of ‘role’ as a means of reconciliation between the two. The ways in which individuals develop their role with respect to notions of inquiry and community is central to the project. Our didactician team has built (common) understandings of inquiry processes and their (theoretical) interpretation in establishing the project. We wish to develop inquiry in mathematical activity in classrooms, and in exploring the teaching approaches to develop classroom activity. So far, these theoretical principles are ‘owned’ by the didacticians. A key issue for didacticians is how collaborating teachers will start to

think in inquiry terms and to use inquiry in classroom work. What roles can didacticians take in drawing teachers towards a community of inquiry?

While the nature of role for any individual is important, we are here focusing on roles from a perspective of social interactions (Mead, 1935, Giddens, 1993) with analysis that focuses on discursive practices (Fairclough 1992; Chouliaraki & Fairclough, 1999). In our study, roles emerge in the discussion taking place during project meetings. Such meetings have the purpose to plan the activity of the project. In doing so, they contribute to knowledge and awareness of didacticians in the project and to an inquiry community of didacticians. The recorded meeting is data for analysis, and analysis of this data leads again to enhanced awarenesses of individuals and growth of knowledge in the community. Thus the developmental research paradigm is fundamental to both empirical research and development within the project.

We see our contribution to the work of Group 11 at CERME4 focusing on *a developmental paradigm*: its nature in revealing relationships between theory and practice, and its power both in offering a critique of the predictive role of theory for practice and in enabling the development of theory for deeper relations with practice.

Data Collection and Analysis

Analysis reported in this paper is of qualitative data from early project meetings of didacticians preparing for mainstream project activity: (a) for workshops between didacticians and teachers; (b) for school groups, where teachers will design activity for the classroom with didactician support. Inquiry in workshops and school activity is intended to lead to inquiry in classroom innovation and experimentation using designed materials. The early data takes the form of meeting notes, audio and video recordings of meetings and personal reflections from didactician/researchers.

We have maintained an events calendar in which we have recorded meetings and other activity, together with details of people involved and related sources of data. A search of this events calendar revealed 15 meetings that could illuminate our research questions for this paper. We made a short factual summary of the content of each of these meetings. From this summary we found 5 meetings that contained elements explicitly related to the topic of 'role', and divided them into episodes through a factual data reduction exercise which showed the topics discussed in each episode. There were 2 meetings where roles were treated explicitly and in some depth; we transcribed those episodes which related directly to our research questions.

In order to investigate how understandings of didacticians' roles are constituted in the flow of the conversation occurring during selected meetings, we have taken a dialogical approach to communication. We focus on *what is made known* and reciprocally *made understood* by *what is said* by participants in the meeting context (Rommetveit, in Linell, 1998). Some dialogical properties included in the present analysis are, following Linell (1998), (a) sequential organisation, (b) joint, social-interactive construction and (c) interdependence between acts (local unities) and

activities (global units and abstract types). The first, (a), refers to the way we approach the empirical material, focusing on how “utterances are both informed by a prior utterance and are consequential for next utterances” (op. cit p. 179). This sequential flow of discourse shows, in a pragmatic way, the social interaction among participants and, at the same time, their joint sense-making (b). In this way, the interdependency, between the local unities of participation and the general activities produced in the conversation, is established (c).

In our analyses, we try to identify ways in which the discussion about roles led to emergent understandings during project meetings. We first present a descriptive account of the flow of discussion revealing perspectives of some participants, exemplified by key quotations from the discussion. We follow this with a rationale taking up the main issues and looking at these through our first experiences of interaction in practice. Finally, we look critically at how the developmental paradigm is manifested in both activity and analysis. Here, the people concerned are all didacticians. For simple anonymity we label them D1, D2 etc in order of contribution, keeping the same label consistently for each person.

Descriptive account of a flow of ideas developing concepts of “role”

In this section we present analysis from conversations occurring during two project meetings between didacticians that focused on our proposed activity in workshops with teachers and forthcoming work with schools. We are interested in

- i. The flow of ideas in the dialogue – tracing how one perspective leads to another;
- ii. Styles of (potential) interaction (between didacticians and teachers) that emerge from discussion (e.g. facilitation, holding back, asking questions).
- iii. Development of concepts for individuals as related to community development.

We have extracted text that relates to the role of didacticians as they consider their (future) interactions with teachers in workshops and school settings. We try to identify the didacticians’ contributions to the understanding of these roles through the content of the conversations. We see clearly here a variety of different individual views, but also the ways in which through the flow of discussion, common understandings emerge. From such discussion and concept development, itself of an inquiry nature, understandings of modes of interaction develop both in community and for individuals acting. We see here key elements of the emergence of an inquiry community.

Focusing on workshops

In the first of the two meetings (WP040603) the Project Director (PD) launched a discussion with the words: “We need to think carefully about our own role in this activity” ... “be aware of drawing teachers into this activity and doing all we can to enable them to be full members”. She used phrases such as “insuring inclusivity”,

“should not patronize”, “involvement has to be sensitively judged”, and, “these things are easy to say aren’t they, but what does it look like when we’re actually there”? The transcript shows colleagues responding differentially. D1: “we have a project ... we will work together, so we have at least one thing in common ... but be clear that we are all different”. D2: “a big thing here is to get everyone to trust each other ... could it be an idea to make a letter, or a note, or *workshop 1 goals*, and give them to the teacher beforehand”?

D2 suggested that if the letter was sent in advance, teachers might not find the workshop so frightening. There was laughter as others said it might be *more* frightening. D3 felt the letter might be less frightening if it focused on “the process when a [small] group works with a [mathematical] problem: what can happen, what opportunities, what different ways to work... [emphasizing] that we are not trying to test their individual knowledge”. D4 suggested that, in producing a letter, “we are going to this meta level too quickly”. It is better to “be brave and get to work ... then afterwards think about what we did”. The word “trust” kept coming up. D5 said, “I agree that it is very important here to build confidence and trust ... also linked to the way we want to develop” and “that’s a big goal as I see it here, really to trust your own thinking”. Several remarks referred to a teacher who had joined us in one planning meeting, and how she had seemed to gain confidence from the way her group had worked. PD suggested this had been encouraged by D5’s contribution in this group. Perhaps D5 had performed a *coordination* role. Thus, PD suggested “presumably we need a group coordinator” and that the coordination job would involve “taking responsibility for inclusiveness”.

D3 suggested that inviting a teacher to relate a problem to children in the classroom would draw overtly on the teacher’s expertise, “here I am, I know something and can share with the group and it is important”. PD responded with “the coordinator has this orchestrating role”. D2 said “I have to, don’t push the mathematician in me so hard, try to be more like a didactician in the group, but I don’t think we need a coordinator ... I think that role should be divided by all group members”. This comment led to a discussion of giving explicit roles to each person in a group, D5 felt that “this would be too technical in the beginning”. He also was “not happy with the coordinator either” ... seeing his name at the head of a group (in a list of groups) made him ask “am I the boss here”?

These remarks on coordination led to a discussion on how to get a group started, avoid individuals “telling the answer”, “we should try to engage”, “what about the way it is important to hold back a bit ... so you don’t do all the work”. “I was just thinking of the word facilitator ... more that kind of role”. The word facilitator was not rejected as coordinator had been. PD used the term “gentle facilitator”. This seemed perhaps to capture the nature of the role that was emerging in the discussion.

Many things could be said about this extraction from the data. Principally we have offered it to demonstrate a flow of ideas, participation of all the eight colleagues present, and to contribute to a perspective of a growth of community knowledge or

awareness in which individuals developed their own perspectives. For example, the word “facilitator” became an agreed word for describing the role of a didactician working in a small group with teachers. D2, who had suggested the letter to teachers, acknowledged his own shift in perspective across this and subsequent meetings.

Focusing on work in schools

In the second of the two meetings (Me040819), the discussion starts with PD summarizing the theoretical perspective, of *inquiry*, and relating to practice in the project. Her words suggest that inquiry is an agreed perspective. However, belief that “inquiry is a process that can be extremely useful for developing mathematical thinking and understanding” (predictive theory) does not ensure practical outcome:

...how we are going to achieve that on the way is part of what we are looking at so we are looking at the process in which this works and we are looking at the *outcomes* there and the outcomes in a sense are going to give us some evidence to go back to the theories that we started off with...

We see here three strands relating to ongoing thinking: (i) there are theoretical notions, such as that of inquiry, which are well rehearsed in the language of the project. (ii) there is the hypothesis that creation of an inquiry community will be beneficial for students’ learning. (iii) the key element here: how are we going to achieve, in practice, what we have set up in theory? The outcomes that we document should take us back to our theories for questioning or strengthening.

D6, introduced a hypothetical situation of a teacher asking him for suggestions to use in the classroom. He expressed his concern about the nature of this relationship.

[D6] ...I think they have an expectation that we will come with something and ... when they plan a lesson they would say er “do you have a good suggestion for er for er probability?” and then you can say okay this is one kind of inquiry from the teacher...

This seems to ask: what if teachers ask us directly for ideas for the classroom, on mathematical content for example? Should we provide something they can take and use directly (uncritically perhaps), or deal with the request in some other way that is more inquiry focused? D6 continued

[D6] ...I am seeing kind of er different sort of inquiry, and do we, how do we facilitate the teachers to have *real* inquiries for themselves? What they *really* inquire into, and I think that’s a question dealing with our roles...

Implied here is that a teacher asking for ideas for the classroom is not engaging in “real” inquiry. It highlights that didacticians have a certain (agreed?) sense of what inquiry is, or what it is not.

[D6] ...I think that the tension me and [D2] had a briefly ((laughter)) over the er, in the morning it’s kind of an important issue because that’s the same thing that the teacher will confront when they are er working with their students...

D6 and D2 had been talking about such issues. Could they act in a way that would be a model for the teachers in working with their students? The issue here is how the role is made explicit and how it addresses fundamental aspects of project theory. PD asked what we might mean by “*real inquiry*”, relating to theoretical concepts of “inquiry as a tool” and “inquiry as a way of being” in interaction with teachers. She exemplified these concepts in practical terms related to teaching activity.

Another didactician, D4, expressed her concern about the need to inform the teachers about concrete materials which are essential for the learning of ICT.

[D4] Well on the other hand er there, it might be some teaching ideas, some tasks and things that could be useful for the teachers to know, and that they don't know from before and also perhaps a computer software that, give some, good opportunities to, to look closer at the concept, so, should we not give away anything?

Here we see a flow of ideas that questions an emergent perspective that didacticians should not offer ideas to teachers: that sometimes offering such an idea might be a helpful act. D2 responded to this as follows:

[D2] Yes, but maybe we should wait for the teacher have er asked some more questions than just how do I, how can I teach probability...how do you think the pupils will learn best by, doing task or by, finding questions themselves...

He points to the pupils' inquiry, indicating a difference between doing tasks and finding questions.

PD then summarized the earlier contributions of different participants with reference to the general theoretical framework of the project. She called attention to processes of establishing an environment where it would be suitable to offer some sort of material, tasks or ideas. The focus is about when to share ideas, where these ideas come from, and the implications of them coming from didacticians. Following this D2 emphasized the importance of sharing expertise in order to establish dialogue. He emphasised how the contribution of the teachers is important in order to establish the inquiry model.

[D2]...Er and of course in the dialog with the teacher er, they will have to understand as I know that I don't have, any, any best sequence or best, er tools for learning statistics er or any other subject and, they have to share with me their expertise on their students and, and they have to, er give me insights in their classroom...

PD responded as devil's advocate, challenging D2 and others to address potential negative response from teachers:

[PD]...what about the teacher who says oh this is pointless you aren't being of any help to me? All you are doing is, asking me questions about you know you're not you're not helping me to develop these things that I want to develop

D2 reacted to this challenge by stressing the importance of knowledge that teachers bring from the classroom, an idea that can be traced back to the previous meeting (see discussion above): “I don’t know er how good an eight, a boy in eighth grade are in probability or anything so, so, he [the teacher] will have to, he will have to share his experience, expertise er before or er while I’m sharing mine”.

It seems important here to remark that all the above data was collected in preparation for the events with respect to which “role” is discussed. Thus, discussion is speculative and reflects potential reality rather than actuality. We have little data yet about the actuality. However, the anecdote below shows early evidence of the nature of development for one individual.

Development of concepts for individuals

In an addendum to analysis of meeting data, we present a short anecdote involving one didactician (D2 in above data) who has acknowledged his own development of ideas as meetings have shifted into early work with teachers in workshops and schools. His reflection after a meeting in one of the project schools shows how he is relating what happened to the issues discussed in earlier meetings. His interpretation of “gentle facilitator” role was significant in his account of the incident. This school (with students in grades 1 to 10) had used, with students, two of the mathematical problems presented at the workshop and one teacher described the outcomes. To facilitate the discussion the didactician asked how they saw the relationship between the problems and their curriculum. Trying to be ‘gentle’ he reported how other teachers in an upper secondary school (grades 11-13) were worried about curriculum issues related to the project. He described the response from teachers as “an uneasy silence”. He writes in reflection on this event:

This is related to earlier discussion on explicitness. Shall we share as much as possible of our concern as researchers? And this relates to the question of when to share. According to previous shared understandings to questions related to teachers asking for teaching materials, should I share these reflections from the meetings with the teachers? Will these reflections interfere with my work with the teachers e.g. will they be afraid or too aware of what to say? My answer, for the moment, is: we expect teachers to ask questions and we will share teaching ideas within some sort of inquiry environment, these meta-reflections and research issues will also be shared when or if teachers ask questions about it and they too can be provided within some sort of inquiry environment.

D2 acknowledges his own development of ideas from the meetings and into current work with teachers. There is abundant evidence in our data of such development occurring widely for individuals—currently didacticians, but we shall be seeking evidence of teachers’ development also. We need to track such development and show how interpretation of theory through such activity leads to clearer understandings of theory-practice relationships. We are exploring the use of an

activity theory framework to explore such questions, and discuss this in another paper (submitted to PME 2005).

Discussion

Reflecting on the focus of this (CERME4) paper, D6 wrote as follows:

I see this paper as dealing with issues before we meet teachers, inquiring into what these meetings could look like and what sort of interactions would emerge, building on our past experiences as teachers and working with teachers. This brings us into issues of educational development that often takes up a model of those-who-know doing something to those-who-don't (top-down model). This would not happen within an inquiry community model. A top-down model has been shown not to be successful and is partly responsible for research ending up as being not of value for educational practices. So when the traditional borderline between researchers and teachers is reconsidered, we have also to reconsider the different roles of didacticians working together with teachers.

We want to study the development of inquiry communities within the project. What we see above is a tracing of ideas, questions and issues across project meetings in the community of didacticians. We have tried to capture flow of ideas, suggested styles of interaction with teachers, and development as it might be seen for individuals. From the first meeting, we see a flow in the discussion towards the "gentle facilitator" role. Important is not so much this end point, but the growth of understanding of role through terms like coordinator or facilitator. PD reflected that she did not see any difference between concepts of coordinator and facilitator, but it was clear that her colleagues did, and it was important to elaborate understandings of these terms. In the second meeting, we see a flow of ideas from offering teachers a sheet of goals, towards ways of encouraging teachers' development of inquiry as a way of being. Discussion on theoretical and practical relationships was lengthy here, and we have been unable to include illustrative evidence.

In the developmental paradigm, it is impossible to plan in a clear and systematic way what will be done and how (except at organisational levels). Although theory suggests ways in which inquiry will enhance practice, there are many stages between a theoretical exposition and the outcomes of practice. Human interaction, and interpretation through interaction, are fundamental to fleshing out theoretical manifestation and growth through practice. In our theoretical perspectives, notions of *inquiry* and *community* are fundamental to our project. The literature provides many insights to theoretical concepts and issues. It cannot, however, tell us how to act. As we act and interact, a study of the activity involved reveals essential issues from practice that theory in its present form cannot predict. We seek ways to enhance theory through our analyses, feeding back subtleties and nuances of meanings and interpretations, to provide a richer theoretical base. Thus, we see the developmental paradigm, linking theory, research and practice, as central to any growth of knowledge that relates to improving practices of learning and teaching.

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DIFFERENT PERSPECTIVES ON COMPUTER-BASED GRAPHS AND THEIR MEANING

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Abstract: *In this paper we present a case study from an activity at secondary school level in which students have to perform motions (walking in front of a sensor), in order to obtain a space-time graph (on a calculator), as close as possible to a given graph. The aim is to analyse empirical data on the students' approach to the two graphs through different theoretical lenses (transparency, fusion and semiotic node), with reference to recent literature. The integration of these lenses provides us with a multi-faceted frame to suitably analyse the activity of our students, thus going beyond a consideration of the mere cognitive processes and embracing the whole learning context in its complexity.*

Introduction

Students' difficulty in constructing graphs using paper and pencil is well documented in the literature. The introduction of new technologies at school made available a lot of graphical settings, allowing for a widespread use of computer-based graphs in math curricula everywhere. Graphs are then now more accessible for the students. But the issue of the construction of their meaning is still an open research problem.

Our study aims at analysing the activity of 9th grade (14 years old) students engaged in reproducing a given graph, by moving in front of a sensor, which is the artefact in use together with a symbolic-graphic calculator. We have previously (Ferrara *et al.*, in press) considered two ways an artefact can get involved in an activity: as a *black box* or a *transparent box*. Here we want to extend this view, taking into account different theoretical lenses recently provided in educational research on graphing, and analysing differences to integrate them in a multi-faceted frame. Graphing is meant in the sense of Ainley (2000): "to encompass a number of related activities: drawing graphs, reading graphs, selecting and customising graphs for particular purposes, and interpreting and using graphs as tools" (Ainley, *ibid.*; p. 365).

Theoretical framework

The first theoretical lens we consider is the notion of *transparency*. As a general notion, it may refer to broader categories, e.g. artefact and sign, of which the graph can be considered a particular case. Lave & Wenger (1991) defined transparency using the

metaphor of a *window*, which is invisible as we look at the view beyond it, and highly visible in contrast to the wall that contains it. Similarly, a graph may be invisible in giving access to features of the phenomenon it represents, and visible to inspection for extracting detailed information. The graph is considered transparent when it has both the features: visibility for itself and invisibility when the student sees beyond it the description of a phenomenon, as a tool or an artefact^[1]. Meira (1998) adopts this viewpoint when he speaks of transparency as an index of access to knowledge: “artefacts become efficient, relevant, and transparent through their use in specific activities and in relation to the transformations that they undergo in the hands of users”. To him, transparency is not an inherent (objective) feature of the tool, but it emerges through the very use of the tool itself. Roth (2003) characterises the notion of transparency within the same perspective, “not as a property of a tool (object) but as a type of relation between user and tool” (Roth, *ibid.*, p. 162). The consciousness and the cultural experience of the individual become relevant, since a graph exists just “in a metonymic relation to the entire research situation and the process that has led to the construction of the graph” (Roth, *ibid.*, p. 164). This is the same assumption of Cobb (2002), who says that graphs do not exist only in terms of the things that they represent (their referent), but also in terms of the work processes that they resulted from.

We want to highlight something essential that remains in the background in the given analysis, namely the fact that graphs appear with a dual nature: as tools and as symbols (or signs^[2]). In this respect, the theoretical frame needs to be enriched with other lenses. Here we consider the notion of *fusion*, and that of *semiotic node*. Fusion means “talking, gesturing, and envisioning in ways that do not distinguish between symbols and referents” (Nemirovsky et al., 1998, p. 141). The correlation between transparency and fusion has already been suggested by Ainley (2000), as “identifying fusion within discussion about a graph offers a clear indication that the graph is being used transparently” (p. 366). Following Nemirovsky, in our case the notion of fusion can be applied when students become able to go back and forth between the graph as a shape and the graph as a response to actions, namely their body motions in front of a sensor. The notion of semiotic node has been developed by Radford to describe those “pieces of the students’ semiotic activity where action, gesture and word work together to achieve knowledge objectification” (Radford et al., 2003, p. 56). “Objectification of knowledge” is meant as the semiotic process that allows the students to successfully construct (mathematical) concepts, starting from their perceptions and interacting with cultural artefacts. This notion is developed by taking into account the integration of different semiotic systems (Radford et al., 2004): body actions, artefacts, graphs and speech.

Based on a specific case study, our analysis is aimed at integrating these three theoretical lenses, in order to analyse the students’ process of construction of meaning in graphical settings.

Methodology

The activity we describe in this paper lasted three hours and is part of a long-term teaching experiment carried out in a classroom of 25 students attending a scientifically oriented high school (9th grade) in Italy. The students worked in *small groups* (of three-four people), using two technological tools: the CBR (a motion sensor that collects space-time data in real time) and the symbolic-graphic calculator TI92.

The main activities of the teaching experiment were:

1. students' motions in front of the sensor and interpretation of space-time graphs obtained on the calculator (details in Ferrara & Robutti, 2002);
2. students' analysis of graphs (given on paper; in the remaining of the article these graphs will be called *paper graphs*) and of motions to be performed, in order to obtain the same graphs on the calculator (*computer-based graphs*), through the movement in front of the sensor.

The aim was the construction of mathematical and physical concepts as function, slope, velocity, acceleration and their change. The second kind of activities (point 2 above) worked at once as a feedback for teachers and researchers with respect to the first one, and as an occasion for students to create a motion in relation to a graph they had decoded. The teaching experiment was part of a National Project funded by the Italian Ministry of Education, called SeT (Science & Technology), where two of the authors were involved^[3].

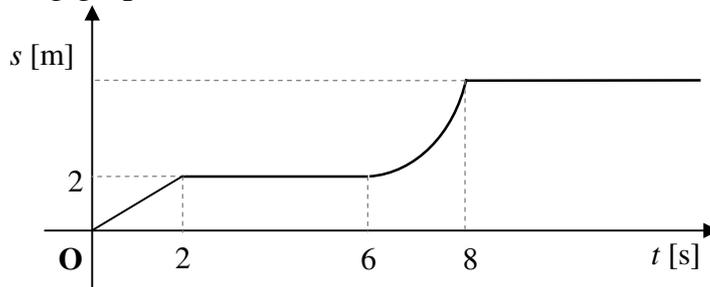
Besides the students, four people were present in the classroom: the Mathematics and the Physics teacher, and two of the authors as observers. A camera video-recorded the activities. The data analysis was carried out by looking at the videos and writing the transcripts, together with field notes taken by observers and teachers.

Each activity of the project was divided in two or more sequences of group work and collective discussion. The group work engaged the students around tasks given by the teacher and described on a paper sheet. The collective discussion was guided by a teacher or an observer, with the aim of sharing ideas, comparing processes and results of the groups and guiding the students to the conceptual knots of the activity.

We focused our observation on a small group of three boys: Filippo, Gabriele, and Fabio. They are all average achievers, but with different natures: Filippo was reserved, studious and thoughtful; Gabriele was an inconstant student, going on with a personal rhythm; Fabio was a bright and intuitive boy.

The activity

Consider the following graph.



- 1) Describe the graph reproduced here, in terms of motion detected by the CBR.
- 2) Perform a motion so that the CBR detects space and time data, providing a graph as close as possible to the given one.
- 3) Compare the graph resulting on the calculator with the given graph; if necessary, repeat the motion, describing what you have modified.

Integrating different lenses

The three theoretical lenses of transparency, fusion and semiotic node, defined above, need to be integrated to suitably analyse the activity of our students. In fact, the analysis of each sort of activity entails not simply attention to students as individuals, but even interest on both the nature of the involved object (in this case, the computer-based graph), and the manner the students make sense of it in light of motion, and quantities as distance, time, velocity, and their changes. All together, the different lenses give us an overall insight on how the situation evolves. The three lenses give us the chance of merging perspectives, going beyond the simple attention to the cognitive processes, and embracing the whole learning context. In order to consider the connections that can be established between the lenses, we provide a description on their use in our context. The notion of semiotic node is framed in a semiotic/cultural approach to students' cognitive processes. It let us see those moments when students introduce new pieces of knowledge objectification. The lens of transparency is more centred on the mediation role of computer-based graphs in the whole activity, taking into account both subjects and context. Visibility and invisibility are features of a graph, in relation to how one looks at it. When the subject is able to 'read' in the graph the phenomenon it represents, then he/she is using the graph transparently. The notion of fusion, being more local, can be seen as a bridge between semiotic node and transparency, since through it the interpretation of cognitive processes is possible in light of the graph they are using. On the one side, it does not distinguish between symbols and their referents, so that (following Nemirovsky) the qualities of the computer-based graphs are merged with the qualities of the represented events. On the other side, it considers this merging looking at students' words, gestures, and glances. The basic idea, which links the two sides, is that of making present in the graph the absent. For example, speaking of a motion-graph as it

were the physical motion (through reading in it the various phases of motion) is an example of a fusion experience, in which the phenomenon is made present in the shape of the graph. Conversely, the shape of the graph mirrors the particular kind of motion performed. Enlarging Ainley's idea, we could then stress that: *fusion experiences within a discussion about a graph are clear clues that the graph is being (or is going to be) used transparently, but also that a process of knowledge objectification for the graph is occurring (or is going to occur), as the presence of a semiotic node can reveal.* The analysis of the activity bearing in mind this idea can shed light on the integrated use of the three lenses.

Protocol analysis

In the first phase of the activity, the group analysed the paper graph in terms of the motion to be performed in front of the sensor. The students discussed on the shape of the given graph, and also on the kind of motion they had to perform, in order to obtain the required graph on the screen of the calculator. Afterwards, one of them (Fabio) walked in front of the CBR, according to the planned features of motion. As a result, they obtained the computer-based graph represented in Figure 1.

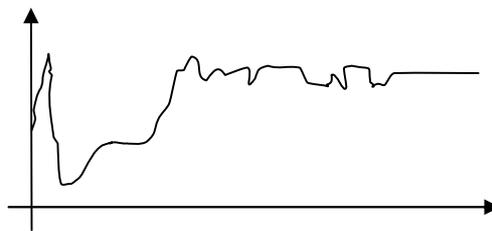


Figure 1

The two axes represent time and space variables with measurement units seconds and meters (horizontally and vertically respectively, as usual). Data gathering last 15 seconds. In the following, the students are comparing the paper graph and the computer-based graph, in relation to Fabio's motion.

52. **Filippo:** *"The motion is similar [his finger is running on the first part of the computer-based graph], it is only here [in the final part of the graph, box E^[4]: Figure 2] that he [Fabio] didn't stop, otherwise..."*
53. **Fabio:** *"But, I don't understand why before, before..."*
54. **Gabriele:** *"And there [he is pointing to the initial peak, box A] it is when he starts [his finger is running on the ascent] and then he goes" [his finger is running on the descent]*
55. **Filippo:** *"Don't care about this" [his finger is running on the initial peak, box A]*
56. **Fabio:** *"But, why does the curve go up and then down?"*
57. **Gabriele:** *"It is this part, here [his finger is running on the peak] that goes up and then down, rather than just going up [he is drawing a small ascent in the air with his pen]... obviously you moved in front of, of..."*

58. **Filippo:** “Did you come back?”
59. **Gabriele:** “CBR”
60. **Fabio:** “No, here [he is pointing to the final horizontal part] it is when I am ...”
61. **Filippo:** “Of a step?”
62. **Fabio:** “Here it is when I am motionless”
63. **Filippo:** “Yeah”
64. **Gabriele:** “There”
65. **Fabio:** “Here [his finger is running on the part in box D] it is when I accelerate”

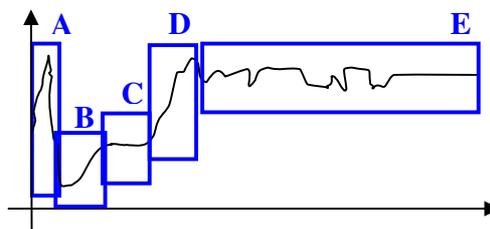


Figure 2

66. **Gabriele:** “There [he is pointing to the first horizontal part, in box C] it is again the other point when you are motionless there [his finger is running that part, from left to right]”
67. **Fabio:** “Then here it is this, here [he is pointing to the part, in box D] it is when I accelerate, here [his finger is running on the final horizontal part, in box E] when I stay motionless and then here it should be a straight line, more or less and...”

The students are endeavouring to read the computer-based graph in terms of motion (#54), but they have a difficulty in interpreting the first peak (#55, #56; box A), which has nothing to do with Fabio’s motion (probably, this peak is due to an external interference). This difficulty comes from students’ expectations when they compare the computer-based graph and the paper graph. In fact they do not expect to see the first peak on the computer-based graph. Gabriele’s attempt to overcome such an obstacle is well expressed in his words and gestures (#57). Fabio makes a step forward to connect his motion to the graph (#62): he recognises the part of the graph that refers to the absence of motion (*Here it is when I am motionless*). This step is also immediately shared by his group mates (#63, #64). Then Fabio proceeds in making sense of the other parts of the computer-based graph, namely those corresponding to the motion (#65, #67). Hence, Fabio comes back to the final part of the computer-based graph which refers to an absence of motion (#67: *when I stay motionless*) and he stresses that he expected it to be horizontal (#67: *here it should be a straight line, more or less*).

Interpretation with the three theoretical lenses. Up to here, the students do *not* yet use the computer-based graph in a really transparent way, although they are progressively constructing a meaning for it. In fact, they are trying to see the features of Fabio’s motion in it, but they are not yet able to see (or at least to express) them clearly. They

are progressively approaching a meaning for the graph, which is not yet transparent. However, they are already able to link some parts of the graph with the corresponding pieces of motion, as marked by gestures and words (#54, #62, #65-#67). The coordinated use of gestures and words allows students to see graph and motion in an indistinguishable way, and for this reason, it constitutes a first example of fusion. On the one hand, locative words (e.g.: *here, there*) indicate precise positions on the graph, as outlined by the *pointing* gestures; on the other hand, the adverb '*when*' refers to the starting points of specific pieces of Fabio's motion. The students reproduce the parts of the graph referring to these pieces of motion through *iconic* gestures (for analyses on gestures in Mathematics Education, see: Edwards, 2003; Arzarello & Robutti, 2004). Particularly, the use of '*when*' allows them to shift between the graph and its referent (motion). The use of the personal pronouns as 'he' (#54), 'you' (#66) and 'I' (#62, #65, #67), all indicating the subject of motion, is significant. It shows how the different parts of the graph are interpreted in terms of Fabio's motion, by making motion present in their shape. The symbolic nature of some parts of the graph is re-constructed through the memory of the corresponding physical actions performed (by Fabio) or seen (by the group mates).

Step by step, a transition begins, from first perceptions of the students to mathematical ideas on the shape of the graph. Words and gestures are coordinated in the semiotic activity of the students, recalling kinaesthetic actions performed both with the sensor (body motion) and on the graph (pointing and iconic gestures), which begins to become transparent. From this coordination (#67) that can be interpreted as index of a semiotic node, the understanding the horizontality of the final part of the graph arises. The discovery that the final part should be a straight line allows students to objectify knowledge about the absence of motion. The use of the word '*then*' is relevant in expressing the causal relation between the motionless state and the horizontality and straightness of the line. Once this relation is made apparent, even the difficulty of understanding that the first peak has nothing to do with motion is overcome. Thus, the initial fusion restricted to single parts of the graph related to specific moments of motion, creates room for the later making sense of the horizontal line, which starts to be seen transparently. From this point on, the graph will turn to be more and more transparent, as the remaining of the analysis will show.

The group continues working, as follows (the teacher arrives to listen to the discussion):

75. **Fabio:** "*So, we consider starting from this point here*" [he is pointing to the lowest point of the computer-based graph]
76. **Gabriele:** "*Yeah, we have to consider starting from that point there*"
77. **Teacher:** "*Where do you consider from?*"
78. **Filippo:** "*From this point*" [he is pointing to the computer-based graph]
79. **Fabio:** "*From, from this point here*" [he is pointing with his pen to the same point of Filippo]

80. **Teacher:** “Which corresponds to, on paper?” [on the paper sheet]
81. **Gabriele:** “To this point here [he is pointing to the origin of the paper graph with his pen]... that is when, at the start, when we were motionless in front of the CBR”
82. **Fabio:** “Yeah, then we have this point here [he is pointing to the final point of the first ascending slanting part with his pen] which is when I stopped, the first time [he is pointing, with the same hand, to the corresponding point on the paper graph], when I stopped the first time, then here [his finger is running on the subsequent horizontal part] when I stopped for four seconds and I didn’t move, from here [he is pointing to the final point of the horizontal part], I start, they are six seconds and I accelerate [his finger is running on the curved part of the paper graph], here I’m accelerating” [he is pointing to the corresponding curved part on the computer-based graph, box D]
83. **Gabriele:** “And after having accelerated [his finger is running on the curved part] you, at the end you stopped [his finger is running on the final horizontal part, box E]... but then you moved and there are interferences along...”
84. **Fabio:** “Hence, here [his finger is running on the slanting part, box B] two seconds passed, here [his finger is running on the first horizontal part, box C] six seconds passed” [...]
89. **Fabio:** “Here eight and then [his finger is running on the final horizontal part, box E] they [the seconds] go on”

The students choose the part of the computer-based graph to be looked at, i.e. that on the right of the first peak (#75, #76; boxes B, C, D and E). The teacher’s input (#80) inserts at this moment and is relevant to the process of construction of meaning for the computer-based graph. In fact, in students’ immediate answer (#81), they disregard the first peak, and they do this considering the paper graph, which gives them a reference for the starting point of the motion. In the next excerpts (#82-#84), Fabio and Gabriele do correctly interpret the computer-based graph, not only blending words and gestures, but even going back and forth from one graph to the other, and simultaneously from the graph as a shape to the graph as a response to an action. In interpreting the computer-based graph, the students call time intervals with their measurements, expressed in seconds (#82, #84, #89) and in this activity they are aware of the corresponding pieces of motion.

Interpretation with the three theoretical lenses. Students now are using both the computer-based graph and the paper graph transparently. In fact, it is interesting to note how they focus (pointing with fingers, gestures that clarify the referent of the deictic words *this*, *here*, *there*) on certain positions on the former useful to locate the beginning of a new action performed during the motion, through the use of ‘*when*’. At the same time, they are able to see beyond the graph the different kinds of actions, introducing a temporal dimension (they planned the parts of Fabio’s motion using this graph; ‘*the first time*’, ‘*six second and I accelerate*’). We can identify in this dimension a rhythm given to motion, through which the paper graph is interpreted. Deictic and iconic gestures on the computer-based graph, and the corresponding gestures on the paper graph are coordinated with locative words, in a semiotic activity we can simultaneously interpret

in terms of fusion and semiotic node. Graph and motion are merged together, their rhythm being beaten by the measures given on the paper-graph. One after the other, the gestures condense the shape of the graph, accompanied by words referring to actions of the motion experiment. The piece of objectified knowledge comes from the introduction of the variable time to beat the rhythm of Fabio's actions, which are made present in the corresponding pieces of the graph (#82-#89). After the symbolic nature of the whole graph is re-constructed through the memory of Fabio's actions (#82, #83; look at the use of the subjects 'I' and 'you'), at the end (#84, #89) attention is uniquely posed on time. It is as if the one important thing is time dimension, since the shape of the graph is clear as a response to particular actions. We can say that the graph is transparent. Fusion experiences and semiotic nodes are both present, to relate the two variables of space and time by means of the horizontal straight line. In fact, at this point students understand the precise meaning (in terms of the relation between variables) of the horizontal straight line (box E): time passes ('*they go on*', #89), even in absence of motion (whereas space remains the same!). The whole graph is finally transparent for the students with respect to both the past phenomenon of motion (being motionless, and the other actions) and the relation between variables.

There is also a social element we want to highlight. In fact, even if only Fabio did actually experience the motion in front of the sensor, all the students refer to it and to the resulting graph as if they had shared the same experience in a very inner way. As a consequence, the entire group adopts the same linguistic structure, the same vocabulary, and performs the same kind of gestures. Lines #54, #62, #81 and #82 are particularly relevant in these terms: at the end, the students describe the motion using the pronoun 'we', thus revealing that they share both the motion experience and the interpretation of the graph.

Final remarks

Analysing data coming from teaching experiments in a technologically rich context as that described above, requires to draw attention to several different aspects: the students' cognitive processes, the nature of the objects involved, and the manner in which the students make sense of them through the artefacts in use.

In an initial study in which we had studied the role of the calculator in a pre-calculus learning context (Ferrara *et al.*, in press), we had used the notion of *transparency* as simply referred to a feature of a technological artefact. Here we wanted to enlarge such a point of view by taking more deeply into account the role of the activity and the user. This attempt arose from the awareness of the complexity of the learning context. To that aim, other two interpretative lenses have been chosen: the *fusion* and the *semiotic node*.

Some connections between the three lenses have also been suggested, pointing out that the notion of fusion, being more local, can be seen as a bridge between those of semiotic node and transparency. The integration of the lenses provided us with a multi-faceted

frame through which we could interpret our data through an in-depth analysis. Such an integrated approach is appeared more suitable than the initial one to deal with the complexity of the learning context.

NOTES

- [1] Tool and artefact are used as synonymous throughout the paper.
- [2] We prefer not to distinguish between signs and symbols in this context.
- [3] The products of this project are teacher-training materials published on-line at the addresses:
<http://www5.indire.it:8080/set/comunicazione/comunicazione.htm>;
http://www5.indire.it:8080/learning_risorse/Castelletto/set/Project2/Unit1/Cycle2/Cluster62/enr_med2.htm.
- [4] The boxes in Figure 2 have been inserted by the authors, in order to let the protocols and their analysis be clear.

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THE ACT OF REMEMBERING AND MATHEMATICAL LEARNING

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Abstract: *What is the link between memories and learning? We are interested in asking memories from the students. Our theoretical background includes Ricœur's work on time and narratives and the studies of the French didactical school on memory and time. Narratives from eleven years old students are examined from several aspects, the triple present, the rearrangement of the experience and the attitude of vigilance. This work is an opportunity to question the relationships between theory and practice and to give an example of the way one can influence the other.*

Key words: Memories, narratives, time, experience, theory and practice.

This paper presents some research about the problem of time and temporality in both the teaching and learning of mathematics. The question we ask roots our work in this working group of CERME 4: what does this work tell us about the relationships between research and practice, between theory and reality? Our answers will be contextualized and we do not claim that our observations can be immediately generalised.

There are various ways of using and linking different theories and there are also various ways to imagine the relationships between theory and didactical reality. A theory can be a tool to produce teaching devices. It can also be a tool to analyse the teaching activity in ordinary classes. On the other way round, the observation of teaching leads to questions, which will open new research developments.

Our research is based on "theoretical interbreeding". This means that we use several theories to imagine, develop and analyse teaching devices. In this work, we will examine how we use some theoretical works (those by Ricœur) that are not in the field of the didactics of mathematics, to present the problem of time in the classroom. This external point of view allows us to reconsider both the didactical theory and the teaching activity. From that aspect, the links between research and teaching are very strong in our work. This interweaving leads to the creation of a prototype of teaching that we will present later on and which we call "multiple device".

At the beginning of our questioning, we find a teaching question: how can one take into account the past mathematical activity of the student? How can the teacher be sure that the student has learned what s/he was supposed to learn before? How can the teacher know about the way the student copes with old mathematical objects? For

some time we used to ask students, in school or at the university, to write about mathematical memories in order to know what was remaining, on the mathematical point of view, after the teaching was over. These memories were very poor, almost void of any mathematical content. We questioned ourselves, as teachers and as researchers, about this poverty. Our first explanation took into account several factors strongly interwoven: it could be that the students had not experienced any mathematical event, or that they did not remember those events, or/and that they did not feel allowed to talk about them in a mathematical classroom. We wanted to go beyond these first explanations.

Theoretical Background: Time, Memory and Narratives

Every learning, seen as a process, has a temporal aspect and the knowledge that emerges from it is the result of a story, however “poor” or short this story might seem to be.

The existence of different kinds of time in a classroom has been noticed and studied by several researchers in didactics. There is first the objective time, the time of the institution, which organises and makes public the time of the teaching system: this time is the “didactical time”. Chevallard & Mercier (1987) and Leutenegger (2000) showed how the didactical time is important in textualising knowledge and regulating the didactical contract. Brousseau & Centeno (1991), and Matheron (2001) studied the importance of didactical memory to remember past events when pupils are learning something new. Apart from this didactical time, the teachers and the students are engaged in a private and subjective time, which is, quite often, silent and implicit, which we call “the students’ temporality”. Following Varela’s works (1993), Arzarello & al. (2002) also distinguished two times: the “physical time” or clock time, and the “inner time”, which emphasises actors’ time, especially pupils’ learning time. Pupils’ learning time was also studied by means of didactic biographies by Mercier (1995), or through other ways such as the “fractions diary” (Sensevy 1996). More recently, Amit & Fried (2004) were interested in the different treatments of time in different school cultures and Assude (2005) studied teachers’ time management strategies.

Our theoretical hypothesis are the following:

- The students’ personal temporality plays an important role in the learning process and therefore it is worth studying it, even if the constraints of the didactical time may appear prominent. This temporality is a tool in the student’s structuring of the mathematical knowledge, in particular in the paradoxical relationship with the features of necessity and permanence of mathematical truth. The difficulty is that it is not easy for a student to express her/his personal subjective time in the classroom, because of the weight of the didactical time.
- The learning temporality is not going straight forward: one has to go back and one also has to be able to anticipate. The difficulty is that the temporality of the student’s activity is organised by the temporality of the various activities in the classroom that are all dependant on the didactical time. The fact that the didactical time is always

going forward induces an absence of chronology in the memories of the students: there is no *before*, no *now*, and no *after*.

Our phenomenological approach of the teaching situation, more in the sense of Brown (1994) than of Freudenthal (1983), taking into account what happens to the subject and the way s/he may be conscious of it, follows Ricœur's works on time and memory.

Ricœur (1983) points two aspects:

- the tension between the three components of what he calls the *triple present*, in other words the coexistence of three intentions: memory of the past, attention to the present, expectation of the future. One must differentiate the time of remembering and the time of the action.

- the undertaking of this tension in the narrative act. Asking for memories can induce, for the student, the dynamic of the triple present: to pay attention to some past event in order to write about it, places the student between the past and the future.

In the didactical context, these analyses led us to work on the subject's consciousness of the temporality of mathematical knowledge, using the production of narratives.

Through their memories, students can pay attention to what is going on in the present, in order to prepare what will happen in the future. For us, to be conscious of the time of learning is not only to remember the past, it is also to pay attention to what remains of this past in order to anticipate the future. Asking for narratives is one part of our device that tries to create, for our students, an attitude of vigilance for their learning. This kind of attitude can be illustrated by a sentence such as: "here, I would have made a mistake, if...".

Ricœur's work permit us to consider that the narratives can have three main functions:

- *to express and to bring to the subject's consciousness* the personal time in the act of learning, as well as its power on the construction of knowledge,
- *to rearrange the personal experience* through stories that can be references for the future,
- *to produce a shared time*, which takes into account the subjective aspects of knowledge and brings together the personal times to build a collective time as well as a common story for the whole class.

The Didactical Device

We have been working on devices that allow the emergence of narratives, which could satisfy the three functions devoted to this "work of remembering". To obtain such narratives (rich, diverse, with a strong mathematical content), one has to modify, more or less explicitly, the didactical contract (Brousseau, 1998). It is certainly, a teaching device. Nevertheless, because of the theoretical elaboration from which it is issued and because of the hypothesis it is supposed to verify, it is also an

experimental device that gives us material to work on. We called it the “multiple device”. It is “multiple” because it integrates many elements co-ordinated by the teacher. We won’t describe this device in many details; one can find in Assude & Paquelier (2005) an analysis of each element, of its purpose in relation with the theory and, when necessary, the changes that occurred following the different experimentations. The teacher first started five years ago and improved the device little by little. Its different elements are the following:

- A questionnaire at the beginning of the school year,
- Weekly chronicles,
- The personal note book,
- True/false questions,
- The narrative of class discussions,
- “Two or three things I remember” (since..., about...),
- The questionnaire at the beginning of the following year.

We’ll give some details about one element of the device, the weekly chronicles. The students volunteer, in turn, one for each week, to be the “chronicler of the mathematical week”. At the beginning of the following week, the chronicler produces a text, one or two pages long, which recalls what s/he thinks has been the most remarkable events of the week, from the mathematical point of view. The command (and the help that is given by the teacher during the first weeks) insists on the fact that the purpose is not to copy the lesson as it was given by the teacher. One has to point at some facts that can help to understand “what is going on in the classroom”. Mathematical events are looked for: a common error driven out during a discussion, an idea that started a research in the class, a drawing that helped to understand,....

The teacher read these texts to correct some mathematical errors, but the style of the writing is respected as well as, of course, the choice of the related events. It happens that these chronicles are discussed in the class either to correct them or to add some more details. They are collected in a book that the students can consult anytime. They install a “memory of the mathematical class” that can be used if necessary. The teacher keeps a copy of the chronicles.

This element of the device has two effects: in parallel and in addition to the official text of knowledge, there is the constitution of a “text of the class”, which puts the students in a situation of creating mathematics. The second effect is more in relation with our theoretical background: the chronicles tend to put one particular student in the position of “the one who will have to tell”, in order that s/he be conscious of the fact that the attention (*now*) doesn’t depend only of the knowledge one has (*past*) but also of what is expected (*future*). By making the student responsible for the narrative of some part of the temporality of the classroom, the teacher invites her/him to pay attention to the perception of this temporality.

The creation of this device is, in itself, an empirical result of our work. It permits us to show how a theory can be used to produce something one can use in the classroom.

Empirical data

We will now be interested by the consequences of the use of this device by a teacher. We haven't yet any global answer to this question. We will examine some data related to the triple present that, for us, give evidence of the effect of the device in the direction we expect.

The work has been conducted in a class of 24 students of 6th form (age 11) from the French Lycée (Secondary School) of Madrid. This is the first year of secondary school and it is expected that the students revisit some notions learned in primary school, such as decimal numbers. In November, the students had to write two narratives, but as some only wrote one, we have 41 narratives to analyse.

In this study, we are not interested in spontaneous souvenirs, but rather in memories that emerge when a student looks for them to produce a narrative. These narratives can have various forms. We will focus here on narratives with very precise rules of production. The structure of these narratives is induced by the device. Our aim is to make the student able to point at the moment when something (an *event*) occurred, bringing to evidence the triple present we talked about in the theoretical part. The memory that is aimed at, is the memory of some passed experience of the student. The structure of the narrative is given by the three initial words "*before – one day – now*". Each student gets a sheet of paper on which these three moments are clearly identified by these words. S/he has to write her/his narrative on that paper.

We will study the following four points: the objects of souvenir, the presence of the triple present, the rearrangement of the experience and the shared time emerging from the personal times that the students recall.

What are the Souvenirs about?

Thirty eight narratives talked about mathematics; twelve were about numbers in general, twenty four about decimal numbers and two about geometry. Decimal numbers appeared in a majority of narratives. The reason for it could be that this subject has been studied just before the teacher asked for the narratives. It could also be that the event the students relate was really important for them, as a break in their former knowledge.

We will focus on the narratives about decimal numbers, thus being able to look at one topic from different points of view. These are the precise contents:

- definition of a decimal number (13)
- density of decimal numbers (7)
- an integer is a decimal number (3)
- transformation from a decimal writing into a fraction (1)
- product by 0.5 (1)

- rounding off a decimal number (1)

The Rearrangement of the Experience

The purpose here is to see how a student tells about her/his personal experience of what happened in the class room. In particular, do students identify the element that has activated the *event* and is this element the same for all of them? We'll study the narratives about the definition of a decimal number.

First of all we'll cite Chloé:

<i>Before</i>	<i>One day</i>	<i>Now</i>
Before, I thought that a decimal number was a number with a comma ¹ .	One day, we explained that the comma in a decimal was only the writing and not the "idea" one has in the head.	Now, I know that if I am asked what is a decimal number, I must not answer: "it is a number with a comma" but "a decimal is the quotient of an integer by a power of ten".

In most narratives, the "*before*" is the same: "I thought that a decimal number was a number with a comma". Only two students write: "I did not know that an integer was a decimal number".

The "*now*" is mainly: "I know that a decimal number is the quotient of a integer by a power of ten". Two students write: "I know it is an integer". Some are very vague: "I know exactly what it is".

The "*one day*" is different: it can be vague: "I understood what it was" or "there were proposals in the classroom". It can be precise like in Chloé's, it can also be wrong: "We said that all numbers were decimal numbers".

The mathematical event is initiated by a debate around a question from the teacher: "what is a decimal number?". After the debate, the students write the result that the class came to in their note books.

The ways the students relate to this are very diverse. Some only evoke the debate and the writing in the note book. There is no mathematics present in what they write. When mathematics is present, we can identify two different elements initiating the event: for some students it is the fact that an integer is a decimal number. Others, like Chloé learned to dissociate a number and the way it is written. All these students had then to rearrange their knowledge to take the new information into account. The rearrangement can also lead to errors ("a decimal number is an integer").

¹ In French writing of numbers, we don't use the point for decimal numbers.

The Presence of the Triple Present and the Attitude of Vigilance

As we said before, one purpose of this work with narratives is to lead the students to a new attitude in their learning of mathematics: they should use their souvenirs of the past to pay attention to the present in order to create an expectation for the future.

Lets start by the precise narrative of Omar:

<i>Before</i>	<i>One day</i>	<i>Now</i>
I thought that a decimal number was a number with a comma BUT	I understood that an integer was a decimal number AND	I know the definition of a decimal for all the days in my life: a decimal number is the quotient of an integer by a power of ten. THE END

Omar anticipates the future: “all the days in my life”.

We found the same anticipation in narratives on other subjects. We’ll read what Claude has written: “the teacher had given us ... the calculation of the sum of the integers from one to one hundred. At home I thought: if I do 10×10 it makes 100. So, if I do $10 \times (1+2+3+4 \dots +10)$, it will be the sum of the first hundred numbers. I had done this work on Saturday for the next Monday. On Sunday, to finish my work, I draw some sort of diagram in order to explain it to the whole class. I then realised that I had forgotten a lot of numbers. I had to do my work again. After that, I never made the mistake again, and I hope I won’t forget it and so never do it again.”

The attitude of vigilance has two components: first understand something new by rearranging one past experience, as we have seen above, second, become conscious of an error in order to avoid it later as Claude tells.

The Shared Time Emerging from the Personal Time

The narratives tell personal stories but they also tell collective stories. We have seen, in the case of Claude, that the project of explaining his result to others made him find a mistake in his work. The presence of the other students was a motivation be more precise in his work.

In all cases, the narratives recall events that had happened in the classroom or, at least have been initiated there. In that way, for each particular student, her/his personal story takes place in the story of the class. By writing about it, the student is no longer in a passive relation to the didactical time. Her/his personal time is linked to it, at least at certain particular moments, the moments when the student has learned something s/he did not know before.

Conclusion

In 6th form, the mathematical curriculum tends mainly to work on former knowledge (from elementary school): numbers, objects in geometry. It is meant to change the relation the students have to these objects of knowledge and to change her/his learning of mathematics.

We think that this work on narratives in the classroom, helps the students to be conscious of the transitions that are organised by the didactical system.

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THE DIDACTICAL TRANSPOSITION OF DIDACTICAL IDEAS: THE CASE OF THE VIRTUAL MONOLOGUE

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Abstract: *This paper is a variation on the theme of didactical transposition, here transposing of a theoretical idea –the virtual monologue– into a reflective tool for practitioners in an in-service teachers workshop. The transposition is effected through what has elsewhere been called aesthetical learning process. The researcher, the teacher educator, and the teachers in the workshop may have different agendas and different practices, but they all work towards their separate goals by reflecting on their practices – they are all reflective practitioners. The virtual monologue, which can be used as a reflection tool at any level, serves to bring out those commonalities.*

Keywords: reflective practitioner, didactical transposition, virtual monologue, in-service teacher education

A. Introduction

This paper is a reflection on the relationships between theory and practice in mathematics education, specifically, the practice of in-service education of teachers. We are looking for ways in which synergy can be created between the practitioner and the theoretician (or researcher) by combining the specific expertise of both. The paper itself is a case in point: though both authors are involved in both teacher education and research, the primary expertise of the first author has been in teacher education and that of the second author in research. However, both see themselves primarily as *reflective practitioners* (Schön, 1983), as will be elaborated below. The paper is essentially a case study in the *didactical transposition* (Brousseau, 1997) of a theoretical idea in mathematics education from the community of researchers to the community of practitioners; the transposition, however, is applied here not to mathematics itself but to mathematical didactics. The general approach is exemplified by studying the case of the *virtual monologue* (Leron & Hazzan, 1997) as a tool for expanding the “scope of reflection” of both communities. The virtual monologue (VM) had initially been introduced by the second author (jointly with Hazzan) as a reflection tool for researchers, but has been adopted and adapted by the first author as a tool for reflection in the professional development of teachers (Ejersbo, 2003).

B. Didactical transposition of didactical ideas

The difficulties of using theory in practice have often been discussed in the research literature (Strauss, 2001; Skott, 2004; Rasmussen, 2004). Why is it so hard, and how

can we better build bridges between theory and practice so that practitioners would be able, so to speak, to ‘walk the talk’? One of the bridges between theory and practice is the *didactical transposition*, which Brousseau (1997, p. 35) describes as follow:

To teach it, then, a teacher must reorganize knowledge so that it fits this description, this “epistemology”. This is the beginning of the process of modification of knowledge that changes its organization, its relative importance, its presentation and its genesis, following the needs of the didactical contract. We called this transformation *didactical transposition*.

Brousseau was referring to the transposition of mathematical knowledge to suit the audience of mathematics students. We propose to extend his idea to the didactical transposition of didactical knowledge itself –the theory– to suit the audience of mathematics teachers. Often the theory presented to teachers continues to be for them ‘just theory’ and is not being implemented in their practice. But when a theory is transposed into a workshop and is experienced as an emotional event, as will be demonstrated below, the participants can then reflect on the event, analyze it, and eventually use it in their practice.

C. We are all reflective practitioners

Schön (1983) talks about various professionals (such as architects, artists and baseball players) when he introduces the concepts knowing-in-practice, reflecting-in-practice, and reflecting-on-practice. We find his ideas very relevant to students, teachers, teacher educators and researchers in mathematics education. Of particular relevance is his distinction between reflection-*in*-action (e.g., by a teacher during an intensive classroom activity) and reflection-*on*-action (e.g., the same teacher reflecting on her classroom activity after school hours).

Our case study takes place in an in-service course and the theme for the transposition is *reflection*. The basis for this choice is the above notion of the *reflective practitioner* (Schön, 1983) which for us unites all level of practice and theory in mathematics education. We see teaching and learning processes as crucially involving reflection *in* and *on* practice (*ibid*), though the specific practices may vary according to the kind of learner: The pupil is a reflective practitioner when she learns mathematical ideas through reflection in and on her mathematical problem-solving or investigations; the teacher is a reflective practitioner when he consolidates his mathematical and didactical knowledge through reflection in and on his teaching practices; the teacher educator learns by reflecting in and on her practice in designing and conducting workshops for teachers; and the math education researcher gains his insights by reflecting in and on all the above practices.

There are several tools to aid the reflective practitioner, and in this paper we will focus on one such tool: the virtual monologue. Leron & Hazzan (1997) introduce the *virtual monologue* (VM) tool, where an experienced teacher or researcher uses the narrative mode (Bruner, 1985; Bruner & Haste, 1987), to *vividly convey his or her view of the student’s mental processes*. Thus the VM is one tool that helps the

reflective practitioner move from practice to theory through reflection. Our experience has been that the VM can be a powerful tool for reflection, but like all such tools it should be used with care and with awareness for its limitations and shortcomings. One obvious limitation is the subjective and ambiguous nature of any particular VM created in a particular situation by a particular person. As will be seen later, this particular limitation can sometimes be turned into an advantage by building on the variety of VMs produced in a group. A second and perhaps more serious weakness is the fact that we are using a verbal medium for describing an essentially non-verbal phenomenon – the student's mental state. A more thorough discussion of the tool's strengths and weaknesses can be found in Leron & Hazzan (1997).

In her workshops with Danish teachers, the first author has created a novel use for the VM and has used it extensively in her practice as teacher educator. In fact, she has effected a *didactical transposition* of the theoretical idea into the practice of a workshop, where the teachers experienced an emotional event that was then used for analyzing the theory and for reflecting on their own practice. Adopting terminology from art education (Horh and Pedersen, 1996), we call this procedure *aesthetical learning process* (ALP). We will explain ALP a bit more below; for now we only mention that the word 'aesthetic' is not meant here to carry connotations of beauty. Rather it is used in its ancient Greek sense of 'aisthesis', meaning 'knowledge which comes through the senses'.

Reflection on practice is clearly a vital task for teachers, but nonetheless, one that many find difficult. For a teacher educator designing an in-service course, this requires creating teaching situations that will help teachers reflect on their actions, beliefs and norms.

D. The practitioner in action

D1. Background. In Denmark, teachers are certified to teach four subjects in grades 1-10. The situation described here takes place at an in-service course for certified teachers, who in addition are specializing to become mathematics teachers. The goal of the course is to develop both mathematical and mathematical-didactical skills. The total course consists of two separate parts of 108 and 120 hours, spread out over a day a week, six hours a day. The following scene occurs half-way through the first part of the course. There are 24 teachers enrolled in the course. This course, and many like it, have been designed and carried out by the first author, who in addition kept a diary containing what she considers her reflections-on-action. The narrative below, written in her first-person voice, is an abridged and edited version of parts of that diary.

D2. Enter VM. I am an experienced teacher through many years both as a math teacher in lower secondary school and as an in-service teacher educator. Therefore one of my main habits is to look for ideas from people and from the research literature, that can be transformed into my teaching. Reading *The world according to Johnny: A coping perspective in mathematics education* (Leron & Hazzan, 1997, henceforth abbreviated L&H), especially their use of VM to interpret the interview

with Dina (p. 269; details below), affected me in different ways. For example, it moved me to write new virtual monologues “in Dina’s voice”, and it gave me ideas on how to design a reflection workshop for teachers in my in-service courses. My own feelings about the interpretation of how Dina might think inspired me to arrange the same situation for the teachers. I wanted to draw their attention to the possible interpretations, both from the student’s and the teacher’s (or the researcher’s in L&H) perspective. I hoped to discuss how they interpret *the need to make sense* (p. 274) and *the need to meet expectations* (p. 275) from the perspective of both the student and the teacher in the interview.

I was excited to adopt a coping perspective, taking an empathic attitude. I assumed that the teachers’ values would be visible through the way they expressed their empathy. Furthermore it would highlight individual differences in how they view the student’s mental processes during the solution process. And as a new direction not taken in the original article, we would try to imagine what kind of mental processes took place in the *teacher’s* (or researcher’s) head. It is easy to criticize the teacher while empathizing with the students, but here was a challenge in my course to focus on both the student’s and the teacher’s inner voice. In the design of the actual teaching, I wanted to work with various kinds of reflections; to select the article on VM was the first step.

D3. Workshop diary (part I): cognitive perspective. Here, then, is how the theoretical ideas in L&H were transposed into my practice.

I translated to Danish the main part of Section 2.2 of L&H: The task on linear equations with a parameter, the researcher’s expectations, the interview with the student (Dina) and its original interpretation, and finally, the authors’ interpretation, as seen through their virtual monologue. I reproduce for the reader three parts of that material, which are needed to understand my story. The task, the “expectations”, and the Dina interview are taken from Sfard & Linchevsky, 1994, pp. 218-220 (henceforth abbreviated S&L). For the complete discussion, including the VM analysis of the Dina interview, cf. L&H, Section 2.2.

The task (S&L): Is it true that the following system of linear equations

$$k - y = 2$$

$$x + y = k$$

has a solution for every value of k ?

The expectations of the researcher (S&L): In a problem like the present one, the objects that the students are supposed to consider are not just numbers – they are functions. To understand the question, one must realize that each of the equations, [...] represent a whole family of linear functions [...].

The interview with Dina (S&L):

(Dina is a tenth-grade student, working on the above task)

D: [reads the question silently] “... has a solution...”

I: What does it mean 'has a solution'?

D: That we can put a number instead of k and it will come out true.

I: When we say that the system has a solution for every value of k , what is the meaning of the word 'solution'? Is it a number or what?

D: Yes, it's a number.

I: One number?

D: Yes, it's the number that when you put instead of k , then the system is true.

[...]

I: This word 'solution' here, to what does it refer? Solution of what?

D: Of the equations, $k - y = 2$ and $x + y = k$.

I: What is a solution of these equations?

D: When we substitute numbers...

I: Instead of what?

D: ... instead of x , y , and k , and it comes out true.

I: So, once more, what are the solutions we are talking about in the question [points to the words 'has a solution']?

D: I think ... I think that I need three numbers: x , y , and k .

I presented the translated materials to the participants on OHP transparencies, together with L&H's first interpretation (the cognitive perspective; not included here for space limitations). The teachers saw the task on the OHP, at this stage without any discussion, but with enough time to read and think about how to solve it. I assumed that some of them would have difficulties with understanding the task just like Dina had; it was a part of my expectations. After a dramatization of the communication we looked at the interpretation about Dina's helplessness and confusion and then I started a discussion with the following question: *What is your opinion on the interview and its interpretation?*

Some reacted quickly and said:

- Dina's answers seem relatively rational and the interviewer seemed to stress her in a way that made it difficult for her to think.
- The interviewer plays the usual teacher's game 'Guess what the teacher is thinking'.

They clearly expressed understanding and sympathy for Dina.

After a while I asked them:

Could we guess what was going on in Dina's head during the interview?

One of the teachers said to me with an angry voice:

- It irritates me that you ask that question 'what goes on in her head'. I am not able to know what is going on in the head of my 24 students.

I was a little surprised but before I could answer her, one of the other teachers said to her:

- Why does it irritate you? Don't we all guess when we communicate with the students? How do *you* listen to them?

After a little discussion about communication and listening, I ended with a quotation from Covey (1989): "Try to understand before you want to be understood."

D4. Workshop diary (part 2): VM and coping perspective. Now I presented the teachers OHP transparencies with the translation of Dina's virtual monologue from L&H (pp. 271-2; The italicized phrases are taken from the actual interview with Dina, as quoted above from S&L):

What do I have here? A system of equations... Oh, well, I know how to do that. You just have to solve it. It does look a bit different, but I can just do the usual solution. [reads the question silently] "... *has a solution... for every value of k ...*" I don't understand this phrase. Why don't they just say 'solve' as they always do? I don't think we had this question before. So how can I solve it? What am I going to do? I really feel I am groping in the dark here. *What does it mean 'has a solution'?* I am not sure, but usually solution means *that we can put a number instead of k and it will come out true.*

I: When we say that the system has a solution for every value of k , what is the meaning of the word 'solution'? Is it a number or what?

I really don't know. I don't even understand the question. What was the question? "Is it a number?" well, what else could it be? I don't know. Oh, well... [performing a leap of faith] *Yes, it's a number.*

I: One number?

Of course, what else? I wish I knew where these questions are leading, I am getting more and more confused. But at least it seems from the question that I was right – it is a number. *Yes, it's the number that when you put instead of k , then the system is true.*

[...]

I: This word 'solution' here, to what does it refer? Solution of what?

What do you mean 'solution of what'? When we do equations in class we never have such questions. We just need to know how to solve them. What was the question? Solution of what? *Of the equations, $k - y = 2$ and $x + y = k$, what else could it be?*

I: What is a solution of these equations?

When we substitute numbers...

I: Instead of what?

What are the letters here? ... *instead of x , y , and k , and it comes out true.*

I: So, once more, what are the solutions we are talking about in the question [points to the words 'has a solution']?

I think ... I think that I need three numbers: x , y , and k .

I then asked:

What are your comments? And why? Where is the pain threshold in this version?

The responses again came immediately:

- No, it is not what she thinks, she thinks...

Different suggestions now filled the air:

- I have three unknown here but only two equations, strange.
- What does the k do here?
- The k must be a letter like x and y – then I just have to find the value.
- Why does she ask that way? I am sure she wants me to say something special. What could it be?

It seemed like their own difficulties made them identify with Dina.

The plenary discussion focused on what might have been going through Dina's head. After a while I turned to ask how the task could have been thought out originally, why the interviewer asked the way she did, and how *they* would have asked, if they were the interviewer. Now they faced some difficulties. It was easy for them to identify with Dina, the student, but much harder to identify with the teacher (here the interviewer), even though it should have been natural for them to think like a teacher. It was easier to criticize the teacher than to understand her. Maybe they felt resistance to the interviewer because they themselves had difficulties in solving the problem. Eventually we of course took the time to solve the problem for ourselves.

D5. Workshop diary (part 3): A VM of their own. The whole group was very engaged, even though some were initially negative. It was easy for everybody to join the discussion. For the next part I chose another transcript. My choice here was a discussion between a teacher and two students at a Danish oral examination concerning percents – an area they all felt safe with. It is a dialogue where it is easy to laugh a little at the teacher.

The participants were split in four smaller groups: Two groups would create a VM for the student and the other two for the teacher. They were given 20 minutes to do this task. Then the 'teacher-groups' and the 'student-groups' presented their VMs at the plenary meeting, followed by lively questions and discussion. It gave some new insights to all of us. Instead of only judging how the teacher asked and how he confused the students, they tried to understand and identify with him. The questions they now asked were:

- How was he caught in that trap?
- How could he come out of it without confusing the student?
- What kind of questions or comments could he make instead?
- Furthermore they started to reflect on their own way of asking, like

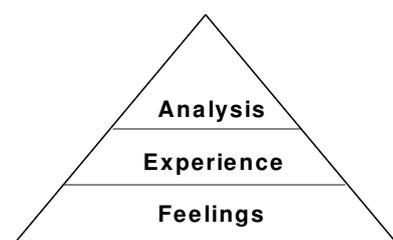
- How do I ask questions myself and what kind of answers do I expect?
- Why are my questions like they are?

One of the ‘students-groups’ gave the interpretation that the students had a clever strategy for asking the teacher questions without answering anything themselves, a strategy they haven’t noticed before, but in retrospect could now recognize in their own communication with their students. Working with VM in this way gave them the time and possibility to become aware of many more details. They were guided by their own emotional involvement and by the communication in the group. The discussion became different from what went on before: it was more balanced and contained more understanding and less criticism of the teacher.

The conclusion of my reflection on that lesson was that we all got a new experience in reflection because the situation was authentic for all of us. A few weeks later when we talked about how the course affected their teaching, I asked specifically about the influence of the VM workshop. Some teachers answered that it influenced their way of listening to themselves; they were more aware of how they asked and listened to the students; they paid more attention to the communication in the classroom; things they didn’t notice before became clearer to them. But at the same time, they also became more uncertain. What they had been doing automatically before, now all of a sudden seemed questionable, and they didn’t yet develop an alternative behaviour. Even though this has not been an easy process, I value it as a first step in learning how to reflect on communication in action.

D6. Workshop diary (part 4): concluding reflections on the practice. The way the teachers experienced the idea of VM has created for them an emotional event. The teachers became involved with their feelings, both positive and negative, and they experienced it before we did any analyzing or theorizing. In reflecting on the workshop and what it has achieved, I was aided by the *Aesthetical Learning Processes* (ALP) theoretical framework, mentioned above. In this framework, Horh and Pedersen (1996) have developed a way to express what cannot easily be communicated verbally. The method tries to shape a room for expressing experiences that are still not completely formed or completely conscious for the person involved. The idea is to create an emotional experience from the beginning, and from that platform let the event be a personal experience, a need, before doing any analyzing.

ALP can be seen as a tripartite model for the process of experiencing a new conception:



The conception arises at the moment the feelings find a conscious form, and becomes later on an experience that can be analyzed. The ALP was used in this situation to create a safe environment for the teachers to express their reflections. The VM idea was the content, the ALP the form. In this process, the teachers became familiar with research ideas and gained ownership and practise in using them.

The task of creating a VM, or trying to express what Dina was thinking and feeling, is an open problem that does not have a single solution, nor even a best one. It has brought up in the teachers many feelings and ideas, and has given them the opportunity to discuss what has come up. It was easy for them to express what they thought she might have been thinking, rather than having to learn an abstract and detached theory. It has started from their knowledge, from their understanding, from what they knew best and felt safe with. They could use experiences from their daily school life. They have acquired a tool for reflection in and on their practice. Developing knowledge in action would come if and when they are ready to use it.

This part was more or less as I planned: I wanted them to be emotionally involved and to reflect in action, and through that let their beliefs come into view. What I couldn't foresee was what kind of discussion would emerge. This is where I myself had to be the reflective practitioner and intensively reflect in action. Working in this way, the workshop facilitator may sense loss of control, having to deal with so many voices that come up from the participants, and being the one that needs to decide what kind of feedback to give, what kind of summary to make, what will be the next step, and what take-home problems to give the participants. The energy comes from all the participants, but the facilitator has to give the direction. No wonder by the end of the 6 hours I felt quite exhausted.

E. Conclusion

The gap between the communities of researchers and practitioners has often been noted and deplored. It has been our repeated experience that theoretical ideas and research papers can be powerful tools in the professional development of teachers, but only after a substantial didactical transposition. At the heart of this paper was a case study of one such didactical transposition – of the virtual monologue from a reflection tool for researchers to a reflection tool by teachers in an in-service course (and eventually, hopefully, in their own classes). The goal of the workshop was to help the teachers reflect on their own communication in the classroom. Nowadays it is demanded that teachers make many more decisions in the classroom related to the individual student, and it is furthermore expected that they be able to explain and justify their decisions. It puts them in a situation of *forced autonomy* (Skott, 2004) together with a demand for a transparent decision process. Therefore it is necessary for teachers to develop a repertoire of ways to reflect in and on their teaching, and be aware of their decisions and actions. The teachers know this; they have talked about it and about how to achieve it. But still many of them are inexperienced in those skills, hence they may resort to their old habits when stressed by the classroom situation.

The VM workshop allowed them under safe conditions to practice reflection on their own practice. Moreover, it gave them the possibility to work at the same time on the reflection process and on the mathematics involved. It turned the abstract theory into a piece of knowledge they could own and use in their practice. It is our hope that many more such transposition efforts can be successful in making theoretical ideas of the research community more accessible and useful for practitioners.

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AN INTEGRATED PERSPECTIVE TO APPROACH TECHNOLOGY IN MATHEMATICS EDUCATION

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Abstract: *This paper concerns a research work developed in an European project. The aim of this work was to produce a document integrating the different theoretical frameworks employed by the project teams. The theoretical constructs of didactical functionalities, and experimental educational cycle, associated to an ICT tool, allowed us to analyse the roles played by technology in the considered set of theoretical frameworks. With this respect, we present examples concerning the theory of didactic situation, the activity theory and the theory of instruments of semiotic mediation.*

Introduction

The project we are reporting on, is being developed in the framework of the Kaleidoscope Network of Excellence¹ which brings together European teams in technology-enhanced learning. Within the activities of the Network we are involved in the TELMA (Technology Enhanced Learning in Mathematics) project, which refers to the use of ICT (Information and Communication Technology) to improve mathematical education at school level.

The research teams involved in TELMA² aim at sharing their studies by discussing the following key themes: research area and goals, theoretical frameworks of reference, tools developed and/or used, contexts of use, work methodologies. A specific aim is to build, by means of a horizontal analysis, a document (IPTA) which represents an integrated in depth presentation of teams' approaches.

In this context, our specific work focuses on the theoretical frameworks of reference, and aims at integrating the different theoretical frameworks employed by the TELMA teams. Our working methodology is that of collecting and analysing ad hoc designed material: each team was required to write a presentation of its theoretical frameworks, and to present some selected papers. Because of the variety of the employed frameworks, an integrated vision was possible only through the definition of a perspective allowing us to analyse each framework pointing out common aspects and differences. In this paper we present such perspective giving examples of how it can

¹ "Kaleidoscope's goal is to integrate 76 research units from around Europe, covering a large range of expertise from technology to education, from academic to private research." (<http://www-kaleidoscope.imag.fr/>).

² Telma teams are the following: MeTAH and Leibniz – IMAG, Grenoble; DIDIREM University Paris 7 Denis Diderot; Istituto per le Tecnologie Didattiche (ITD) – C.N.R. of Genova; University of London (UNILON) - Institute of Education; Educational Tech Lab – NKUA University Athens.

be employed as a tool for analyzing different theoretical frameworks concerning the use of ICT in mathematics education.

1 Technology and mathematics educations

A first analysis of the collected material revealed that, the variety of theoretical frameworks depends on the involved ICT tools, and on the educational objectives addressed by each single research.

Two main kinds of ICT are involved in TELMA team's researches: those (e.g. Aplusix, l'Algebrista, ARI-LAB-2) which have been realized for explicit educational purposes (which we may call educational ICT), and those (e.g. CAS and Spreadsheet) that have been realized for professional purposes (professional ICT).

The researches involving educational ICT, in some cases, focus only on the use of ICT in educational practices, in other cases they consider the whole lifecycle of the tools, from the design to the actual use in educational practices and evaluation. In the case of professional ICT, TELMA teams have been focusing only on the educational use of the software, but not in their development.

Moreover it turns out that the teams address specific educational goals (for instance introducing pupils to symbolic manipulation, to geometry, to algebra, to proofs, etc.), referring to different theoretical frameworks and employing different ICT tools. In particular, we observed that a theoretical framework influences how a given ICT tool is employed in order to achieve a given educational goal, or in other cases it influences how an ICT tool is designed and developed to be used to achieve a given educational goal. This suggested us to consider the following primitives for our work: ICT tools, specific educational goals, how the ICT tools can be employed in order to achieve the given educational goals. We present a perspective, based on the concept of *didactical functionalities*, where we can define the relationships among such primitives.

2 Didactical functionalities of ICT tools

Given an ICT tool, and an educational goal, it is possible to identify its *didactical functionalities*:

With didactical functionalities we mean those properties (or characteristics) of a given ICT, and/or its (or their) modalities of employment, which may favor or enhance teaching/learning processes according to a specific educational goal.

The three key elements of the definition of the *didactical functionalities* of an ICT tool are:

1. a set of *features/characteristics of the tool*;
2. a *specific educational goal*;
3. a set of *modalities of employing* the tool in a teaching/learning process referred to the chosen educational goal.

For what concerns the features and characteristics of ICT tools, we focus on the distinction between professional and educational ICT.

An educational ICT tool provides, because of its nature, a set of such functionalities. In fact we assume that the producers of the tool, not only design it with respect to a set of specific educational goals, but we assume that they also consider the possible modalities of employment of the tools in order to achieve such goals. In other words educational ICT tools are designed together with a set of *didactical functionalities*.

On the other hand professional ICT tools are not designed considering a possible educational goal and related modalities of employment: they are designed without a set of *didactical functionalities*. Nevertheless professional ICT tools may provide features that can be interpreted in terms of didactical functionalities, that is, we can identify modalities of employment of such tools aiming at the achievement of a given educational goal.

In general, the didactical functionalities can be defined/individuated either at the level of the design phase, or at the educational use phase. Thus in the case of professional ICT, the definition of didactical functionalities occurs only in the utilization phase, whilst in the case of educational ICT, they surely occur in the design phase, but may occur also in the educational use phase.

In the perspective we are proposing, in order to exploit a given ICT tool as a mean for achieving a given educational goal, it is needed to define the *modalities of employment* of the tool, which depend on the chosen theoretical framework of reference. In fact, in the researches of TELMA teams not only we found different theoretical frameworks, but we found also that ICT tools are employed in different phases of teaching/learning processes, and with different aims. For this reason we built a model allowing us to characterize such phases in which an ICT tool can be employed. The model is to be intended as a tool for classifying the modalities of employment defined by TELMA teams.

3 A model to classify the *modality of employment* of ICT tools

With respect to the definition of *didactical functionalities*, we shall observe that, given an ICT tool, the definition involves at least the tool itself, one learner and an interaction among them oriented toward a specific educational goal. However in the considered teaching/learning process other factors may play crucial roles. For instance, among the factors allowing an effective exploitation of the *didactical functionalities* of an ICT, we may consider: the context (is it on line, in class, or in a laboratory and so on), the proposed educational activities, the teacher and his/her strategies, national curricula etc.

TELMA teams employ ICT tools according to quite different modalities of employment. For this reason we developed a model, named *Educational Experiment Cycle* (EEC), to help us to classify such modalities (See Fig. 1).

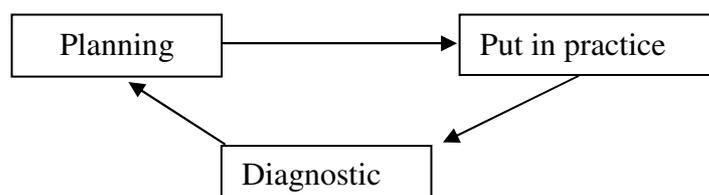


Fig. 1: Educational Experiment Cycle

The model attempts to describe the basic phases of a teaching/learning activity individuating three phases: the *planning* of the teaching/learning activity; the *put in practice* of the teaching/learning activity; the *diagnostic* phase.

Given an educational goal, the *planning* phase consists of the design and setting up of an activity (or sequence of activities) aiming at reaching the educational goal. The *put in practice* phase consists of the actual implementation of the planned activity (or sequence). The *diagnostic* phase consists of some evaluation of the actors involved in the cycle, could they be learners or teachers, with respect to the assumed educational goal.

4 Influence of theoretical frameworks on ICT tools didactical functionalities and on the Educational Experiment Cycle

In our perspective, the specific theoretical frameworks can be interpreted as instruments for defining the relationships among the primitives that characterise the concept of *didactical functionalities*. In this section we exemplify how the choice of a given theoretical framework can influence the definition of the didactical functionalities of ICT tools, either in terms of the design of the tools, or in terms of design of the modalities of employment. Moreover we show where the considered theoretical frameworks have been employed in different phases of the *EEC*.

The choice of the tools, and of the *modalities of employment* depend on the chosen framework of reference. Here we will consider (among the set of frameworks of TELMA teams) the theory of *didactic situations* (TDS) (Brousseau, 1986), the *Activity theory* (AT) (Nardi, 1996), and the theory of *instruments of semiotic mediation* (TISM) (Mariotti, 2002; Cerulli, 2004; Cerulli & Mariotti, 2003), and we will consider the example of symbolic manipulators employed to introduce pupils to symbolic manipulation. A comprehensive description of the three theories is beyond the scope of this paper, thus we limit ourselves to point out some key ideas and show how they can influence the definition of the didactical functionalities of ICT tools, and of symbolic manipulators in particular.

4.1 Defining didactical functionalities of an ICT according to the theory of didactic situations, in order to introduce pupils to symbolic manipulation

According to the TDS, learning happens by means of a continuous interaction between subject and milieu: each action of the subject in the milieu, is followed by a retro-action of the milieu itself, and learning happens through a spontaneous adaptation of the subject to the milieu, which is considered to be “milieu antagoniste” (Brousseau, 1986). One way of applying this key idea to the domain of educational ICT, is that of considering an ICT tool as an element of the milieu, and as such, its retroactions become a source for learning (by means of interaction with the ICT tool) in terms of the subject’s adaptation to the milieu.

Within this perspective, if we are given an ICT tool, a first *modality of employment* of the tool to achieve a given educational goal, is that of setting up a situation in which learners interact with the tool receiving a relevant feedback. In this case, the tool could be employed either during the *planning phase* or during the *put in practice* phase of the *EEC*.

For instance, suppose that a teacher wants to set up a situation, involving a symbolic manipulator, where the student is required to transform an algebraic expression into another one, producing a chain of transformations. Following the TDS, the teacher may a-priori analyze the possible actions performed by the learner and the consequent retroactions of the symbolic manipulator. In this *planning phase*, of the *EEC*, the he/she may employ the ICT tool in order to investigate its retroactions. The teacher may thus individuate those retroactions that can be particularly relevant/effective for his/her specific educational goal, and, in the *put in practice phase*, he/she may submit to pupils a task that involves such particular retroactions.

For instance, if the focus is on the role of the brackets in algebraic expressions, and if the considered symbolic manipulator gives a particular feedback when the user tries to remove brackets from an expression, then the teacher may set up a task that involves removing of brackets in order to exploit the feedback provided by the software.

In summary, the TDS can be used in order to individuate *didactical functionalities* of an ICT tool with respect of a given educational goal, by defining its the *modalities of employment* in terms of setting up an ad hoc designed situation that exploit users' interaction with the ICT tool and the provided feedback.

If we want to design an educational ICT tool to be employed according to this perspective, special attention has to be paid to interaction issues and to the feedback offered. In the example that we discussed, the feedback could be very trivial or more complex; it could just inform the user if removing a couple of brackets is correct or incorrect, or it could also explain why the removal of a couple of brackets is correct or incorrect; it could allow incorrect removing of brackets signaling the error (or signaling nothing!), or it could simply not allow incorrect removing. Each of these different kinds of feedback could be exploited by setting up different kinds of situations. Actually among the researches of TELMA teams we find examples in which symbolic manipulators are developed within the perspective of the theory of didactic situations, and particular attention is paid to the feedback offered by the developed symbolic manipulators (Nicaud, 1994). In particular we find the example of Aplusix, developed within the framework of didactic situations by the IMAG team, which, in the case of incorrect removal of brackets, signals the error by means of a visual feedback (See Fig. 2).

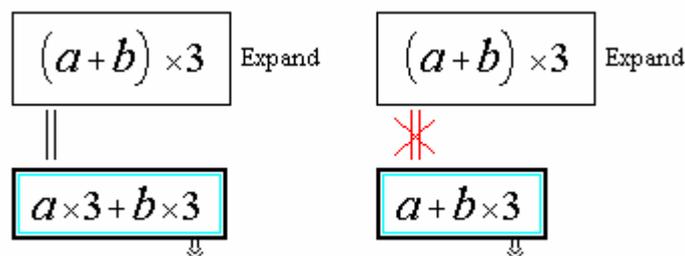


Fig. 2: Feedback in Aplusix in the case of incorrect removal of brackets a red cross appears between the old and the new expression.

4.2 Defining didactical functionalities of an ICT according to Activity Theory (AT), in order to introduce pupils to symbolic manipulation

The key concept of AT is the notion of *activity*, which is interpreted as a form of doing directed to an object. This theory provides a model to describe the structure of any human activity together with the transformations it undergoes during its evolution. The model, proposed by Engestrom and Cole (Nardi, 1996), concerns human activities in general, but can be used also to describe the system of relationships characterizing a teaching/learning activity. This model assigns a crucial mediation role to the instruments, the rules, and the division of labour in the three relationships characterizing any human activity, that is the relationships between subject and object, between subject and community, between community and object. According to this theory an activity can evolve, during its development, when contradictions or breakdowns occur, forcing a change of focus in the activity, thus forcing a transformation of its structure. In other words, during the development of an activity, pupil's actions, teacher's actions, or other events can cause a change of the object or of the relationships characterising the activity itself; in this sense the teacher, which is a co-actor of the activity, can administrate/control/cause such changes, thus guiding the development of the activity according to his/her educational goal or to the exigencies of the class.

In this perspective, an ICT tool is not considered as antagonist to the subjects (as in the case of the *mileu antagoniste* of the TDS), on the contrary, it is considered to be a cooperative environment. When a learner uses an instrument for achieving an objective within an activity, the learning outcomes are considered to be structured by the nature of the activity itself and by roles played by all its components. Consequently, given an educational goal, the AT can be used to define the *modalities of employment* of a given ICT tool in terms of setting up an activity, involving the tool, and based on the cooperation of all participants. In other words, the *didactical functionalities* of the tool are defined in terms of how it can structure activities, rather than in terms of the retroactions given to the user as in the case of TDS; of course also such retroactions are to be considered, because they influence the relation between learner and tool, but they are not the main focus.

If we want to design an educational ICT tool to be employed according to the AT perspective, special attention has to be paid to the tool's potentialities of interaction, communication and visualization.

Among the researches of TELMA teams we find examples in which symbolic manipulators are developed within the perspective of AT. Here we refer to the system of ARI-LAB-2, a software for the arithmetic problem solving, developed by the ITD (Istituto per le Tecnologie Didattiche - CNR Genova) team.

In this case the ICT tool is used by the team in the *planning* and *put in practice* phase. ARI-LAB-2 consists of a set of microworlds and two modes of interaction, the *teacher mode*, and the *pupil mode*. In the pupils mode it is possible to interact with the software solving tasks within one, or more, of the available microworlds. Not all the developed microworlds are always available to the pupils, in fact in the teacher mode it is possible

to set up tasks to be submitted to pupils, and for each task it is possible to choose which specific microworlds shall be accessible to the pupil in order to solve the task. In other words the modalities of employment of the ITC tool involve both, the *planning* and the *put in practice phase*.

In particular, in the *planning phase* the ARI-LAB-2 can be used by the teacher to set up an activity (aimed at developing certain arithmetical competencies) in terms of defining the characteristics of the microworlds available to the user (See Fig. 3).

In the *put in practice phase*, learning is assumed to be an outcome of the planned activity which involves among other elements, the pupils and the ICT tool. As a consequence, configuring the tool, is a way, for the teacher, to define specific *didactical functionalities* as means for achieving her specific educational goals. In other words, the *didactical functionalities* are individuated in terms of the activities that can be set up and managed by the teacher.

For instance in order to introduce rules for adding fractions, the teacher can direct the focus back and forth between the fraction microworld, where the rules are explored dynamically and geometrically, and the symbolic manipulator microworld, where the rules are proven using a given set of axioms (See Fig. 3).

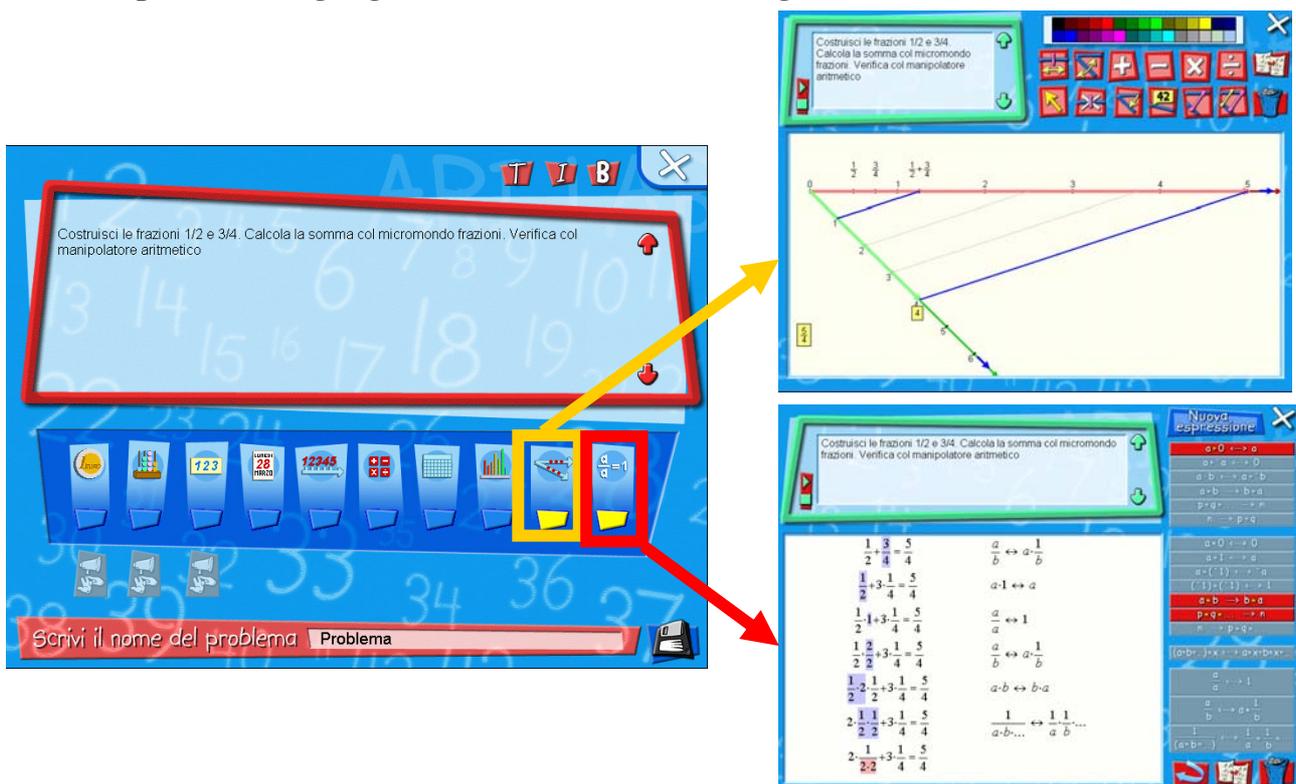


Fig. 3: In the teacher mode (on the left) the fraction microworld (top right) and the symbolic manipulator microworld (bottom left) are selected and are available for pupils' problem solving.

4.3 Defining didactical functionalities of an ICT according to the theory of instruments of semiotic mediation, in order to introduce pupils to symbolic manipulation

A different perspective is that of the theory of the *instruments of semiotic mediation* (TISM), which, like the AT, is derived from the theories of Vigotskij. The key hypothesis of this theory is that meanings are rooted in the phenomenological experience, but they can evolve, under the guidance of the teacher, by means of special communication strategies (Mariotti, 2002), such as for instance that of the mathematical class discussions (Bartolini Bussi, 1996). Without going deeply in detail in the description of this theory, we observe that it assumes that a part of the teaching/learning process happens at the semiotic level, and that it depends strictly on the signs that can be derived from the considered ICT tool, and can be employed by the teacher as means for orchestrating relevant mathematical class discussions. In other words the *modalities of employing* an ICT tool within this perspective consist on the one hand of setting up ad hoc designed activities involving the tool, and on the other hand of orchestrating mathematical discussions using signs derived from the ICT tool. Consequently for this theory, it is particularly important to study what kinds of signs (words, formulas, gestures, etc.) can be derived from a given ICT tool in order to orchestrate a mathematical discussion relevant for the chosen educational goal. For instance, if we take the example we discussed previously in the case of TDS, we considered the issue of removing brackets in algebraic expressions. Such an issue has been addressed by the ITD team of TELMA when they developed the symbolic manipulator L'Algebrista (Cerulli 2004), and the chosen strategy was that of providing the software with a button to be used to remove brackets; such a button does not check if the operation is correct or not, it just executes it, thus it may produce incorrect transformations of algebraic expressions, giving no feedback. However the interface of the software associates a formula to the button (" $(a+b) \rightarrow a+b$ "), together with a peculiar name "risky button" which is used by the teacher in the *put in practice phase*, during mathematical discussions, as a means for focusing pupils attention on the "risks" of removing brackets from algebraic expressions. In this case the provided feedback is not the most important element contributing to the achievement of the educational goal. In fact the most important element is the communication strategy that can be developed by the teacher with reference to the ICT tool.

4.4 Employing ICT tools in the *diagnostic phase* of the *EEC*

ICT tools can be employed for educational purposes at any stage of an *EEC*, exploiting their *educational functionalities* as means for reaching a given educational goal. The examples we presented concern the *planning* and the *put in practice phase* of the *EEC*, however, among TELMA researches we find also examples concerning the *diagnostic phase*.

For instance in the Lingot project (<http://pepite.univ-lemans.fr/>), the DIDIREM team research aims at developing diagnostic and remedial tools in elementary algebra,

testing them with students and also studying how teachers appropriate the use of such tools. The hypothesis is that there exists some coherence in student's behavior. Thus understanding this coherence and how it can evolve is a necessity for developing effective diagnostic and didactic strategies based on this diagnostic. Then, the TDS is used for supporting the conception of tasks linked to the diagnostic. In this case, the definition of the *modalities of employment*, of the used ICT tool, is based on the idea that the teacher submits to pupils a *diagnostic* activity based on the tool, and the feedback received by the teacher is used as a basis for *planning* (according to the TDS) the tasks to be submitted to pupils in the *put in practice* activity. In other words the ICT tools is employed in the diagnostic phase of the *EEC*, and the provided feedback contributes to the setting up (*planning phase*) of the situations to be submitted to pupils, in order to achieve the given educational goal in the *put in practice* phase. The peculiarity of this perspective is that the ICT tools are employed by the teacher as sources of information rather than as mediators directly fostering pupils learning: the didactic (or adidactic) situations, planned for fostering learning may even not include an ICT tool at all, even if they have been planned with the aid of a diagnostic ICT tool.

4.5 Some remarks and conclusions

We observe that the designer of an educational ICT tool, provides it with a certain set of *didactical functionalities* according to a given theoretical framework of reference. However it may happen that someone else decides to employ the same tool to achieve the same educational goal, but taking the perspective of another framework of reference. If that is the case, the individuated *didactical functionalities* will be different from those implemented by the designer. An example is the research brought forward by the University of Siena team where the Cabri-Geometry software for introducing pupils to geometrical constructions, designed within the framework of TDS, is used by the team according to the TISM (Mariotti 2002). In this case the *didactical functionalities* defined by the developer of the software are different from the *didactical functionalities* defined by the TELMA team because even if educational goal and ICT tool coincide, the *modalities of employment* are different.

It is interesting to observe that in this example, like in the other examples we presented, ICT tools are provided with very different didactical functionalities, depending on the different theoretical frameworks that assign very different roles to the tool itself, to the learners, and to the teacher.

We considered the case of TDS that is based on Piaget's theories, according to which, the cognitive development of each individual follows biological stages driving the movement from one stage to the next. In this context, in order to teach a mathematical concept, it is important that the teacher, in the *planning phase*, sets up a fundamental situation (adidactic situation) which will be the point of departure to create an antagonist system for pupils, the milieu, which includes the ICT tool. The role of the teacher is to construct the condition under which the responsibility of the solution of the task is entirely submitted to the student in the *put in practice phase*; in

this phase the interaction between student and tool (included in the milieu) is the main source of learning.

On the other hand AT and the TISM, are both based Vigotskij's socio-historical theory. In this theory the student's cognitive development has to be understood as taking place in the interaction with other members of the society, in particular with the teacher and other members of the class. In this perspective, the teacher assumes a key role in the *put in practice phase*, for instance in the TISM, the teacher plays the central role of orchestrating mathematical discussions arising from students interaction with the ICT tool.

In all these cases, the learning outcomes depend strongly on the tools used, but in the case of Vigotskian theories we find a strong dependence on cultural settings which may not be so relevant in the case of Piagetian theories such as TDS. A direct implication is that when defining the didactical functionalities of a tool, a different attention is put on the social context according to the theoretical framework used.

In conclusion, we showed how the constructs of didactical functionalities and the *EEC*, allowed us to analyse the roles played by technology in some examples. We hypothesize that the introduced constructs can be used to extend the analysis and comparison of researches, including also researches outside the TELMA project but concerning educational use of ICT tools.

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